UNIQUENESS OF CONFORMAL MEASURES AND LOCAL MIXING FOR ANOSOV GROUPS.

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Abstract. In the late seventies, Sullivan showed that for a convex cocompact subgroup $\Gamma$ of $\text{SO}^\circ(n,1)$ with critical exponent $\delta > 0$, any $\Gamma$-conformal measure on $\partial H^n$ of dimension $\delta$ is necessarily supported on the limit set $\Lambda$ and that the conformal measure of dimension $\delta$ exists uniquely. We prove an analogue of this theorem for any Zariski dense Anosov subgroup $\Gamma$ of a connected semisimple real algebraic group $G$ of rank at most 3.

1. Introduction

Let $(X, d)$ be a Riemannian symmetric space of rank one and $\partial X$ the visual boundary of $X$. Let $G = \text{Isom}^+ X$ and $\Gamma < G$ a non-elementary discrete subgroup. Fixing $o \in X$, a Borel probability measure $\nu$ on $\partial X$ is called a $\Gamma$-conformal measure of dimension $s > 0$ if for all $\gamma \in \Gamma$ and $\xi \in \partial X$,\[ \frac{d\gamma_* \nu}{d\nu}(\xi) = e^{s \beta_\xi(o, \gamma o)} \]
where $\beta_\xi(x, y) = \lim_{z \to \xi} d(x, z) - d(y, z)$ denotes the Busemann function.

Let $\delta > 0$ denote the critical exponent of $\Gamma$. The well-known construction of Patterson and Sullivan ([8], [12]) provides a $\Gamma$-conformal measure of dimension $\delta$ supported on the limit set $\Lambda$, called the Patterson-Sullivan (PS) measure. A discrete subgroup $\Gamma < G$ is called convex cocompact if $\Gamma$ acts cocompactly on some nonempty convex subset of $X$.

Theorem 1.1 (Sullivan). [12] If $\Gamma$ is convex cocompact, any $\Gamma$-conformal measure on $\partial X$ of dimension $\delta$ is necessarily supported on $\Lambda$. Moreover, the PS-measure is the unique $\Gamma$-conformal measure of dimension $\delta$.

In this paper, we extend this result to Anosov subgroups, which is a higher rank analogue of convex cocompact subgroups of rank one groups. Let $G$ be a connected semisimple real algebraic group and $P$ a minimal parabolic subgroup of $G$. Let $\mathcal{F} := G/P$ be the Furstenberg boundary, and $\mathcal{F}^{(2)}$ the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$. In the whole paper, we let $\Gamma$ be a Zariski dense Anosov subgroup $G$ with respect to $P$. This means that there exists a representation $\Phi: \Sigma \to G$ of a Gromov hyperbolic group $\Sigma$ with $\Gamma = \Phi(\Sigma)$, which induces a continuous equivariant map $\zeta$ from the Gromov boundary

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\[ \partial \Sigma \text{ for } \mathcal{F} \text{ such that } (\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)} \text{ for all } x \neq y \in \partial \Sigma. \] This definition is due to Guichard-Wienhard [5].

Let \( A \) be a maximal real split torus of \( P \) and \( \mathfrak{a} := \text{Lie}(A) \). Given a linear form \( \psi \in \mathfrak{a}^* \), a Borel probability measure \( \nu \) on \( \mathcal{F} \) is called a \((\Gamma, \psi)\)-conformal measure if, for any \( \gamma \in \Gamma \) and \( \xi \in \mathcal{F} \),

\[ \frac{d\gamma \ast \nu}{d\nu}(\xi) = e^{\psi(\beta(e, \gamma))} \] (1.2)

where \( \beta \) denotes the \( \mathfrak{a} \)-valued Busemann function (see (2.1) for the definition). Let \( \Lambda \subset \mathcal{F} \) denote the limit set of \( \Gamma \), which is the unique \( \Gamma \)-minimal subset. A \((\Gamma, \psi)\)-conformal measure supported on \( \Lambda \) will be called a \((\Gamma, \psi)\)-PS measure. Finally, a PS measure means a \((\Gamma, \psi)\)-PS measure for some \( \psi \in \mathfrak{a}^* \).

Let \( \mathcal{L}_\Gamma \subset \mathfrak{a}^+ \) denote the limit cone of \( \Gamma \). Benoist [1] showed that \( \mathcal{L}_\Gamma \) is a convex cone with non-empty interior, using the well-known theorem of Prasad [9] on the existence of an \( \mathbb{R} \)-regular element in any Zariski dense subgroup of \( G \). Let \( \psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\} \) denote the growth indicator function of \( \Gamma \) as defined in (2.2). Set

\[ D_\Gamma^+ := \{ \psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma, \ \psi(u) = \psi_\Gamma(u) \text{ for some } u \in \mathcal{L}_\Gamma \cap \text{int} \mathfrak{a}^+ \} \] (1.3)

As \( \Gamma \) is Anosov, for any \( \psi \in D_\Gamma^+ \), there exists a unique unit vector \( u \in \text{int} \mathcal{L}_\Gamma \), such that \( \psi(u) = \psi_\Gamma(u) \), and a unique \((\Gamma, \psi)\)-PS measure \( \nu_\psi \). Moreover, this gives bijections among

\[ D_\Gamma^+ \simeq \{ u \in \text{int} \mathcal{L}_\Gamma : \|u\| = 1 \} \simeq \{ \text{PS measures on } \Lambda \} \] (see [1], [6]). When \( G \) has rank one, \( D_\Gamma^+ = \{ \delta \} \). Therefore the following generalizes Sullivan’s theorem [1.1]

**Theorem 1.4.** Let \( \text{rank } G \leq 3 \) and \( \psi \in D_\Gamma^+ \). Any \((\Gamma, \psi)\)-conformal measure on \( \mathcal{F} \) is necessarily supported on \( \Lambda \). Moreover, the PS measure \( \nu_\psi \) is the unique \((\Gamma, \psi)\)-conformal measure on \( \mathcal{F} \).

