COUNTING TOTALLY REAL UNITS AND EIGENVALUE PATTERNS IN $\mathrm{SL}_n(\mathbb{Z})$ AND $\mathrm{Sp}_{2n}(\mathbb{Z})$ ALONG THIN TUBES

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ABSTRACT. Fix a vector $v = (v_1, \dots, v_n)$ with $v_1 > \cdots > v_n$ and $\sum v_i = 0$, and let $B_\varepsilon(Tv)$ be the ε-ball around Tv for $T > 1$. As $T \to \infty$,

(1) the number of degree-n totally real units whose logarithmic embeddings lie in $B_\varepsilon(Tv)$, and
(2) the number of eigenvalue patterns of $\mathrm{SL_n}(\mathbb{Z})$ whose logarithmic embeddings lie in $B_\varepsilon(Tv)$

both grow like

$$\exp\big(\rho_{\mathrm{SL_n}}(v)\, T\big)$$

where $\rho_{\mathrm{SL_n}}(v) = \sum_{i=1}^{n-1} (n-i)\, v_i$ is the half-sum of positive roots of $\mathrm{SL_n}(\mathbb{R})$. Hence the two arithmetic problems share the same *directional entropy.*

Because each eigenvalue pattern determines an $\mathrm{SL_n}(\mathbb{R})$-conjugacy class, this yields a lower bound of order $\exp\big(\rho_{\mathrm{SL_n}}(v)T\big)$ for the number of $\mathrm{SL_n}(\mathbb{Z})$-conjugacy classes with a prescribed eigenvalue pattern; we also obtain an upper bound of order $\exp\big(2\rho_{\mathrm{SL_n}}(v)T\big)$.

A parallel argument for the symplectic lattice $\mathrm{Sp_{2n}}(\mathbb{Z})$ and

$$v = (v_1, \dots, v_n, -v_n, \dots, -v_1), \quad v_1 > \cdots > v_n > 0,$$

gives growth $\exp\big(\rho_{\mathrm{Sp_{2n}}}(v)T\big)$ with $\rho_{\mathrm{Sp_{2n}}}(v) = \sum_{i=1}^{n} (n+1-i)v_i$, the half-sum of positive roots of $\mathrm{Sp_{2n}}(\mathbb{R})$.

1. INTRODUCTION

Classical arithmetic asks how many objects of a given kind—ideals, points, matrices, geodesics—fit inside a region that grows without bound. In higher rank, the natural "size" of an object is rarely a single number; instead it is a vector that records growth rates in several directions at once. When we restrict our attention to a thin tube around a fixed ray, the leading exponent of an exponential growth can be viewed as a *directional entropy*: it measures how densely the arithmetic set populates that ray.

This paper pinpoints an explicit linear functional that governs the directional entropy of the following two collections:

- the logarithmic embeddings of *all* totally real units of fixed degree n;
- the ordered eigenvalue data, equivalently, Jordan projections, of elements in $\mathrm{SL}_n(\mathbb{Z})$.

We prove that both collections exhibit the same entropy along every ray in the positive Weyl chamber. Going further, we count $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes that share a prescribed eigenvalue pattern. The Jordan-projection entropy yields an immediate lower bound for this count, and we provide an upper bound, which is conjecturally a true order of magnitude. We also address the analogous problem for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Totally real algebraic units. For an integer $n \geq 2$, let $\mathcal{K}_n^\star$ denote the set of totally real number fields K of degree n. For each $K \in \mathcal{K}_n^\star$, let Σ_K denote the set of all *ordered* embeddings of K into $\mathbb{R}$. Define

$$\mathcal{K}_n = \{(K,\sigma) : K \in \mathcal{K}^\star,\ \sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma_K\}.$$

For $(K,\sigma) \in \mathcal{K}_n$, define the logarithmic map

$$\Lambda_{K,\sigma} : K - \{0\} \to \mathbb{R}^n,\ \ \Lambda_{K,\sigma}(\mathsf{u}) = \big(\log|\sigma_1(\mathsf{u})|, \ldots, \log|\sigma_n(\mathsf{u})|\big).$$

Denote by O_K the ring of integers of K and by $O_K^\times$ its unit group. Consider the hyperplane

$$\mathsf{H} = \{v = (v_1, \cdots, v_n) \in \mathbb{R}^n : \sum v_i = 0\}$$

and note that, by Dirichlet's unit theorem, $\Lambda_{K,\sigma}(O_K^\times)$ is a lattice in H (cf. [18]).

We extend $\Lambda_{K,\sigma}$ coordinate-wise to a map $\Lambda : \mathcal{K}_n - \{0\} \to \mathbb{R}^n$: $\Lambda(u) = \Lambda_{K,\sigma}(u)$ if $u \in (K,\sigma)$. Collect all totally real units of degree n in the disjoint union

$$O_n^\times = \bigcup_{(K,\sigma)\in\mathcal{K}_n} (O_K^\times, \sigma).$$

We are interested in the asymptotic distribution of $\Lambda(O_n^\times)$ inside radial tubes in the following H^+:

$$\mathsf{H}_+ = \Big\{v = (v_1, \ldots, v_n) \in \mathsf{H} : v_1 > \cdots > v_n\Big\}.$$

Fix any norm $\|\cdot\|$ on $\mathbb{R}^n$.

Theorem 1.1. *Fix the maximum norm $\|\cdot\|_{\max}$ on $\mathbb{R}^n$ and let $v \in \mathsf{H}_+$. For all sufficiently small $\varepsilon > 0$, we have*

$$\#\big\{\mathsf{u} \in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\big\} \asymp_\varepsilon {}^{1} \ \exp\left(\sum_{i=1}^{n-1}(n-i)v_iT\right).$$

More precisely,

$$2\left(\frac{4\varepsilon}{(n-1)3^n}\right)^{n-1} \leq \liminf_{T\to\infty} \frac{\#\big\{\mathsf{u} \in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\big\}}{\exp\left(\sum_{i=1}^{n-1}(n-i)v_iT\right)}$$

$$\leq \limsup_{T\to\infty} \frac{\#\big\{\mathsf{u} \in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\big\}}{\exp\left(\sum_{i=1}^{n-1}(n-i)v_iT\right)} \leq 2(4\varepsilon)^{n-1}n!.$$

[1]We write $f(T) \asymp g(T)$ if there exist $C_1, C_2 > 0$ such that $C_1\, g(T) \leq f(T) \leq C_2\, g(T)$ for all $T \geq 1$. The notation $f(T) \asymp_\varepsilon g(T)$ has the same meaning, except that C_1 and C_2 may depend on ε.

See Proposition 3.2 for exponential error terms in the upper and lower bounds for the number $\#\{\mathsf{u} \in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\}$; in particular, these errors terms can be taken uniformly over all v in a fixed compact subset of H_+.

We propose the following notion of directional entropies of $O_n^\times$:

Definition 1.2 (Directional entropy)**.** Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$. For $v \in \mathsf{H}_+$, define the *upper* and *lower* directional entropies of $O_n^\times$ in the direction v by

$$\overline{\mathsf{E}}_n(v) := \|v\| \cdot \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log N_\varepsilon(T, v),$$
$$\underline{\mathsf{E}}_n(v) := \|v\| \cdot \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T} \log N_\varepsilon(T, v)$$

where

$$N_\varepsilon(T, v) := \#\{u \in O_n^\times : \ \|\Lambda(u)\| \leq T, \ \|\Lambda(u) - \mathbb{R}_+ v\| < \varepsilon\}.$$

These quantities lie in $\{-\infty\} \cup [0, \infty)$, are independent of the choice of a norm, and are homogeneous of degree one in v. When they coincide, we write $\mathsf{E}_n(v)$ for their common value.

Theorem 1.3. *For each $v \in \mathsf{H}_+$, we have*

$$\mathsf{E}_n(v) = \sum_{i=1}^{n-1} (n-i) v_i.$$

This entropy value can be expressed in terms of the discriminant of the model polynomial $q_{Tv}(x) = \prod_{i=1}^n (x - e^{Tv_i})$:

$$\lim_{T\to\infty} \frac{1}{T} \log \sqrt{\operatorname{Disc}(q_{Tv})} = \sum_{i=1}^{n-1} (n-i) v_i.$$

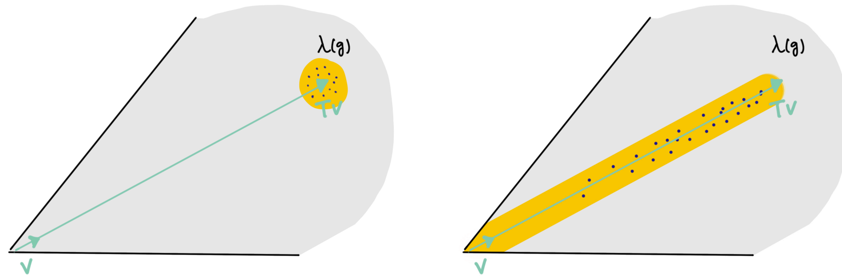
Remark 1.4. Observe that for every $v \in \mathsf{H}$, $\sum_{i=1}^{n-1}(n-i)v_i = \frac{1}{2}\sum_{i=1}^n (n+1-2i)v_i$. On the unit sphere for the max-norm $\{v \in \mathsf{H}_+ : \|v\|_{\max} = 1\}$, the entropy functional E_n reaches its supremum $\lfloor \frac{n^2}{4} \rfloor$ in the direction of $v = (1, \cdots 1, 0, -1, \cdots, -1)$ for n odd and $v = (1, \cdots, 1, -1, \cdots, -1)$ for n even, where the first $\lfloor n/2 \rfloor$-coordinates are 1. For instance,

$$\sup\{\mathsf{E}_4(v) : v \in \mathsf{H}_+, \|v\|_{\max} = 1\} = 4.$$

On the Euclidean unit sphere $\{\|v\|_{\mathrm{Euc}} = 1\}$, the maximum value of E_n is $\sqrt{\frac{n(n^2-1)}{12}}$, attained in the direction $(n-1, n-3, \cdots, -(n-3), -(n-1))$.

Eigenvalue patterns in $\mathrm{SL}_n(\mathbb{Z})$. An element $g \in \mathrm{SL}_n(\mathbb{R})$ is called *loxodromic* if its eigenvalues have pairwise distinct moduli; in particular, they are all real. For such $g \in \mathrm{SL}_n(\mathbb{R})$, write its eigenvalues as

$$\mathcal{E}(g) = \Big(m_1(g) e^{\lambda_1(g)}, \ldots, m_n(g) e^{\lambda_n(g)}\Big) \tag{1.1}$$



with signs $m_i(g) \in \{\pm 1\}$ and ordering given by $\lambda_1(g) \geq \cdots \geq \lambda_n(g)$. Set

$$\lambda(g) := \big(\lambda_1(g), \ldots, \lambda_n(g)\big), \qquad m(g) := \big(m_1(g), \ldots, m_n(g)\big). \tag{1.2}$$

The vector $\lambda(g)$ is called the *Jordan projection* of g. Define the linear functional

$$\rho_{\mathrm{SL}_n}(v) = \tfrac{1}{2} \sum_{1 \leq i < j \leq n} (v_i - v_j) = \sum_{i=1}^{n-1} (n-i) v_i$$

which is the half-sum of all positive roots of $\mathrm{SL}_n(\mathbb{R})$. For $v \in \mathsf{H}_+$ and $\varepsilon > 0$, if T is sufficiently large, then any $\gamma \in \mathrm{SL}_n(\mathbb{R})$ with $\|\lambda(\gamma) - Tv\| < \varepsilon$ is loxodromic.

Theorem 1.5. *Let $v \in \mathsf{H}_+$ and let $m = (m_1, \cdots, m_n) \in \{\pm 1\}$ be a sign pattern with $\prod_{i=1}^n m_i = 1$. Fix $\varepsilon > 0$.*

(1) *We have*

$$\#\Big\{\lambda(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z}), \|\lambda(\gamma) - Tv\| \leq \varepsilon, m(\gamma) = m\Big\} \asymp_\varepsilon e^{\rho_{\mathrm{SL}_n}(v)T},$$

where explicit upper and lower multiplicative constants are given in Theorem 4.1.

(2) *There exist constants $C_1, C_2 > 0$ such that for all sufficiently large $T > 1$,*

$$C_1 e^{\rho_{\mathrm{SL}_n}(v)T} \leq \#\Big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \leq \varepsilon, m(\gamma) = m\Big\} \leq C_2 e^{2\rho_{\mathrm{SL}_n}(v)T}$$

where $[\mathrm{SL}_n(\mathbb{Z})]$ denotes the set of all $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes.

Observe that $2\rho_{\mathrm{SL}_n}(v)$ is precisely the volume growth exponent for the thin tubes around the ray $\mathbb{R}_+ v$;

$$\mathrm{Vol}\Big\{g \in \mathrm{SL}_n(\mathbb{R}) : \|\lambda(g) - Tv\| \leq \varepsilon\Big\} \asymp_\varepsilon e^{2\rho_{\mathrm{SL}_n}(v)T},$$

with volume taken with respect to a Haar measure of $\mathrm{SL}_n(\mathbb{R})$ (cf. the proof of Theorem 6.4). Consequently, the Jordan–projection count in part (1) grows like the *square root* of this ambient volume growth.

Because each eigenvalue pattern determines an $\mathrm{SL}_n(\mathbb{R})$-conjugacy class, this yields a lower bound of order $e^{\rho_{\mathrm{SL}_n}(v)T}$, as stated in part (2), for the number of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes with prescribed eigenvalue pattern; we also obtain an upper bound of order $e^{2\rho_{\mathrm{SL}_n}(v)T}$.

Denote by $\mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}$ (resp. $[\mathrm{SL}_n(\mathbb{Z})]_{\mathrm{lox}}$) the set (resp. the set of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes) of loxodromic elements of $\mathrm{SL}_n(\mathbb{Z})$. For $\gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}$, write

$$[\gamma]_{\mathbb{R}} \subset \mathrm{SL}_n(\mathbb{Z}) \qquad \text{and} \qquad [\gamma]_{\mathbb{Q}} \subset \mathrm{SL}_n(\mathbb{Z})$$

for its $\mathrm{SL}_n(\mathbb{R})$- and $\mathrm{SL}_n(\mathbb{Q})$-conjugacy classes, respectively. Because the centralizer of $\gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}$ is a maximal $\mathbb{Q}$-split torus and all such tori are conjugate under $\mathrm{SL}_n(\mathbb{Q})$, we have

$$[\gamma]_{\mathbb{R}} = [\gamma]_{\mathbb{Q}}.$$

Define the "class number"

$$h(\gamma) = \#\{\mathrm{SL}_n(\mathbb{Z})\text{-conjugacy classes inside } [\gamma]_{\mathbb{Q}}\}. \tag{1.3}$$

Since the eigenvalue pattern of a loxodromic element uniquely determines its $\mathrm{SL}_n(\mathbb{R})$-conjugacy class, the map $\gamma \mapsto \mathcal{E}(\gamma)$ gives a bijection

$$\{[\gamma]_{\mathbb{R}} : \gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}\} \Leftrightarrow \{\mathcal{E}(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}\}.$$

Hence for any region $R \subset \mathbb{R}^n$,

$$\#\big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})]_{\mathrm{lox}} : \mathcal{E}(\gamma) \in R\big\} \;=\; \sum_{\mathcal{E}(\gamma)\in R} h(\gamma).$$

In other words, the number of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes whose eigenvalue pattern lies in R equals the count of those patterns, each weighted by its class number $h(\gamma)$.

Remark 1.6. In view of Theorem 1.5, one may expect that for any $\varepsilon > 0$,

$$h(\gamma) \ll_{\varepsilon} e^{\rho_{\mathrm{SL}_n}(\lambda(\gamma))(1+\varepsilon)}$$

for all loxodromic $\gamma \in \mathrm{SL}_n(\mathbb{Z})$.

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $D = \mathrm{tr}(\gamma)^2 - 4$ square-free, the quantity $h(\gamma)$ coincides with the classical class number $h_K = \#\,\mathrm{Cl}(O_K)$ of the quadratic field $K = \mathbb{Q}(\sqrt{D})$ (see [19], [30], [18]). Moreover, the conjugacy classes $[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})]_{\mathrm{lox}}$ correspond bijectively to closed geodesics C_γ on the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^2$, with length given by $2\lambda_1(\gamma)$ [28]. Hence the prime geodesic theorem on modular surface ([29], [15]) implies

$$\#\Big\{[\gamma] \in [\mathrm{SL}_2(\mathbb{Z})]_{\mathrm{lox}} : T-\varepsilon \le \|\lambda(\gamma)\| < T+\varepsilon\Big\} \asymp_{\varepsilon} \frac{e^{2T}}{2T}.$$

In this case, Theorem 1.5 gives

$$\#\Big\{\mathcal{E}(\gamma) : \gamma \in \mathrm{SL}_2(\mathbb{Z})_{\mathrm{lox}} : T-\varepsilon \le \|\lambda(\gamma)\| \le T+\varepsilon\Big\} \asymp_{\varepsilon} e^{T},$$

which also follows from the elementary fact that $e^{\|\lambda(\gamma)\|}$ is essentially the size of the (integral) trace of γ.