Our proof of Theorem [1.4] is obtained by combining the rank dichotomy theorem established by Burger, Landesberg, Lee, and Oh [2] and the local mixing property of a generalized Bowen-Margulis-Sullivan measure associated to any \((\Gamma, \psi)\)-conformal measures on \( \mathcal{F} \), which generalizes the work of Edwards, Lee and Oh [4]. Indeed, our proof yields that under the hypothesis of Theorem [1.4] any \((\Gamma, \psi)\)-conformal measure on \( \mathcal{F} \) is supported on the \( u \)-directional radial limit set \( \Lambda_u \) (see (3.3)) where \( \psi(u) = \psi_\Gamma(u) \).

**Open problem:** Is Theorem [1.4] true without the hypothesis \( \text{rank } G \leq 3 \)?

**2. Local mixing of Generalized Bowen-Margulis-Sullivan measures**

In this section, let \( \Gamma \) be a Zariski dense discrete subgroup of a semisimple real algebraic group \( G \). Let \( P = MAN \) be the Langlands decomposition so that \( A \) is a maximal real split torus, \( M \) is the centralizer of \( A \) and \( N \) is the unipotent radical of \( P \).
In [4] Prop. 6.8, we proved that local mixing of a BMS-measure implies local mixing of the Haar measure on $\Gamma \backslash G/M$. In this section, we provide a generalized version of this statement, where we replace the Haar measure by any generalized BMS-measure and also work on the space $\Gamma \backslash G/M$. We refer to [4] for a more detailed description of a generalized BMS-measure, but briefly recall its definition here.

Let $a = \log A$ and fix a positive Weyl chamber $a^+$. We also fix a maximal compact subgroup $K < G$ so that the Cartan decomposition $G = K(\exp a^+)K$ holds. Denote by $\mu : G \to a^+$ the Cartan projection, i.e., for $g \in G$, $\mu(g) \in a^+$ is the unique element such that $g \in K \exp(\mu(g))K$. Denote by $L_\Gamma \subset a^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$. The $a$-valued Busemann function $\beta : F \times G \times G \to a$ is defined as follows: for $\xi \in F$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi) \tag{2.1}$$

where $\sigma(g^{-1}, \xi) \in a$ is defined by the relation $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for $\xi = kP$, $k \in K$.

The growth indicator function $\psi_T : a^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_T(tu) = t\psi_T(u)$ for all $t > 0$, such that for any unit vector $u \in a^+$,

$$\psi_T(u) := \inf_{u \in C, \text{open cones} \subset C} \tau_C \tag{2.2}$$

where $\tau_C$ is the abscissa of convergence of $\sum_{\gamma \in \Gamma, \mu(\gamma) \in C} e^{-t\|\mu(\gamma)\|}$ and the norm $\| \cdot \|$ on $a$ is the one induced from the Killing form on $g$. Denote by $w_0 \in K$ a representative of the unique element of the Weyl group $N_K(A)/M$ such that $\text{Ad}_{w_0} a^+ = -a^+$. The opposition involution $i : a \to a$ is defined by $i(u) = -\text{Ad}_{w_0}(u)$. Note that $i$ preserves $\text{int} L_\Gamma$.

For $g \in G$, we set $g^+ = gP \in F$ and $g^- = gw_0P \in F$. Then map

$$gM \to (g^+, g^-, \beta_{g^+}(e, g))$$

gives a homeomorphism $G/M \simeq F(2) \times a$, called the Hopf parametrization of $G/M$.

The generalized BMS-measure $m_{\nu_1, \nu_2}$. For a pair of linear forms $\psi_1, \psi_2 \in a^*$, and a pair of $(\Gamma, \psi_1)$ and $(\Gamma, \psi_2)$ conformal measures $\nu_1$ and $\nu_2$ respectively, define a locally finite Borel measure $\tilde{m}_{\nu_1, \nu_2}$ on $G/M$ as follows: for $g = (g^+, g^-, b) \in F(2) \times a$,

$$d\tilde{m}_{\nu_1, \nu_2}(g) = e^{\psi_1(\beta_{g^+}(e, g)) + \psi_2(\beta_{g^-}(e, g))} \, d\nu_1(g^+) d\nu_2(g^-) db. \tag{2.3}$$

where $db = d\ell(b)$ is the Lebesgue measure on $a$. By abuse of notation, we also denote by $\tilde{m}_{\nu_1, \nu_2}$ the $M$-invariant measure on $G$ induced by $\tilde{m}_{\nu_1, \nu_2}$. This is always left $\Gamma$-invariant and we denote by $m_{\nu_1, \nu_2}$ the $M$-invariant measure on $\Gamma \backslash G$ induced by $\tilde{m}_{\nu_1, \nu_2}$.
PS-measures on $gN^\pm$. Let $N^- = N$ and $N^+ = w_0NW_0^{-1}$. To a given $(\Gamma, \psi)$-conformal measure $\nu$ and $g \in G$, we define the following associated measures on $gN^\pm$: for $n \in N^+$ and $h \in N^-$,
\[
d\mu_{gN^+\nu}(n) := e^{\psi(\beta(gn)^+)(e,gn))}d\nu((gn)^+),
\]
\[
d\mu_{gN^-\nu}(h) := e^{\psi(\beta(gh)^-)(e,gh))}d\nu((gh)^-).
\]
Note that these are left $\Gamma$-invariant; for any $\gamma \in \Gamma$ and $g \in G$, $\mu_{\gamma g N^{\pm}\nu} = \mu_{gN^{\pm}\nu}$. For a given Borel subset $Y \subseteq \Gamma \backslash G$, define the measure $\mu_{gN^{\pm}\nu}|_Y$ on $N^+$ by
\[
d\mu_{gN^{\pm}\nu}|_Y(n) = \mathbf{1}_Y([g]n)d\mu_{gN^{\pm}\nu}(n);
\]
note that here the notation $|_Y$ is purely symbolic, as $\mu_{gN^{\pm}\nu}|_Y$ is not a measure on $Y$. Set $P^\pm := MAN^\pm$. The following were obtained in [4]:

**Lemma 2.4.** [4] Lem. 5.6, Cor. 5.7] We have:

1. For any fixed $\rho \in C_c(N^\pm)$ and $g \in G$, the map $N^\pm \to \mathbb{R}$ given by $n \mapsto \mu_{gN^\pm\nu}(\rho)$ is continuous.
2. Given $\varepsilon > 0$ and $g \in G$, there exist $R > 1$ and a non-negative $\rho_{g,\varepsilon} \in C_c(N_R)$ such that $\mu_{gN^\pm\nu}(\rho_{g,\varepsilon}) > 0$ for all $n \in N^\pm$.