Remark 1.7. Eskin-Mozes-Shah studied a *transversal* counting problem in [11]. Fix a loxodromic element $\gamma_0 \in \mathrm{SL}_n(\mathbb{Z})$ and let $p \in \mathbb{Z}[x]$ be its characteristic polynomial. Write $K = \mathbb{Q}(\alpha)$ for a root α of p. Assume that p is irreducible over $\mathbb{Q}$ and $\mathbb{Z}[\alpha] = \mathcal{O}_K$. By [11, Theorem 1.1], as $T \to \infty$,

$$\#\Big\{\gamma \in [\gamma_0]_{\mathbb{R}} : \|\gamma\| < e^T\Big\} \sim^{2} c_n \frac{h(\gamma_0)\, R_K}{\sqrt{\mathrm{Disc}(p)}} \exp\left(\tfrac{1}{2}(n^2 - n)T\right),$$

where

- $c_n > 0$ depends only on n;
- $h(\gamma_0)$ is the class number defined in (1.3);
- R_K is the regulator of K, i.e. the volume of $\mathsf{H}/\left(\Lambda_{K,\sigma}(O_K^\times)\right)$;

Thus [11] counts *integral matrices lying inside a* fixed $\mathrm{SL}_n(\mathbb{R})$*–conjugacy class*, whereas our results count *the number of* distinct $\mathrm{SL}_n(\mathbb{Z})$*–conjugacy classes* whose Jordan projections fall into a given tube.

Definition 1.8 (Directional entropy for $\mathrm{SL}_n(\mathbb{Z})$)**.** Let $v \in \mathsf{H}_+$ and let $m = (m_1, \cdots, m_n) \in \{\pm 1\}$ satisfy $\prod_{i=1}^n m_i = 1$. Define the *upper* and *lower* directional entropies by

$$\overline{\mathsf{E}}_{\mathrm{SL}_n(\mathbb{Z})}(v, m) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log N_\varepsilon(T, v, m)}{T};$$

$$\underline{\mathsf{E}}_{\mathrm{SL}_n(\mathbb{Z})}(v, m) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log N_\varepsilon(T, v, m)}{T}$$

where

$$N_\varepsilon(T, v, m) = \#\big\{\lambda(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z}) : \|\lambda(\gamma) - \mathbb{R}_+ v\| \le \varepsilon, \|\lambda(\gamma)\| \le T, m(\gamma) = m\big\}.$$

Similarly, set

$$\overline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v, m) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log M_\varepsilon(T, v, m)}{T};$$

$$\underline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v, m) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log M_\varepsilon(T, v, m)}{T}$$

where

$$M_\varepsilon(T, v, m) := \#\big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma) - \mathbb{R}_+ v\| \le \varepsilon, \|\lambda(\gamma)\| \le T, m(\gamma) = m\big\}.$$

As before, these quantities in $\{-\infty\} \cup [0, \infty)$ are norm-independent and homogeneous of degree one. When the lower and upper limits agree, we write $\mathsf{E}_{\mathrm{SL}_n(\mathbb{Z})}(v)$ and $\mathsf{E}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v)$, respectively.

As an immediate consequence of Theorem 1.5, we get

Theorem 1.9. *Let $v \in \mathsf{H}_+$ and $m \in \{\pm 1\}^n$ with $\prod_{i=1}^n m_i = 1$. Then*

$$\mathsf{E}_{\mathrm{SL}_n(\mathbb{Z})}(v, m) = \rho_{\mathrm{SL}_n}(v);$$

$$\rho_{\mathrm{SL}_n}(v) \le \underline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v, m) \le \overline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v, m) \le 2\rho_{\mathrm{SL}_n}(v).$$

[2] We write $f(T) \sim g(T)$ if $\lim_{T \to \infty} f(T)/g(T) = 1$.

We think that $\mathsf{E}^{\star}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) = 2\rho_{\mathrm{SL}_n}(v)$ should be true, although we do not know how to prove this; see Conjecture 1.12 for a more general formulation.

Eigenvalue patterns in $\mathrm{Sp}_{2n}(\mathbb{Z})$. We also carry out a parallel analysis for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$, obtaining analogous counting results and entropy estimates. Fix the symplectic form in (5.3) so that

$$\mathfrak{a}^+ \;=\; \Big\{ v = \mathrm{diag}(v_1,\dots,v_n,-v_n,\dots,-v_1) :\; v_1 \ge \cdots \ge v_n \ge 0 \Big\}$$

is a positive Weyl chamber of $\mathrm{Sp}_{2n}(\mathbb{R})$. An element $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ is *loxodromic* precisely when its Jordan projection

$$\lambda(g) \;=\; \big(\lambda_1(g),\dots,\lambda_n(g),-\lambda_n(g),\dots,-\lambda_1(g)\big) \in \mathrm{int}\,\mathfrak{a}^+.$$

For such g, set

$$m(g) = \big(m_1(g),\dots,m_n(g)\big) \in \{\pm 1\}^n,$$

so that, for each i, the two real eigenvalues of g are $m_i(g)\,e^{\pm\lambda_i(g)}$.

Let

$$\rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^{n} (n+1-i)\,v_i$$

be the half-sum of all positive roots of $(\mathfrak{sp}_{2n}(\mathbb{R}),\mathfrak{a})$.

Theorem 1.10. *Let $v \in \mathrm{int}\,\mathfrak{a}^+$ and $m \in \{\pm1\}^n$. Fix small $0<\varepsilon<1$.*

(1) We have

$$\#\Big\{ (\lambda(\gamma), m(\gamma)) : \gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}),\ \|\lambda(\gamma) - Tv\| \le \varepsilon,\ m(\gamma) = m \Big\} \;\asymp_\varepsilon\; e^{\rho_{\mathrm{Sp}_{2n}}(v) T},$$

where explicit upper and lower multiplicative constants are given in Theorem 5.7.

(2) There exist constants $C_1, C_2 > 0$ such that for all $T \ge T_0(v,\varepsilon)$ such that

$$C_1 e^{\rho_{\mathrm{Sp}_{2n}}(v)T} \le \#\Big\{ [\gamma] \in [\mathrm{Sp}_{2n}(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \le \varepsilon, m(\gamma) = m \Big\} \le C_2 e^{2\rho_{\mathrm{Sp}_{2n}}(v)T}.$$

Define the directional entropies $\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m)$ and $\mathsf{E}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m)$ exactly as in Definition 1.8, with $\mathrm{SL}_n(\mathbb{Z})$ replaced everywhere by $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Corollary 1.11. *For all $v \in \mathrm{int}\,\mathfrak{a}^+$ and $m \in \{\pm1\}^n$, we have*

$$\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) = \rho_{\mathrm{Sp}_{2n}}(v);$$

$$\rho_{\mathrm{Sp}_{2n}}(v) \;\le\; \underline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) \;\le\; \overline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) \;\le\; 2\rho_{\mathrm{Sp}_{2n}}(v).$$

On the proof: We outline the proof of Theorem 1.5. The proof of Theorem 1.1 is entirely analogous; one simply uses the bijection between *primitive* units and their minimal polynomials. Let $v \in \mathsf{H}_+$ and m a sign pattern. We translate the geometric condition "$\lambda(\gamma)$ lies in $B_\varepsilon(Tv)$ with sign pattern m" into a purely arithmetic statement about integral polynomials, and then we count those polynomials. For a loxodromic element $\gamma \in \mathrm{SL}_n(\mathbb{Z})$, its eigenvalue pattern $\mathcal{E}(\gamma)$ is equivalent to its characteristic polynomial $p_\gamma(x)$. Requiring $\lambda(\gamma) \in B_\varepsilon(Tv)$ and $m(\gamma) = m$ forces the roots of p_γ to satisfy $m_i e^{Tv_i} + O(\varepsilon e^{Tv_i})$, $1 \le i \le n$. Let $\mathcal{Q}_T(v, m; \varepsilon)$ denote the collection of all monic integral polynomials with this property. Using Rouché's theorem, we observe that $p \in \mathcal{Q}_T(v, m; \varepsilon)$ iff each coefficient lies in an interval of length $(1 + O(\varepsilon))e^{T(v_1+\cdots+v_i)}$. Hence $\mathcal{Q}_T(v, m; \varepsilon)$ coincides with an expanding box $\mathcal{P}_T(v, m; \varepsilon)$ inside $\mathbb{Z}^{n-1}$ whose side-lengths grow at precisely those exponential rates. Counting integral points in this expanding box is governed by the square-root of the discriminant of the model polynomial $q_{Tv,m}(x) = \prod_{i=1}^n (x - m_i e^{Tv_i})$ with $\mathrm{Disc}(q_{Tv,m}) \asymp e^{2\rho_{\mathrm{SL}_n}(v)T}$. Exactly the same reasoning works for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$. Here one exploits the fact that the characteristic polynomials of $\mathrm{Sp}_{2n}(\mathbb{Z})$ matrices are *precisely* the integral monic reciprocal (palindromic) polynomials of degree $2n$ ([33], [20]). Because the reciprocal property simply folds the coefficient box in half, the counting again reduces to a volume estimate and the resulting exponent is $\rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^n (n + 1 - i)v_i$. For other arithmetic groups, no tidy description is available for the integral polynomials that arise as characteristic polynomials. Even in the case of integral orthogonal groups, a clean criterion necessary for this approach to work does not seem to be known.