**Lemma 2.5.** [4] Lem. 4.2] For any $g \in G$, $a \in A, n_0, n \in N^+$, we have
\[
d\mu_{gN^+\nu}(an_0a^{-1}) = e^{-\psi(\log a)}d\mu_{gna_0N^+\nu}(n).
\]

**Lemma 2.6.** [4] Lem. 4.4 and 4.5] For $i = 1, 2$, let $\psi_i \in a^*$ and $\nu_i$ a $(\Gamma, \psi_i)$-conformal measure. Then

1. For $g \in G$, $f \in C_c(gN^+P^-)$, and $nham \in N^+NAM$,
\[
\tilde{m}_{\nu_1, \nu_2}(f) = \int_{N^+} \left( \int_{NAM} f(gnham)e^{(\psi_1-\psi_2)\log a}dm \right) d\mu_{gN^+\nu_2}(h) d\mu_{\nu_1}(n).
\]
2. For $g \in G$, $f \in C_c(gP^-N^+)$, and $hamn \in NAMN^+$,
\[
\tilde{m}_{\nu_1, \nu_2}(f) = \int_{NAM} \left( \int_{N^+} f(ghamn)d\mu_{ghamnN^+, \nu_1}(n) \right) e^{-\psi_2\log a}dm d\mu_{gN^+\nu_2}(h).
\]

**Local mixing.** Let $P^0$ denote the identity component of $P$ and $\mathcal{F}_\Gamma$ denote the set of all $P^0$-minimal subsets of $\Gamma \backslash G$. While there exists a unique $P$-minimal subset of $\Gamma \backslash G$ given by $\{g \in \Gamma \backslash G : g^+ \in \Lambda\}$, there may be more than one $P^0$-minimal subset. Note that $\#\mathcal{F}_\Gamma \leq [P : P^0]$.

We fix a unit vector $u \in L_\Gamma \cap \text{int } a^+$, and set $a_t = \exp(tu)$ for $t \in \mathbb{R}$. For each $i = 1, 2$, we also fix a $(\Gamma, \psi_i)$-conformal measure $\nu_i$ on $\mathcal{F}$ for $\psi_i \in a^*$.

In the rest of the section, we assume that the associated BMS-measure $m = m_{\nu_1, \nu_2}$ satisfies the local mixing property for the $\{\exp tu\}$-action in the following sense: there exists a proper continuous function $\Psi : (0, \infty) \to (0, \infty)$ such that for all $f_1, f_2 \in C_c(\Gamma \backslash G)$,
\[
\lim_{t \to +\infty} \Psi(t) \int_{\Gamma \backslash G} f_1(xa_t)f_2(x)dm(x) = \sum_{Y \in \mathcal{F}_\Gamma} m|_Y(f_1)m|_Y(f_2). \tag{2.7}
\]
The main goal in this section is to obtain the local mixing property for a generalized BMS-measure from that of \( m \):  

**Theorem 2.8.** For \( i = 1, 2 \), let \( \lambda_i \) be a \((\Gamma, \varphi_i)\)-conformal measure on \( \mathcal{F} \) for \( \varphi_i \in A^s \). Then for all \( f_1, f_2 \in C_c(\Gamma\backslash G) \), we have 

\[
\lim_{t \to +\infty} \Psi(t) e^{(\varphi_1-\varphi_2)(tu)} \int_{\Gamma\backslash G} f_1(xa_t) f_2(x) \, dm_{\lambda_1, \lambda_2}(x) = \sum_{Y \in \mathcal{Y}_t} m_{\nu_1, \nu_2} |Y(f_1)| m_{\lambda_1, \nu_2} |Y(f_2)|.
\]

In order to do this, we first deduce equidistribution of translates of \( \mu_{gN, \nu_1} \) from the local mixing property of \( m \) (Proposition 2.9), and then convert this into equidistribution of translates of \( \mu_{gN, \lambda_1} \) (Proposition 2.13). Let \( G_{\varepsilon} \) denote the \( \varepsilon \)-neighborhood of \( e \) in \( G \). We also use the notation \( S_{\varepsilon} := S \cap G_{\varepsilon} \) for any subset \( S \) of \( G \).

**Proposition 2.9.** For any \( x = [g] \in \Gamma\backslash G \), \( f \in C_c(\Gamma\backslash G) \), and \( \phi \in C_c(N^+) \),

\[
\lim_{t \to \infty} \Psi(t) \int_{N^+} f(xna_t) \phi(n) \, d\mu_{\nu_1}(n) = \sum_{Y \in \mathcal{Y}_t} m_{\nu_1} |Y(f)| \mu_{gN, \nu_1} |Y(\phi)|.
\]

**Proof.** Let \( x = [g] \), and \( \varepsilon_0 > 0 \) be such that \( \phi \in C_c(N_{\varepsilon_0}^+) \). For simplicity of notation, we write \( d\mu_{\nu_1} = d\mu_{gN, \nu_1} \) throughout the proof. By Lemma 2.4 we can choose \( R > 0 \) and a nonnegative \( \rho_{g, \varepsilon_0} \in C_c(N_R) \) such that 

\[
\mu_{ghN, \nu_1}(\rho_{g, \varepsilon_0}) > 0 \quad \text{for all} \quad h \in N_{\varepsilon_0}^+.
\]