On the other hand, the upper bound for the conjugacy-class count in Theorem 1.5 is a special case of Theorem 6.2, which applies to any lattice in a semisimple real algebraic group. The proof proceeds by relating the Jordan projection to the Cartan projection and by applying the standard orbital-counting technique of Eskin-McMullen [10], which exploits the mixing of the G-action on $\Gamma\backslash G$ and the strong wavefront lemma ([14, Theorem 3.7]).

We conclude the introduction by formulating the following conjecture:

Conjecture 1.12. *Let Γ be an arithmetic lattice of a connected simple real algebraic group G. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and let ρ_G be the half-sum of all positive roots of $(\mathfrak{g}, \mathfrak{a}^+)$, where $\mathfrak{g} = \mathrm{Lie}\, G$. For $v \in \mathrm{int}\, \mathfrak{a}^+$, define directional entropies $\mathsf{E}_\Gamma(v)$ and $\mathsf{E}^\star_\Gamma(v)$ as in Definition 6.1. Then*

$$\mathsf{E}_\Gamma(v) = \rho_G(v) \quad \textit{and} \quad \mathsf{E}^\star_\Gamma(v) = 2\,\rho_G(v).$$

If G has rank-one, the prime geodesic theorem for rank-one locally symmetric manifolds (see, for instance, [29], [15], [21], [12], [27], [23], etc.) implies that $\mathsf{E}^\star_\Gamma(v) = 2\rho_G(v)$. As we shall see in Theorem 6.4, the corresponding entropy $\mathsf{E}^\star_\Gamma(v)$ defined via the Cartan projection is always $2\rho_G(v)$; it seems plausible that the Jordan and Cartan counts differ only by a polynomial factor, in which case the second equality in the above would indeed hold.

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2. Root separations and Proof of Theorem 1.1

Let $n \geq 2$. As $T \to \infty$, the number of monic integral polynomials of degree n whose roots are bounded by e^T grows in the order of $e^{n(n+1)T/2}$. If we additionally require the constant term to be ± 1, the growth rate drops to the order $e^{n(n-1)T/2}$. These orders remain unchanged when we restrict to totally real polynomials [3].

In this section, we fix a vector $v \in \mathsf{H}_+$ and a sign pattern

$$m = (m_1, \cdots, m_n) \in \{\pm 1\}^n,$$

and count those polynomials whose roots lie near the prescribed points

$$m_i e^{Tv_i} \quad 1 \leq i \leq n,$$

up to an additive error order $O(\varepsilon e^{Tv_i})$ for a fixed $\varepsilon > 0$. The proof relies on translating the information about the roots into precise size constraints on the polynomial's coefficients.

Definition 2.1. For $\varepsilon > 0$ and $T > 1$, denote by

$$\mathcal{Q}_T(v, m; \varepsilon) \quad (\text{resp. } \mathcal{Q}_T^{\mathrm{irr}}(v, m; \varepsilon))$$

the set of all monic integral (resp. irreducible[3]) polynomials with roots $x_1, \cdots, x_n$ such that

$$|x_i - m_i e^{Tv_i}| \leq \varepsilon e^{Tv_i} \qquad \text{for all } i = 1, \cdots, n. \tag{2.1}$$

Set

$$\delta_v := \min_{1 \leq i \leq n-1} (v_1 + \cdots + v_i) \tag{2.2}$$

Theorem 2.2. *Let $\varepsilon > 0$. As $T \to \infty$,*

$$\#\mathcal{Q}_T(v, m; \varepsilon) \asymp_\varepsilon \exp \tfrac{1}{2} \sum_{i<j} (v_i - v_j) T.$$

More precisely, there exist absolute constants $c_1, c_2 > 0$ such that for all T large enough depending on n and ε,

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 - c_1 \tfrac{n}{\varepsilon} e^{-\delta_v T}\right) \leq \#\mathcal{Q}_T(v, m; \varepsilon)$$
$$\leq (2\varepsilon)^{n-1}\, n!\, e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 + c_2 \tfrac{n}{\varepsilon} e^{-\delta_v T}\right)$$

and

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 - c_1 \tfrac{2^n}{\varepsilon} e^{-\min(\delta_v, \eta_v)T}\right) \leq \#\mathcal{Q}_T^{\mathrm{irr}}(v, m; \varepsilon)$$
$$\leq (2\varepsilon)^{n-1}\, n!\, e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 + c_2 \tfrac{2^n}{\varepsilon} e^{-\min(\delta_v, \eta_v)T}\right)$$

where $\eta_v > 0$ is defined in (2.15).

[3] Throughout the paper, irreducible means irreducible over $\mathbb{Z}$

This theorem follows from three lemmas 2.4, 2.5 and 2.8 below.

To motivate Definition 2.3, let us first examine the size of each coefficient of the following reference polynomial

$$q_T(x) = q_{Tv,m}(x) \coloneqq \prod_{i=1}^{n}(x - m_i e^{Tv_i}). \tag{2.3}$$

Writing $q_T(x) = \sum_{k=0}^{n}(-1)^{n-k} b_{n-k} x^k = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^n b_n$, Vieta's formulas give:

$$b_i = \sum_{\substack{S\subset\{1,\ldots,n\}\\ |S|=i}} \Big(\prod_{j\in S} m_j\Big) e^{T\sum_{j\in S} v_j} \quad 1 \le i \le n.$$

Therefore, for any $0 < \varepsilon < 1$, there exists $T_1 = T_1(v, \varepsilon) > 0$ such that for all $T \ge T_1$ and all $1 \le i \le n$,

$$(1-\varepsilon)e^{T(v_1+\cdots+v_i)} \;\le\; b_i M_i \;\le\; (1+\varepsilon)e^{T(v_1+\cdots+v_i)}$$

where $M_i = \prod_{j=1}^{i} m_j$.

Definition 2.3. For $0 < \varepsilon < 1$, let $\mathscr{P}_T(v, m; \varepsilon)$ be the set of all monic integral polynomials

$$p(x) = \sum_{i=0}^{n} (-1)^{n-i} a_{n-i} x^i$$

such that for all $1 \le i \le n$,

$$(1-\varepsilon)e^{T(v_1+\cdots+v_i)} \le a_i M_i \le (1+\varepsilon)e^{T(v_1+\cdots+v_i)}. \tag{2.4}$$

We also define $\mathscr{P}'_T(v, m; \varepsilon)$ to be the set of all monic integral polynomials $p(x) = \sum_{i=0}^{n}(-1)^{n-i} a_{n-i} x^i$ such that for all $1 \le i \le n$,

$$(1-(i+1)\varepsilon)e^{T(v_1+\cdots+v_i)} \le a_i M_i \le (1+(i+1)\varepsilon)e^{T(v_1+\cdots+v_i)}. \tag{2.5}$$

Note that for all $0 < \varepsilon < 1/(n+1)$, any $p \in \mathscr{P}'_T(v, m; \varepsilon)$ satisfies $a_n = M_n$. Throughout the paper, we repeatedly use the following simple identity:

$$\sum_{i=1}^{n-1}(v_1 + \cdots + v_i) = \tfrac{1}{2}\sum_{1\le i<j\le n}(v_i - v_j) = \sum_{i=1}^{n-1}(n-i)v_i \tag{2.6}$$

where $\sum_{i=1}^{n} v_i = 0$ is used.

We immediately get the following by counting integral vectors in the axis-parallel boxes given by (2.4) and (2.5) from the classical theorem of Davenport [8]. Recall the constant $\delta_v > 0$ from (2.2).

Lemma 2.4. *For any $0 < \varepsilon < \frac{1}{n+1}$ and $T > 1$ large enough, we have we have*

$$(2\varepsilon)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T}\left(1 - \tfrac{n}{\varepsilon}e^{-\delta_v T}\right) \le \#\mathscr{P}_T(v, m; \varepsilon)$$
$$\le (2\varepsilon)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T}\left(1 + \tfrac{n}{\varepsilon}e^{-\delta_v T}\right)$$

and

$$(2\varepsilon)^{n-1} n! e^{\frac{1}{2}\sum_{i<j}(v_i-v_j)T}\left(1-\tfrac{n}{\varepsilon}e^{-\delta_v T}\right) \leq \#\mathscr{P}'_T(v,m;\varepsilon) \leq (2\varepsilon)^{n-1} n! e^{\frac{1}{2}\sum_{i<j}(v_i-v_j)T}\left(1+\tfrac{n}{\varepsilon}e^{-\delta_v T}\right)$$

Proof. Davenport's theorem [8] gives that for any axix-parallel box $B = \prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$,

$$|\#(B\cap \mathbb{Z}^d) - \mathrm{Vol}(B)| \leq \sum_{i=1}^d \prod_{j\neq i}(b_j - a_j) = \sum_{i=1}^d \frac{\mathrm{vol}(B)}{b_i - a_i}. \tag{2.7}$$

Setting $E_i = e^{(v_1+\cdots+v_i)T}$ and $B = \prod_{i=1}^{n-1}[(1-\varepsilon)E_i, (1+\varepsilon)E_i]$, the number $\#\mathscr{P}_T(v,m;\varepsilon)$ is same as $\#(\mathbb{Z}^{n-1}\cap B)$ and hence the first claim follows by (2.6) and (2.7). The second claim follows similarly. □

Lemma 2.5 (Root approximation)**.** *Let $c_n = (n-1)3^n$. For any $0 < \varepsilon < 1/(4c_n)$, there exists $T_0 = T_0(v,\varepsilon) \geq 1$ such that for all $T \geq T_0$, every polynomial $p \in \mathscr{P}_T(v,m;\varepsilon)$ has n-distinct real roots $x_1, \cdots, x_n$ with*

$$|x_i - m_i e^{Tv_i}| \leq c_n \varepsilon e^{Tv_i} \qquad \textit{for all } i = 1, \cdots, n. \tag{2.8}$$

Conversely, any monic polynomial $p \in \mathbb{Z}[x]$ with roots $x_1, \cdots, x_n$ satisfying (2.8) belongs to $\mathscr{P}'_T(v,m;\varepsilon)$.

In other words, for all T sufficiently large,

$$\mathscr{P}_T(v,m;\frac{\varepsilon}{c_n}) \subset \mathscr{Q}_T(v,m;\varepsilon) \subset \mathscr{P}'_T(v,m;\varepsilon).$$

Proof. The second statement is a simple consequence of Vieta's formulas. Let $w_i = \sum_{j=1}^i v_j$ for each $1 \leq i \leq n-1$. Let q_T be as in (2.3) so that $q_T(x) = \sum_{i=0}^n (-1)^{n-i} b_{n-i} x^i$. So for all $T \geq T_1(v,\varepsilon)$ and for all $1 \leq i \leq n-1$,

$$e^{Tw_i - \varepsilon} \leq b_i M_i \leq e^{Tw_i + \varepsilon}.$$

By increasing T_1 if necessary, we may assume that $e^{Tv_i} \geq 3e^{Tv_{i+1}}$ and $e^{Tw_i} \geq 3e^{Tw_{i+1}}$ for all i. Consequently $|e^{Tv_i} - e^{Tv_{i+1}}| \geq \left(\frac{2}{3}\right)e^{Tv_i}$.

Fix $0 < \varepsilon < (4c_n)^{-1}$. Then we have

$$3(n-1)(1+c_n\varepsilon)^{n-1} < c_n(2/3 - c_n\varepsilon)^{n-1}. \tag{2.9}$$

To check this, we note that $f(\varepsilon) = \frac{3(n-1)(1+c_n\varepsilon)^{n-1}}{c_n(2/3-c_n\varepsilon)^{n-1}}$ is a strictly increasing function on the interval $(0, (4c_n)^{-1})$ and $f((4c_n)^{-1}) = 1$.

For each $1 \leq j \leq n$, consider the discs

$$D_j = \{x \in \mathbb{C} : |x - m_j e^{Tv_j}| \leq c_n \varepsilon e^{Tv_j}\}.$$

Since $c_n\varepsilon \leq 1/4$, we have $(1+c_n\varepsilon)e^{Tv_{j+1}} < (1-c_n\varepsilon)e^{Tv_j}$ for all j and hence these discs are pairwise disjoint.

Let $p_T(x) = \sum_{i=0}^{n}(-1)^{n-i}a_{n-i}x^i$ be a polynomial in $\mathscr{P}_T(v,m;\varepsilon)$. We claim that for all T sufficiently large, $p_T(x)$ has precisely one root inside each disc D_j. Write

$$\Delta_T(x) \coloneqq q_T(x) - p_T(x) = \sum_{i=1}^{n-1}(-1)^{n-i}(b_{n-i} - a_{n-i})x^i.$$

From (2.4), $|b_{n-i} - a_{n-i}| \le 3\varepsilon\, e^{Tw_{n-i}}$. Hence for all $x \in \partial D_j$, we have

$$|\Delta_T(x)| \le 3\varepsilon\big(1 + c_n\varepsilon\big)^{n-1}\sum_{i=1}^{n-1} e^{Tw_{n-i}+iTv_j}. \tag{2.10}$$

On the other hand,

$$|q_T(x)| = \prod_{i=0}^{n}|x - m_i e^{Tv_i}| = c_n\varepsilon e^{Tv_j}\prod_{i<j}|x - m_i e^{Tv_i}|\cdot\prod_{i>j}|x - m_i e^{Tv_i}|.$$

For $i < j$,

$$|x - m_i e^{Tv_i}| \ge |m_i e^{Tv_i} - m_j e^{Tv_j}| - |x - m_j e^{Tv_j}| \ge \frac{2}{3}e^{Tv_i} - c_n\varepsilon e^{Tv_j} \ge (\frac{2}{3} - c_n\varepsilon)e^{Tv_i}.$$

For $i > j$, we similarly have

$$|x - m_i e^{Tv_i}| \ge (\frac{2}{3} - c_n\varepsilon)e^{Tv_j}.$$

Hence

$$|q_T(x)| \ge c_n\varepsilon\left(\frac{2}{3} - c_n\varepsilon\right)^{n-1} e^{T(\sum_{i<j} v_i)+(n-j+1)Tv_j}. \tag{2.11}$$

Let

$$S(T,j) = \sum_{i=1}^{n-1} e^{Tw_{n-i}+iTv_j} \quad\text{and}\quad R(T,j) = e^{T(\sum_{k<j} v_k)+(n-j+1)Tv_j}.$$

Let

$$\Delta_{i,j} = \left(\sum_{k<j} v_k + (n-j+1)v_j\right) - (v_1 + \cdots + v_{n-i} + iv_j)$$

so that

$$\frac{S(T,j)}{R(T,j)} = \sum_{i=1}^{n-1} e^{-T\Delta_{i,j}}. \tag{2.12}$$

We check that $\Delta_{i,j} \ge 0$ by writing

$$\Delta_{i,j} = \begin{cases}(n-i-(j-1))v_j - (v_j + \cdots + v_{n-i}) \ge 0 & \text{if } n-i \ge j-1,\\ (v_{n-i+1} + \cdots + v_{j-1}) - ((j-1)-(n-i))v_j \ge 0 & \text{otherwise.}\end{cases} \tag{2.13}$$

Hence by (2.12), we have

$$\frac{S(T,j)}{R(T,j)} \le n-1.$$

Therefore by (2.10) and (2.11), and since ε satisfies (2.9), we get that for all $x \in \partial D_j$,

$$|\Delta_T(x)| \leq 3\varepsilon(1+c_n\varepsilon)^{n-1}S(T,j) \leq (n-1)3\varepsilon(1+c_n\varepsilon)^{n-1}R(T,j) < c_n\varepsilon(2/3-c_n\varepsilon)^{n-1}R(T,j) \leq |q_T(x)|,$$

and hence

$$|\Delta_T(x)| < |q_T(x)|.$$

Hence by Rouché's theorem (cf. [2]), two polynomials $q_T(x)$ and $p_T(x)$ have the same number of zeros (counted with multiplicity) inside each D_j. Since D_j are pairwise disjoint and q_T has exactly one root in each D_j, the same holds for p_T. Since p_T has real coefficients and each D_j is invariant under complex conjugation, $p_T(x)$ has one *real* root x_j such that

$$|x_j - m_j e^{Tv_j}| \leq c_n\varepsilon e^{Tv_j}.$$

Hence $p_T \in \mathscr{Q}_T(v, m; c_n\varepsilon)$. This finishes the proof. □

Denote by $\operatorname{Disc}(p) = \prod_{i \neq j}(x_i - x_j) = \prod_{i<j}(x_i - x_j)^2$ the discriminant of a polynomial p with roots $x_1, \cdots, x_n$. For the polynomial $q_{Tv,m}(x) = \prod_{i=1}^n (x - m_i e^{Tv_i})$, its discriminant $\operatorname{Disc}(q_{Tv,m})$ satisfies

$$\operatorname{Disc}(q_{Tv,m}) = e^{\left(\sum_{1\leq i<j\leq n} v_i\right)T}\left(1 + O(e^{-\eta T})\right) \qquad \text{for some } \eta > 0$$

and hence

$$\lim_{T\to\infty} \tfrac{1}{T}\log \operatorname{Disc}(q_{Tv,m}) = \sum_{i<j}(v_i - v_j).$$