Given arbitrary \( \varepsilon > 0 \), choose a non-negative function \( q_\varepsilon \in C_c(A_\varepsilon M_\varepsilon) \) satisfying \( \int_{A M} q_\varepsilon(am) \, da \, dm = 1 \). Then

\[
\int_{N^+} f(xna_t) \phi(n) \, d\mu_{\nu_1}(n) = \quad (2.11)
\]

\[
\int_{N^+} f(xna_t) \phi(n) \left( \frac{1}{\mu_{gN, \nu_2}(\rho_{\varepsilon_0})} \right) \int_{N^+} \rho_{g, \varepsilon_0}(h) q_\varepsilon(am) \, da \, dm \mu_{gN, \nu_1}(h) \, d\mu_{\nu_1}(n)
\]

\[
= \int_{N^+} \left( \int_{N^+} f(xna_t) \phi(n) \rho_{g, \varepsilon_0}(h) q_\varepsilon(am) \, da \, dm \mu_{gN, \nu_1}(h) \right) d\mu_{\nu_1}(n).
\]

We now define \( \Phi_\varepsilon \in C_c(gN_{\varepsilon_0}^+N_R A_\varepsilon M_\varepsilon) \subset C_c(G) \) and \( \Phi_\varepsilon \in C_c(\Gamma\backslash G) \) by

\[
\tilde{\Phi}_\varepsilon(g_0) := \begin{cases} 
\frac{(n)\rho_{g, \varepsilon_0}(h)q_\varepsilon(am)}{\mu_{gN, \nu_2}(\rho_{g, \varepsilon_0})} & \text{if } g_0 = g_{hN}, \\
0 & \text{otherwise},
\end{cases}
\]

and \( \Phi_\varepsilon(g_0) := \sum_{\gamma \in \Gamma} \tilde{\Phi}_\varepsilon(\gamma g_0) \). Note that the continuity of \( \tilde{\Phi}_\varepsilon \) follows from Lemma 2.4. We now assume without loss of generality \( f \geq 0 \) and define, for all \( \varepsilon > 0 \), functions \( f_\varepsilon^+ \) as follows: for all \( z \in \Gamma\backslash G \),

\[
f_\varepsilon^+(z) := \sup_{b \in N_{\varepsilon_0}^+ N_\varepsilon A_\varepsilon M_\varepsilon} f(zb) \quad \text{and} \quad f_\varepsilon^-(z) := \inf_{b \in N_{\varepsilon_0}^+ N_\varepsilon A_\varepsilon M_\varepsilon} f(zb).
\]
Since $u \in \text{int} \, a^+$, for every $\varepsilon > 0$, there exists $t_0(R, \varepsilon) > 0$ such that

$$a_{t_0}^{-1} N^+_R a_t \subset N_\varepsilon \quad \text{for all } t \geq t_0(R, \varepsilon).$$

Then, as $\text{supp}(\tilde{\Phi}_\varepsilon) \subset gN^+_0 N^+_R A_\varepsilon M_\varepsilon$, we have

$$f(xna_t)\tilde{\Phi}_\varepsilon(gnham) \leq f^+_3(xnhama_t)\tilde{\Phi}_\varepsilon(gnham) \tag{2.12}$$

for all $nham \in N N^+_0 A^+_0 M_\varepsilon$ and $t \geq t_0(R, \varepsilon)$. We now use $f^+_3$ to give an upper bound on the limit we are interested in; $f^+_3$ is used in an analogous way to provide a lower bound. Entering the definition of $\Phi_\varepsilon$ and the above inequality (2.12) into (2.11) gives

$$\limsup_{t \to \infty} \Psi(t) \int_{N^+} f(xna_t)\phi(n) \, d\mu_{\nu_1}(n) \leq \limsup_{t \to \infty} \Psi(t)$$

$$\int_{N^+} \int_{NA^+_M} f^+_3(xnhama_t)\tilde{\Phi}_\varepsilon(gnham) \, d\mu_{g^+_0 N^+_0 N^+_R A^+_0} \, d\mu_{\nu_1}(n)$$

$$\leq \limsup_{t \to \infty} \Psi(t) e^{\varepsilon \|\psi_1 - \psi_2\|} \int_{N^+} \int_{NA^+_M} f^+_3(xnhama_t)\tilde{\Phi}_\varepsilon(gnham)$$

$$e^{(\psi_1 - \psi_2)/(\log a)} \, d\mu_{g^+_0 N^+_0 N^+_R A^+_0} \, d\mu_{\nu_1}(n)$$

$$= \limsup_{t \to \infty} \Psi(t) e^{\varepsilon \|\psi_1 - \psi_2\|} \int_G f^+_3([g_0]a_t)\tilde{\Phi}_\varepsilon(g_0) \, d\tilde{m}(g_0)$$

$$= \limsup_{t \to \infty} \Psi(t) e^{\varepsilon \|\psi_1 - \psi_2\|} \int_{\Gamma \setminus G} f^+_3([g_0]a_t)\tilde{\Phi}_\varepsilon([g_0]) \, dm([g_0]),$$

where Lemma 2.6 was used in the second to last line of the above calculation. By the standing assumption (2.7), we have

$$\limsup_{t \to \infty} \Psi(t) \int_N f(xna_t)\phi(n) \, d\mu_{g^+_0 N^+_0 N^+_R A^+_0}(n)$$

$$\leq e^{\varepsilon \|\psi_1 - \psi_2\|} \sum_{Y \in \mathcal{A}_G} m|_Y(f^+_3)^\varepsilon \, m|_Y(\tilde{\Phi}_\varepsilon)$$

$$= e^{\varepsilon \|\psi_1 - \psi_2\|} \sum_{Y \in \mathcal{A}_G} m|_Y(f^+_3)^\varepsilon \, m|_Y(\tilde{\Phi}_\varepsilon),$$

where $\tilde{Y} \subset G$ is a $\Gamma$-invariant lift of $Y$. Using Lemma 2.6 for all $0 < \varepsilon < 1$,

$$\tilde{m}|_\gamma(\tilde{\Phi}_\varepsilon)$$

$$= \int_{N^+} \int_{NA^+_M} \tilde{\Phi}_\varepsilon \tilde{Y}(gnham)e^{(\psi_1 - \psi_2)/(\log a)} \, dm \, d\mu_{g^+_0 N^+_0 N^+_R A^+_0}(h) \, d\mu_{\nu_1}(n) \leq$$

$$e^{\varepsilon \|\psi_1 - \psi_2\|} \int_{N^+} \phi(n) 1_{Y(gn)} \left( \int_{N^+} \rho_{g^+_0 N^+_0 N^+_R A^+_0}(h) q_{\varepsilon}(a) \, dm \, d\mu_{g^+_0 N^+_0 N^+_R A^+_0}(h) \right) \, d\mu_{\nu_1}(n)$$

$$\leq e^{\varepsilon \|\psi_1 - \psi_2\|} \int_{N^+} \phi(n) \, d\mu_{\nu_1}(n) \leq$$

$$\int_{N^+} \phi(n) \, d\mu_{\nu_1}(n) \leq$$

$$\int_{N^+} \phi(n) \, d\mu_{\nu_1}(n).$$
where we have used that $\tilde{Y}$ is invariant under the right translation of identity component $M^0$ of $M$. Since $\varepsilon > 0$ was arbitrary, taking $\varepsilon \to 0$ gives

$$\limsup_{t \to \infty} \Psi(t) \int_{N^+} f(xna_t)\phi(n)\,d\mu_{\nu_1}(n) \leq \sum_{Y \in \mathcal{Y}_r} m|Y(f)\mu_{\nu_1}|Y(\phi).$$

The lower bound given by replacing $f^+_{3\varepsilon}$ with $f^-_{3\varepsilon}$ in the above calculations completes the proof. □

**Proposition 2.13.** For any $x = [g] \in \Gamma \setminus G$, $f \in C_c(\Gamma \setminus G)$ and $\phi \in C_c(N)$,

$$\lim_{t \to \infty} \Psi(t)e^{(\varphi_1 - \varphi)(tu)} \int_{N^+} f(xna_t)\phi(n)\,d\mu_{\lambda_1}(n) = \sum_{Y \in \mathcal{Y}_r} m_{\lambda_1,\nu_2}|Y(f)\mu_{gN^+,\nu_1}|Y(\phi).$$

**Proof.** For $\varepsilon_0 > 0$, set $B_{\varepsilon_0} = N_{\varepsilon_0}A_{\varepsilon_0}M_{\varepsilon_0}N_{\varepsilon_0}^{+}$. Given $x_0 \in \Gamma \setminus G$, let $\varepsilon_0(x_0)$ denote the maximum number $r$ such that the map $G \to \Gamma \setminus G$ given by $h \mapsto x_0h$ for $h \in G$ is injective on $B_r$. By using a partition of unity if necessary, it suffices to prove that for any $x_0 \in \Gamma \setminus G$ and $\varepsilon_0 = \varepsilon_0(x_0)$, the claims of the proposition hold for any non-negative $f \in C(x_0B_{\varepsilon_0})$, non-negative $\phi \in C(N_{\varepsilon_0}^{+})$, and $x = [g] \in x_0B_{\varepsilon_0}$. Moreover, we may assume that $f$ is given as

$$f([g]) = \sum_{\gamma \in \Gamma} \tilde{f}(\gamma g)$$

for some non-negative $\tilde{f} \in C_c(g_0B_{\varepsilon_0})$. For simplicity of notation, we write $\mu_{\lambda_1} = \mu_{gN^+\lambda_1}$. Note that for $x = [g] \in [g_0]B_{\varepsilon_0}$,

$$\int_{N^+} f([g]na_t)\phi(n)\,d\mu_{\lambda_1}(n) = \sum_{\gamma \in \Gamma} \tilde{f}(\gamma gna_t)\phi(n)\,d\mu_{\lambda_1}(n). \tag{2.14}$$

Note that $\tilde{f}(\gamma gna_t) = 0$ unless $\gamma gna_t \in g_0B_{\varepsilon_0}$. Together with the fact that supp$(\phi) \subset N_{\varepsilon_0}$, it follows that the summands in (2.14) are non-zero for only finitely many elements $\gamma \in \Gamma \cap g_0B_{\varepsilon_0}a_{-t}N_{\varepsilon_0}^{+}g^{-1}$.

Suppose $\gamma gN_{\varepsilon_0}^{+}a_{t} \cap g_0B_{\varepsilon_0} \neq \emptyset$. Then $\gamma gna_t \in g_0N_{\varepsilon_0}A_{\varepsilon_0}M_{\varepsilon_0}N^+$, and there are unique elements $p_{t,\gamma} \in N_{\varepsilon_0}A_{\varepsilon_0}M_{\varepsilon_0}$ and $n_{t,\gamma} \in N^+$ such that

$$\gamma gna_t = g_0p_{t,\gamma}n_{t,\gamma} \in g_0P_{\varepsilon_0}N^+.\gamma$$

Let $\Gamma_t$ denote the subset $\Gamma \cap g_0(N_{\varepsilon_0}A_{\varepsilon_0}M_{\varepsilon_0}N^+)a_{-t}^{-1}g^{-1}$. Note that although $\Gamma_t$ may possibly be infinite, only finitely many of the terms in the sums we
consider will be non-zero. This together with Lemma 2.5 gives
\[
\int_{N^+} f([g]na_t) \phi(n) \, d\mu_{\lambda_1}(n) = \sum_{\gamma \in \Gamma} \int_{N^+} \tilde{f}(\gamma g a_t) \phi(n) \, d\mu_{\lambda_1}(n)
\]
\[
= \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(\gamma g a_t(a_t^{-1}na_t)) \phi(n) \, d\mu_{\lambda_1}(n)
\]
\[
= e^{-\varphi_1(\log a_t)} \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(\gamma g a_t) \phi(a_t^{-1}na_t^{-1}) \, d\mu_{\lambda_1}(n)
\]
\[
= e^{-\varphi_1(\log a_t)} \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(g_0 p_{t, \gamma} n_{t, \gamma}) \phi(a_t^{-1}na_t^{-1}) \, d\mu_{\lambda_1}(n)
\]
\[
= e^{-\varphi_1(\log a_t)} \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(g_0 p_{t, \gamma} n_{t, \gamma}) \phi(a_t^{-1}na_t^{-1}) \, d\mu_{\lambda_1}(n).
\]