The following is a simple consequence of Lemma 2.5:

Corollary 2.6. *For all small $\varepsilon > 0$, there exist $T_0 = T_0(v,\varepsilon) > 0$ such that for all $T \geq T_0$, every polynomial $p_T \in \mathscr{P}_T(v, m; \varepsilon)$ satisfies*

$$(1 - C_n\varepsilon)e^{\sum_{i<j}(v_i - v_j)T} \leq \operatorname{Disc}(p_T) \leq (1 + C_n\varepsilon)e^{\sum_{i<j}(v_i - v_j)T}$$

where $C_n = n(n-1)/2$. In particular,

$$\lim_{T\to\infty} \tfrac{1}{T}\log \operatorname{Disc}(p_T) = \sum_{i<j}(v_i - v_j).$$

We will need the following estimates in the next lemma 2.8:

Lemma 2.7. *Let $\{1, \ldots, n\} = S_1 \sqcup S_2$ be a partition into two non-empty subsets so that $\sum_{i\in S_j} v_i = 0$ for $j = 1, 2$. Writing $S_1 = \{i_1 < \cdots < i_{\ell_1}\}$ and $S_2 = \{j_1 < \cdots < j_{\ell_2}\}$ with $\ell_1 + \ell_2 = n$, we have*

$$\sum_{k=1}^{\ell_1-1}(v_{i_1} + \cdots + v_{i_k}) \; + \; \sum_{k=1}^{\ell_2-1}(v_{j_1} + \cdots + v_{j_k}) \; < \; \sum_{k=1}^{n-1}(v_1 + \cdots + v_k)\,.$$

Proof. We first rewrite the right hand side (RHS) as

$$\sum_{k=1}^{n-1}(v_1+\cdots+v_k)=\sum_{k=1}^{n-1}\sum_{i=1}^{k}v_i=\sum_{i=1}^{n-1}(n-i)\,v_i=\sum_{1\le i<j\le n}v_i.$$

Similarly, the left hand side, for each $j=1,2$,

$$\sum_{k=1}^{\ell_j-1}(v_{i_1}+\cdots+v_{i_k})=\sum_{i<j,i,j\in S_j}v_i,$$

and hence

$$\text{LHS}=\sum_{i<j,i,j\text{ in the same }S_*}v_i.$$

Set

$$D_v(S_1,S_2):=\sum_{k=1}^{n-1}(v_1+\cdots+v_k)-\left(\sum_{k=1}^{\ell_1-1}(v_{i_1}+\cdots+v_{i_k})\;+\;\sum_{k=1}^{\ell_2-1}(v_{j_1}+\cdots+v_{j_k})\right). \tag{2.14}$$

Hence

$$D_v(S_1,S_2)=(\text{RHS})-(\text{LHS})=\sum_{i<j,S(i)\neq S(j)}v_i,$$

where $S(i)$ denotes the block containing i. Add the same pairs with the complementary index:

$$\sum_{i<j,S(i)\neq S(j)}(v_i+v_j)=|S_2|\sum_{i\in S_1}v_i\;+\;|S_1|\sum_{j\in S_2}v_j=0,$$

because each block has total sum 0. Hence $D_v(S_1,S_2)=-\sum_{i<j,S(i)\neq S(j)}v_j$; so

$$2D_v(S_2,S_2)=\sum_{i<j,S(i)\neq S(j)}(v_i-v_j).$$

Since $v_1>\cdots>v_n$, every difference v_i-v_j $(i<j)$ is strictly positive. There is at least one cross pair (the blocks are non-empty), so $D_v(S_1,S_2)>0$. □

Define

$$\eta_v:=\min D_v(S_1,S_2)>0 \tag{2.15}$$

where $D_v(S_1,S_2)$ is as in (2.14) and the minimum is taken over all non-trivial partitions of $\{1,\cdots,n\}=S_1\sqcup S_2$. The function $v\mapsto\eta_v$ is clearly continuous on H_+ and hence $\min_{v\in Q}\eta_v>0$ for any compact subset $Q\subset\mathsf{H}_+$.

Lemma 2.8. *Let $0<\varepsilon<1$. As $T\to\infty$, the proportion of irreducible polynomials in $\mathscr{P}_T(v,m;\varepsilon)$ tends to 1 exponentially fast: there exists $T_0=T_0(n,\varepsilon)$ such that for all $T\geq T_0$,*

$$\#\{p\in\mathscr{P}_T(v,m;\varepsilon)\text{ irreducible}\}=\#\mathscr{P}_T(v,m;\varepsilon)\cdot\left(1+O(2^n\varepsilon^{-1}e^{-\eta_vT})\right)$$

where the implied constant is an absolute constant. The same type of estimate holds for $\mathscr{P}'_T(v,m;\varepsilon)$.

Proof. Let $T \geq T_0(v, \varepsilon)$ be as in Lemma 2.5. For any $p \in \mathscr{P}_T(v, m; \varepsilon)$, by Lemma 2.5, p has distinct roots $x_1, \cdots, x_n$ such that $|x_i - m_i e^{Tv_i}| \leq c_n \varepsilon e^{Tv_i}$ for each $1 \leq i \leq n$. Suppose that $p \in \mathscr{P}_T(v, m; \varepsilon)$ is reducible over $\mathbb{Z}$. We then have a partition of $\{1, \cdots, n\}$ as the disjoint union $S_1 \sqcup S_2$ of non-empty subsets such that $p(x) = f_1(x) f_2(x)$ where $f_j(x) = \prod_{k \in S_j} (x - x_k) \in \mathbb{Z}[x]$ for $j = 1, 2$. List elements of S_j as $j_1 > j_2 > \cdots > j_{\ell_j}$. Let $u_j = (v_{j_1}, \cdots, v_{j_{\ell_j}})$ and $M'_j = (m_{j_1}, \cdots, m_{j_{\ell_j}})$. It follows that

$$f_j \in \mathscr{P}_T(u_j, M'_j; \varepsilon).$$

Hence for all sufficiently large T, by Lemma 2.4, for all sufficiently large $T \geq 1$, we have

$$\#\mathscr{P}_T(u_j, M'_j; \varepsilon) \leq 2(2\varepsilon)^{\ell_j - 1} e^{T \sum_{k=1}^{\ell_j - 1} (v_{j_1} + \cdots + v_{j_k})}.$$

Since the constant terms of f_1 and f_2 are ± 1, we have $\sum_{i \in S_j} v_i = 0$. By Lemma 2.7, we have

$$\sum_{k=1}^{\ell_1 - 1} (v_{1_1} + \cdots + v_{1_k}) + \sum_{k=1}^{\ell_2 - 1} (v_{2_1} + \cdots + v_{2_k}) < \sum_{k=1}^{n-1} (v_1 + \cdots + v_k),$$

that is, $D_v(S_1, S_2) > 0$. Therefore we have

$$\frac{\#\mathscr{P}_T(u_1, M'_1; \varepsilon) \cdot \#\mathscr{P}_T(u_2, M'_2; \varepsilon)}{\#\mathscr{P}_T(v, m; \varepsilon)} \leq 8\varepsilon^{-1} e^{-D_v(S_1, S_2)T}.$$

Since this holds for any non-trivial partition of $\{1, \cdots, n\}$ into two non-empty subsets and there are at most 2^{n-1} number of such partitions, this proves the first claim by the definition of η_v. The proof for $\mathscr{P}'_T(v, m; \varepsilon)$ is similar. □

Proof of Theorem 2.2. The first claim follows from Lemma 2.4 and Lemma 2.5. By Lemma 2.5 and Lemma 2.8, for all T sufficiently large, the cardinality of the set of *reducible* polynomials in $\mathscr{Q}_T(v, m; \varepsilon)$ is at most $c 2^n \varepsilon^{-1} e^{-\eta_v T} \cdot \#\mathscr{Q}_T(v, m; \varepsilon)$ for some absolute constant $c > 0$. Hence the second claim follows from this and the first claim. □