Since \(\text{supp}(\tilde{f}) \subset g_0 B_{\varepsilon_0}\), we have
\[
\sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(g_0 p_{t, \gamma} n_{t, \gamma}) \phi(a_t^{-1}na_t^{-1}) \, d\mu_{\lambda_1}(n)
\]
\[
\leq \sum_{\gamma \in \Gamma_t} \left( \sup_{n \in N_{\varepsilon_0}^+} \phi(a_t^{-1}na_t^{-1}) \right) \cdot \int_{N^+} \tilde{f}(g_0 p_{t, \gamma} n) \, d\mu_{\lambda_1}(n).
\]

Since \(u \in \text{int} \mathcal{L}_t\), there exist \(t_0 > 0\) and \(\alpha > 0\) such that
\[
a_t N_0^+ a_t^{-1} \subset N_r^{+} \quad \text{for all } r > 0 \text{ and } t > t_0.
\]
Therefore, for all \(n \in N_{\varepsilon_0}^+ \) and \(t > t_0\), we have
\[
\phi(a_t^{-1}na_t^{-1}) \leq \phi^+_{\varepsilon_0}(a_t^{-1}na_t^{-1}),
\]
where
\[
\phi^+_{\varepsilon}(n) := \sup_{b \in N_0^+} \phi(nb) \quad \text{for all } n \in N^+, \ v > 0.
\]

We now have the following inequality for \(t > t_0\):
\[
e^{-\varphi_1(\log a_t)} \int_{N^+} f([g]na_t) \phi(n) \, d\mu_{\lambda_1}(n)
\]
\[
\leq \sum_{\gamma \in \Gamma_t} \phi^+_{\varepsilon_0}(a_t^{-1}na_t^{-1}) \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 p_{t, \gamma} n) \, d\mu_{\lambda_1}(n).
\]

By Lemma 2.4, we can now choose \(R > 0\) and \(\rho \in C_c(N_0^+)\) such that
\(\rho(n) \geq 0\) for all \(n \in N^+\), and \(\mu_{g_0 p N^+, \nu_1}(\rho) > 0\) for all \(p \in N_{\varepsilon_0} A_{\varepsilon_0} M_{\varepsilon_0}\).
Define \(\tilde{F} \in C_c(g_0 N_{\varepsilon_0} A_{\varepsilon_0} M_{\varepsilon_0} N_0^+)\) by
\[
\tilde{F}(g) = \frac{\rho(n)}{\mu_{g_0 p N^+, \nu_1}(\rho)} \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 pv) \, d\mu_{g_0 p N^+, \lambda_1}(v)
\]
Similarly as before, we have, for all $t > t_0$, and $N_+^+$, and $\tilde{F}(g) = 0$ otherwise. The key property of $\tilde{F}$ we will use is the following: for all $p \in P_{t_0}$,

$$\int_{N_+^+} \tilde{F}(g_0p) \, d\mu_{g_0pN^+} \nu_1(n) = \int_{N_0^+} \tilde{F}(g_0p) \, d\mu_{g_0pN^+} \nu_1(n) = \int_{N_0^+} \tilde{f}(g_0p) \, d\mu_{g_0pN^+} \nu_1(n).$$

Returning to (2.16), we now give an upper bound. We observe:

$$e^{\phi_1(\log a_t)} \int_{N_+^+} f([g]n \alpha_t) \phi(n) \, d\mu_{\lambda_1}(n)$$

$$\leq \sum_{n \in \pi_t^+} \phi_\varepsilon e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}) \int_{N_0^+} \tilde{F}(g_0p_{t,\gamma}n) \, d\mu_{\lambda_1}(n)$$

$$= \sum_{n \in \pi_t^+} \phi_\varepsilon e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}) \int_{N_0^+} \tilde{F}(g_0p_{t,\gamma}n) \, d\mu_{g_0p_{t,\gamma}N^+} \nu_1(n)$$

$$= \sum_{n \in \pi_t^+} \int_{N_0^+} \tilde{F}(g_0p_{t,\gamma}n) \phi_\varepsilon e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}) \, d\mu_{g_0p_{t,\gamma}N^+} \nu_1(n).$$

Similarly as before, we have, for all $t > t_0$ and $n \in N_0^+$,

$$\phi_\varepsilon e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}) = \phi_\varepsilon e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}) \leq \phi_\varepsilon (R+\varepsilon_0)e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}).$$

Hence (2.16) is bounded above by

$$\leq \sum_{n \in \pi_t^+} \int_{N_0^+} \tilde{F}(g_0p_{t,\gamma}n) \phi_\varepsilon (R+\varepsilon_0)e^{-\alpha t} (n \alpha_t^{-1} \alpha_t^{-1}) \, d\mu_{g_0p_{t,\gamma}N^+} \nu_1(n)$$

$$= \sum_{n \in \pi_t^+} \int_{N_0^+} \tilde{F}(g_0p_{t,\gamma}n) \phi_\varepsilon (R+\varepsilon_0)e^{-\alpha t} (n) \, d\mu_{g_0p_{t,\gamma}N^+} \nu_1(n) \nu_1(n).$$

By Lemma 2.5,

$$d\mu_{g_0p_{t,\gamma}N^+} \nu_1(n \alpha_t^{-1} \alpha_t^{-1} \alpha_t^{-1}) = e^{\psi_1(\log a_t)} \, d\mu_{g_0p_{t,\gamma}N^+} \nu_1(n) \nu_1(n).$$