3. Directional entropy of totally real algebraic units

We now apply our polynomial analysis to compute the directional entropy for totally real units of degree n. Fix $v \in \mathsf{H}_+$ and a sign pattern $m = (m_1, \cdots, m_n) \in \{\pm 1\}^n$. We use the notation $\mathcal{K}_n, O_n^\times, \Sigma_K$, etc. from the introduction. For simplicity, for $\mathsf{u} \in (O_K^\times, \sigma)$, we write $\|\sigma(\mathsf{u}) - m e^{Tv}\| \leq \varepsilon e^{Tv}$ to mean that $|\sigma_i(\mathsf{u}) - m_i e^{Tv_i}| < \varepsilon e^{Tv_i}$ $\forall i$. For each $T > 1$, define

$$\mathscr{U}_T(v, m; \varepsilon) = \bigcup_{(K,\sigma) \in \mathcal{K}_n} \left\{ \mathsf{u} \in (O_K^\times, \sigma) : \|\sigma(\mathsf{u}) - m e^{Tv}\| \leq \varepsilon e^{Tv} \right\}.$$

Let $\mathcal{U}_T^{\text{prim}}(v,m;\varepsilon)$ be the set of all $\mathsf{u} \in (O_K^\times,\sigma)$, $(K,\sigma)\in\mathcal{K}_n$, such that the field $\mathbb{Q}(\mathsf{u})$ has degree n, or equivalently, $p(x)=\prod_{i=1}^n(x-\sigma_i(\mathsf{u}))$ is irreducible over $\mathbb{Z}$ for $\sigma=(\sigma_1,\cdots,\sigma_n)$.

Lemma 3.1. *Let $0\le\varepsilon<1/2$. Then for all sufficiently large $T>1$, we have*

$$\mathcal{U}_T(v,m;\varepsilon)=\mathcal{U}_T^{\text{prim}}(v,m;\varepsilon).$$

Proof. If $\mathsf{u}\in\mathcal{U}_T(v,m;\varepsilon)\cap(O_K^\times,\sigma)$ is non-primitive, then there is a subfield K_0 of K of degree $1<m<n$ such that $\mathsf{u}\in O_{K_0}^\times$ and each of the m-embeddings $K_0\hookrightarrow\mathbb{R}$ extends to precisely n/m-embeddings to K into $\mathbb{R}$. Therefore for some $i<j$, $\sigma_i(\mathsf{u})=\sigma_j(\mathsf{u})$. This implies that $|m_ie^{Tv_i}-m_je^{Tv_j}|\le\varepsilon e^{Tv_i}$. Since $v_i>v_j$ and hence $|m_ie^{Tv_i}-m_je^{Tv_j}|=e^{Tv_i}(1\pm e^{T(v_j-v_i)})\ge e^{Tv_i}/2$ for all sufficiently large T, u cannot be non-primitive if T is large enough. ☐

Proposition 3.2. *Let $v\in\mathsf{H}_+$. For all small $\varepsilon>0$, we have*

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1}\le\liminf_{T\to\infty}\frac{\#\mathcal{U}_T(v,m;\varepsilon)}{e^{\frac12\sum_{i<j}(v_i-v_j)T}}\le\limsup_{T\to\infty}\frac{\#\mathcal{U}_T(v,m;\varepsilon)}{e^{\frac12\sum_{i<j}(v_i-v_j)T}}\le(2\varepsilon)^{n-1}\,n!\,.$$

More precisely, there exist absolute constants $c_1,c_2>0$ such that for all large $T>1$ depending on n and ε, we have

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1}e^{\frac12\sum_{i<j}(v_i-v_j)T}\left(1-c_12^n\varepsilon^{-1}e^{-\min(\delta_v,\eta_v)T}\right)\le\#\mathcal{U}_T(v,m;\varepsilon)$$
$$\le(2\varepsilon)^{n-1}\,n!\,e^{\frac12\sum_{i<j}(v_i-v_j)T}\left(1+c_22^n\varepsilon^{-1}e^{-\min(\delta_v,\eta_v)T}\right)$$

where $v\mapsto\delta_v$ and $v\mapsto\eta_v$ are positive continuous functions given in (2.2) and (2.15) respectively.

Proof. Let $\varepsilon>0$ and $T\ge T_0(v,\varepsilon)$ as in Lemma 2.5. Let $p\in\mathcal{Q}_T^{\text{irr}}(v,m;\varepsilon)$, and let K be its splitting field, which must be a totally real number field of degree n. Let $x_1,\cdots,x_n$ be the roots of p ordered so that $|x_1|>\cdots>|x_n|$. Since $x_i\in O_K$ and $\prod_{i=1}^n x_i=\prod_{i=1}^n m_i=\pm1$, there exists a unit $\mathsf{u}\in O_K^\times$ and $\sigma=(\sigma_1,\cdots,\sigma_n)\in\Sigma_K$ such that $K=\mathbb{Q}(u)$ and $x_i=\sigma_i(\mathsf{u})$. Hence

$$\#\mathcal{Q}_T^{\text{irr}}(v,m;\varepsilon)\le\#\mathcal{U}_T^{\text{prim}}(v,m;\varepsilon).$$

Conversely, let $\mathsf{u}\in\mathcal{U}_T^{\text{prim}}(v,m;\varepsilon)\cap(O_K^\times,\sigma)$. Setting $p(x)=\prod(x-\sigma_i(\mathsf{u}))$, we have $p\in\mathcal{Q}_T^{\text{irr}}(v,m;O(\varepsilon))$. Moreover, this map is injective for T large enough. To see this, suppose that there exist $\mathsf{u}\in\mathcal{U}_T(v,m;\varepsilon)^{\text{prim}}\cap(O_K^\times,\sigma)$ and $\mathsf{u}'\in\mathcal{U}_T(v,m;\varepsilon)^{\text{prim}}\cap O_{K'}^\times,\sigma')$ such that $p(x)=q(x)$ where $p(x)=\prod(x-\sigma_i(\mathsf{u}))$ and $q(x)=\prod(x-\sigma_i'(\mathsf{u}'))$. Since K and K' must be the splitting fields of p and q respectively, $K=K'$ and $\{\sigma_i'(\mathsf{u}'):i=1,\cdots,n\}=\{\sigma_i(\mathsf{u}):i=1,\cdots,n\}$. Since the intervals $(Tv_i-\varepsilon,Tv_i+\varepsilon)$ are pairwise disjoint once T is sufficiently big, and $\log\sigma_i(\mathsf{u}),\log\sigma_i'(\mathsf{u}')\in(Tv_i-\varepsilon,Tv_i+\varepsilon)$

for all T sufficiently large, we must have $\sigma_i(\mathsf{u}) = \sigma_i'(\mathsf{u}')$ for all $1 \le i \le n$. Hence for all sufficiently large $T \gg 1$,

$$\#\mathcal{U}_T^{\text{prim}}(v, m; \varepsilon) \le \#\mathcal{Q}_T^{\text{irr}}(v, m; \varepsilon).$$

Therefore the claim follows from Lemma 3.1 and Theorem 2.2. □

Observe that once T is sufficiently large depending only on v and ε, the sets $\mathcal{U}_T(v, m; \varepsilon)$, $m \in \{\pm 1\}^n$, are pairwise disjoint. Since

$$\{u \in O_n^\times : \|\Lambda(u) - Tv\| \le \varepsilon\} = \bigsqcup_{m \in \{\pm 1\}^n} \mathcal{U}_T(v, m; \varepsilon),$$

Theorem 1.1 follows from Proposition 3.2.

Define

$$\mathsf{E}_n(v, m) = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \log \#\mathcal{U}_T(v, m; \varepsilon),$$

if the limit exists. As an immediate consequence of Proposition 3.2, we have:

Theorem 3.3. *We have*

$$\mathsf{E}_n(v, m) = \tfrac{1}{2} \sum_{i<j} (v_i - v_j).$$

4. Eigenvalue entropy of $\mathrm{SL}_n(\mathbb{Z})$

In this section, we count the eigenvalue patterns of $\mathrm{SL}_n(\mathbb{Z})$ that lie in a thin tube around a fixed ray, invoking Theorem 2.2.