Since $g_0p_{t,\gamma}n \alpha_t^{-1} \alpha_t^{-1} = \gamma g$, it follows that for all $t > t_0$,

$$e^{(\phi_1-\psi_1)(\log a_t)} \int_{N_+^+} f([g]n \alpha_t) \phi(n) \, d\mu_{\lambda_1}(n)$$

$$\leq \sum_{n \in \pi_t^+} \int_{N_0^+} \tilde{F}(\gamma gn \alpha_t) \phi_\varepsilon (R+\varepsilon_0)e^{-\alpha t} (n) \, d\mu_{gN^+} \nu_1(n)$$

$$\leq \int_{N_+^+} \left\{ \sum_{n \in \pi_t^+} \tilde{F}(\gamma gn \alpha_t) \phi_\varepsilon (R+\varepsilon_0)e^{-\alpha t} (n) \right\} \, d\mu_{\nu_1}(n).$$
Define a function $F$ on $\Gamma \setminus G$ by

$$F([g]) := \sum_{\gamma \in \Gamma} \tilde{F}(\gamma g).$$

Then for any $\varepsilon > 0$ and for all $t > t_0$ such that $(R + \varepsilon_0)e^{-\alpha t} \leq \varepsilon$,

$$\Psi(t)e^{(\varphi_1 - \psi_1)(\log a t)} \int_{N^+} f([g]na_t)\phi(n) \, d\mu_{\lambda_1}(n) \leq \Psi(t) \int_{N^+} F([g]na_t)\phi_+^+(n) \, d\mu_{\nu_1}(n).$$

By Proposition 2.9, letting $\varepsilon \to 0$ gives

$$\limsup_{t \to \infty} \Psi(t)e^{(\varphi_1 - \psi_1)(\log a t)} \int_{N^+} f([g]na_t)\phi(n) \, d\mu_{\lambda_1}(n) \leq \sum_{Y \in \mathfrak{F}} Y \in \mathfrak{C} \int_{N^+} F([g]na_t)\phi(n) \, d\mu_{\nu_1}(n).$$

By the definition of $F$, together with Lemma 2.6, we have

$$m|_Y(F) = \tilde{m}|_Y(\tilde{F}) = \int_P \left( \int_{N^+} \tilde{F}(g_0 h, m n) \, d\mu_{g_0 h, m n}^{PS, \lambda_1}(n) \right) e^{-\psi_2 \circ (\log a) \, dm \, da \, d\mu_{g_0 h, m n}(h)} = \int_P \left( \int_{N^+} \tilde{F}(g_0 h, m n) \, d\mu_{g_0 h, m n}^{\lambda_1}(n) \right) e^{-\psi_2 \circ (\log a) \, dm \, da \, d\mu_{g_0 h, m n}(h)} = \tilde{m}_{\lambda_1, \nu_2}|_Y(\tilde{F}) = m_{\lambda_1, \nu_2}|_Y(F).$$

This gives the desired upper bound. The lower bound can be obtained similarly, finishing the proof. □

With the help of Proposition 2.9, we are now ready for

**Proof of Theorem 2.8.** By the compactness hypothesis on the supports of $f_i$, we can find $\varepsilon_0 > 0$ and $x_i \in \Gamma \setminus G, i = 1, \cdots, \ell$ such that the map $G \to \Gamma \setminus G$ given by $g \to x_i g$ is injective on $R_{\varepsilon_0} = N_{\varepsilon_0}^- A_{\varepsilon_0} M_{\varepsilon_0} N_{\varepsilon_0}^+$, and $\bigcup_{i=1}^\ell x_i R_{\varepsilon_0/2}$ contains both supp $f_1$ and supp $f_2$. We use continuous partitions of unity to write $f_1$ and $f_2$ as finite sums $f_1 = \sum_{i=1}^\ell f_{1,i}$ and $f_2 = \sum_{j=1}^\ell f_{2,j}$ with supp $f_{1,i} \subset x_i R_{\varepsilon_0/2}$ and supp $f_{2,j} \subset x_j R_{\varepsilon_0/2}$. Writing $p = \text{ham} \in NAM$ and using Lemma 2.6

$$dm_{\lambda_1, \lambda_2}(\text{ham} n) = d\mu_{\text{ham},N^+, \lambda_1}(n) e^{-\psi_2 \circ (\log a) \, dm \, da \, d\mu_{N^+, \lambda_2}(h)}.$$
We have
\[
\int_{\Gamma \setminus G} f_1(x a_t) f_2(x) \, dm_{\lambda_1, \lambda_2}(x) = \quad (2.18)
\]
\[
\sum_{i,j} \int_{R_{\epsilon_0}} f_{1,i}(x_j p m a_t) f_{2,j}(x_j p m) \, d\mu_{ham N^+, \lambda_1}(n) \, e^{-\psi_0 i (\log a)} \, dm \, d\mu_{N^+, \lambda_2}(h)
\]
\[
= \sum_{i,j} \int_{N_{\epsilon_0} A_{\epsilon_0} M_{\epsilon_0}} \left( \int_{N_{\epsilon_0}^+} f_{1,i}(x_j p m a_t) f_{2,j}(x_j p m) \, d\mu_{ham N^+, \lambda_1}(n) \right) \times e^{-\psi_0 i (\log a)} \, dm \, d\mu_{N^+, \lambda_2}(h).
\]
Applying Proposition 2.13 it follows:
\[
\lim_{t \to \infty} \Psi(t) e^{(\varphi_1 - \psi_1)(\log a)} \int_{\Gamma \setminus G} f_1(x a_t) f_2(x) \, dm_{\lambda_1, \lambda_2}(x)
\]
\[
= \sum_{Y \in \mathcal{P}} m_{\lambda_1, \nu_2} |Y(f_2)| \sum_j \sum_{Y \in \mathcal{P}} m_{\lambda_1, \nu_2} |Y(f_2)| \sum_i \int_{N_{\epsilon_0} A_{\epsilon_0} M_{\epsilon_0}} \mu_{x_i p N^+, \nu_1} |Y(f_{1,i}(x_j p \cdot))| \times e^{-\psi_0 i (\log a)} \, dm \, d\mu_{N^+, \lambda_2}(h)
\]
\[
= \sum_{Y \in \mathcal{P}} m_{\lambda_1, \nu_2} |Y(f_2)| \sum_i \int_{N_{\epsilon_0} A_{\epsilon_0} M_{\epsilon_0}} \mu_{x_i p N^+, \nu_1} |f_{1,i} \cdot Y(x_j p \cdot)| \times e^{-\psi_0 i (\log a)} \, dm \, d\mu_{N^+, \lambda_2}(h)
\]
\[
= \sum_{Y \in \mathcal{P}} m_{\lambda_1, \nu_2} |Y(f_2)| \sum_i \sum_{m_{\lambda_1, \nu_2}} m_{\lambda_1, \lambda_2} |Y(f_1) m_{\lambda_1, \nu_2} |Y(f_2)|
\]
where the second last equality is valid by Lemma 2.6. This completes the proof. □

3. PROOF OF THEOREM 1.3

Let \( \Gamma < G \) be a Zariski dense Anosov subgroup with respect to \( P \).

\( u \)-balanced measures. Fix \( u \in \text{int} \, \mathcal{L}_\Gamma \). There exists a unique \( \psi \in D^*_\Gamma \) such that \( \psi(u) = \psi_T(u) \) [6] Prop. 4.4]. Let \( \nu_\psi \) denote the unique \((\Gamma, \psi)\)-PS measure [6] Thm. 1.3]. Similarly, \( \nu_{\psi_0 i} \) denotes the unique \((\Gamma, \psi \circ i)\)-PS-measure.

We recall the following theorem of Chow and Sarkar:

**Theorem 3.1.** [3] There exists \( \kappa_u > 0 \) such that for any \( f_1, f_2 \in C_0(\Gamma \setminus G) \), we have
\[
\lim_{t \to +\infty} t^{(r-1)/2} \int_{\Gamma \setminus G} f_1(x \exp(tu)) f_2(x) \, dm_{\nu_\psi, \nu_{\psi_0 i}}(x)
\]
\[
= \kappa_u \sum_{Y \in \mathcal{P}} m_{\nu_\psi, \nu_{\psi_0 i}} |Y(f_1) m_{\nu_\psi, \nu_{\psi_0 i}} |Y(f_2)|.
\]
Following [2], we say that a locally finite Borel measure $m$ on $\Gamma \setminus G$ is $u$-balanced if
\[
\limsup_{T \to \infty} \frac{\int_0^T m(O_1 \cap O_1 \exp(tu)) \, dt}{\int_0^T m(O_2 \cap O_2 \exp(tu)) \, dt} < \infty,
\]
for all bounded Borel subsets $O_i \subset \Gamma \setminus G$ with $m(\text{int} O_i) > 0$.

By Theorems 3.1 and 2.8, we get

**Theorem 3.2** (Local mixing). For $i = 1, 2$, let $\lambda_{\varphi_i}$ be any $(\Gamma, \varphi_i)$-conformal measure on $F$. For any $f_1, f_2 \in C_c(\Gamma \setminus G)$, we have
\[
\lim_{t \to +\infty} t^{(r-1)/2} e^{(\varphi_1 - \psi)(tu)} \int_{\Gamma \setminus G} f_1(x \exp(tu)) f_2(x) \, dm_{\lambda_{\varphi_1}, \lambda_{\varphi_2}}(x)
= \kappa_u \sum_{Y \in G} m_{\nu_{\psi}, \lambda_{\varphi_2}} |Y_f 1 m_{\lambda_{\varphi_1}, \nu_{\psi}}| Y_f
\]
where $\kappa_u$ is as in Theorem 3.1. In particular, $m_{\lambda_{\varphi_1}, \lambda_{\varphi_2}}$ is $u$-balanced.

**Proof of Theorem 1.4.** The main ingredient is the higher rank Hopf-Tsuji-Sullivan dichotomy established in [2]. The main point is that all seven conditions of Theorem 1.4 of [2] are equivalent to each other for Anosov groups and $u \in \text{int} L_{\Gamma}$, since all the measures considered there are $u$-balanced by Theorem 3.2. For the proof of Theorem 1.4, we only need the equivalence of (6) and (7), which we now recall.

Consider the following $u$-directional conical limit set of $\Gamma$:

\[
\Lambda_u := \{ g^+ \in \Lambda : \gamma_i \exp(t_i u) \text{ is bounded for some } t_i \to +\infty \text{ and } \gamma_i \in \Gamma \}.
\]

(3.3)

Applying the dichotomy [2, Thm. 1.4] to a $u$-balanced measure $m_{\lambda_{\varphi}, \nu_{\psi}}$, we deduce

**Proposition 3.4.** Let $\psi \in D^*_\Gamma$ be such that $\psi(u) = \psi_T(u)$ for $u \in \text{int} L_{\Gamma}$. The following conditions are equivalent for any $(\Gamma, \psi)$-conformal measure $\lambda_{\psi}$ on $F$:

1. $\lambda_{\psi}(\Lambda_u) = 1$;
2. $\sum_{\gamma \in \Gamma_u} e^{-\psi(\mu(\gamma))} = \infty$ for some $R > 0$

where $\Gamma_u := \{ \gamma \in \Gamma : \| \mu(\gamma) - u \| < R \}$.

On the other hand, if rank $G \leq 3$, we have
\[
\sum_{\gamma \in \Gamma_u} e^{-\psi(\mu(\gamma))} = \infty
\]
for some $R > 0$ [2, Thm. 6.3]. Therefore, by Proposition 3.4, we have $\lambda_{\psi}(\Lambda_u) = 1$ and hence $\lambda_{\psi}$ is supported on $\Lambda$ in this case. This finishes the proof of the first part of Theorem 1.4. The second claim follows from the first one by [6, Thm. 1.3].
References


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