Fix $v \in \mathsf{H}_+$ and a sign pattern $m = (m_1, \cdots, m_n) \in \{\pm 1\}^n$ with $\prod_{i=1}^n m_i = 1$. Let

$$\rho_{\mathrm{SL}_n}(v) = \frac{1}{2} \sum_{i<j} (v_i - v_j);$$

be the half-sum of all positive roots of $\mathrm{SL}_n(\mathbb{R})$. For each loxodromic element $g \in \mathrm{SL}_n(\mathbb{R})$, let $\mathcal{E}(g)$, $\lambda(g)$ and $m(g)$ be its eigenvalue pattern, Jordan projection and sign pattern as defined in (1.1) and (1.2). Set

$$J_T(v, m; \varepsilon) := \#\Big\{(\lambda(\gamma), m(\gamma)) : \gamma \in \mathrm{SL}_n(\mathbb{Z}), \|\lambda(\gamma) - Tv\|_{\max} \le \varepsilon, m(\gamma) = m\Big\}.$$

Theorem 4.1. *For all small $\varepsilon > 0$, we have*

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1} \le \liminf_{T \to \infty} \frac{\#J_T(v, m; \varepsilon)}{e^{\rho_{\mathrm{SL}_n}(v) T}} \le \limsup_{T \to \infty} \frac{\#J_T(v, m; \varepsilon)}{e^{\rho_{\mathrm{SL}_n}(v) T}} \le (2\varepsilon)^{n-1} n!.$$

In particular,

$$\mathsf{E}_{\mathrm{SL}_n(\mathbb{Z})}(v, m) = \rho_{\mathrm{SL}_n}(v). \tag{4.1}$$

Proof. There exists $T_1 = T_1(v, \varepsilon) > 0$ such that for all $T \ge T_1$ and for each $(\lambda(\gamma), m(\gamma)) \in J_T(v, m; \varepsilon)$, the polynomial $p(x) = \prod (x - m_i(\gamma) e^{\lambda_i(\gamma)})$ belongs to $\mathcal{Q}_T(v, m; \varepsilon)$. Since this gives an injective map, we have $\#J_T(v, m; \varepsilon) \le \#\mathcal{Q}_T(v, m; \varepsilon)$.

Let $f \in \mathcal{Q}_T(v,m;\varepsilon) = \sum_{i=0}^n (-1)^{n-i} a_{n-i} x^i$. Consider the companion matrix of f:

$$C_f = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n+1}a_n \\ 1 & 0 & \cdots & 0 & (-1)^n a_{n-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_2 \\ 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}$$

Since $\det C_f = a_n = \prod_{i=1}^n m_i = 1$, we have $C_f \in \mathrm{SL}_n(\mathbb{Z})$. If $x_1, \cdots, x_n$ are distinct real roots of f ordered so that $|x_1| > \cdots > |x_n|$, then

$$|x_i - m_i e^{Tv_i}| \le \varepsilon e^{Tv_i}, \qquad 1 \le i \le n.$$

Therefore $\|\lambda(C_f) - Tv\| \le \varepsilon$ and $m(C_f) = m$. Hence the assignment $f \mapsto (\lambda(C_f), m(C_f))$ gives a map from the set $\mathcal{Q}_T(v,m;\varepsilon)$ to $J_T(v,m;\varepsilon)$. Since $(\lambda(C_f), m(C_f))$ describes all roots of f, this map is also injective. Therefore $\#J_T(v,m;\varepsilon) \ge \#\mathcal{Q}_T(v,m;\varepsilon)$. Hence the claim follows from Theorem 2.2. ∎

The lower bound stated below follows directly from Theorem 4.1 and the corresponding upper bound will be proved in Theorem 6.2 in a more general setting.

Theorem 4.2. *For each $v \in \mathsf{H}_+$, each sign pattern $m \in \{\pm 1\}$ with $\prod_{i=1}^n m_i = 1$, and $\varepsilon > 0$, there exist $C_1, C_2 > 0$ such that*

$$C_1 e^{\rho_{\mathrm{SL}_n}(v)T} \le \#\Big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \le \varepsilon, m(\gamma) = m\Big\} \le C_2 e^{2\rho_{\mathrm{SL}_n}(v)T}.$$

In particular,

$$\rho_{\mathrm{SL}_n}(v) \le \underline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) \;\le\; \overline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) \le 2\rho_{\mathrm{SL}_n}(v). \qquad (4.2)$$

Corollary 4.3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Let $v = v_{\|\cdot\|} \in \mathsf{H}_+$ be a unit vector such that $\max_{\|v\|=1} \rho_{\mathrm{SL}_n}(v) = \rho_{\mathrm{SL}_n}(v_{\|\cdot\|})$. There exists $C > 0$ such that that for all $T > 1$,*

$$\#\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma)\| < T, \gamma \in \mathrm{SL}_n(\mathbb{Z})\} \ge C e^{\rho_{\mathrm{SL}_n}(v_{\|\cdot\|})T}.$$

(1) *For the Euclidean norm $\|\cdot\|_{\mathrm{Euc}}$, $\rho_{\mathrm{SL}_n}(v_{\|\cdot\|_{\mathrm{Euc}}}) = \sqrt{\frac{n(n^2-1)}{12}}$.*
(2) *For the maximum norm $\|\cdot\|_{\max}$, $\rho_{\mathrm{SL}_n}(v_{\|\cdot\|_{\max}})$ is $\lfloor n^2/4 \rfloor$.*

Proof. Let $N(T) := \#\{\lambda(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z}), \|\lambda(\gamma)\| < T\}$. Since

$$N(T) \ge \#J_{m,T-\varepsilon}(v_{\|\cdot\|}, \varepsilon)$$

for any sign pattern m with $\prod m_j = 1$, the desired lower bound for $N(T)$ follows from Theorem 4.1.

Since $2\rho_{\mathrm{SL}_n}(v) = \sum_{k=1}^n (n+1-2k)v_k$, its maximum on the unit sphere (for the Euclidean norm) is attained in the direction of $(n+1-2k)_{k=1}^n$. Hence if we write $v_{\|\cdot\|_{\mathrm{Euc}}} = (v_1^*, \cdots, v_n^*)$, then

$$v_k^* = \frac{n+1-2k}{\sqrt{n(n^2-1)/3}}, \quad \text{and} \quad \rho_{\mathrm{SL}_n}(v_{\|\cdot\|_{\mathrm{Euc}}}) = \sqrt{\frac{n(n^2-1)}{12}}. \tag{4.3}$$

For the maximum norm, $v_{\|\cdot\|_{\max}}$ is given by $v_k = 1$ whenever $n+1-2k > 0$ and $v_k = -1$ whenever $n+1-2k < 0$ and $v_k = 0$ if $2k = n+1$. Then for $n = 2m$, $2\rho_{\mathrm{SL}_n}(\|v\|_{\max}) = \sum_{k=1}^{m}(2m+1-2k) + \sum_{k=m+1}^{2m}(2m+1-2k)(-1) = 2m^2 = n^2/2$, and for $n = 2m+1$, $2\rho_{\mathrm{SL}_n}(\|v\|_{\max}) = m(m+1) = (n^2-1)/2$. □

5. Reciprocal polynomials and Counting for $\mathrm{Sp}_{2n}(\mathbb{Z})$

In this section, we investigate directional entropies for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$. Our estimates rely on the anaylsis of reciprocal polynomials.

Reciprocal polynomials. A monic polynomial $p \in \mathbb{R}[x]$ of degree $2n$ is called *reciprocal* (also called *palindromic*) if

$$p(x) = x^{2n} p(x^{-1}).$$

Equivalently,

$$p(x) = \sum_{k=0}^{2n} (-1)^{2n-k} a_{2n-k}\, x^k, \qquad a_0 = a_{2n} = 1, \quad a_i = a_{2n-i} \ \ (1 \le i \le n);$$

or

$$p(x) = \prod_{i=1}^{n} (x - x_i)(x - x_i^{-1}), \qquad x_1, \ldots, x_n \in \mathbb{C} - \{0\}.$$

Let

$$\mathfrak{a}^+ = \{v = \mathrm{diag}(v_1, \cdots, v_n, -v_n, \cdots, -v_1) : v_1 \ge \cdots \ge v_n \ge 0\}.$$

Fix

$$v \in \mathrm{int}\, \mathfrak{a}^+ \quad \text{and} \quad m = (m_1, \cdots, m_n) \in \{\pm 1\}^n.$$

Definition 5.1. Let $\varepsilon > 0$ and $T > 1$. Let $\mathfrak{Q}_T^*(v, m; \varepsilon)$ (resp. $\mathfrak{Q}_T^{*,\mathrm{irr}}(v, m; \varepsilon)$) be the set of all monic integral (resp. irreducible) reciprocal polynomials with roots $x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}$ such that for all $i = 1, \cdots, n$,

$$|x_i - m_i e^{Tv_i}| \le \varepsilon e^{Tv_i}, \ \ |x_i^{-1} - m_i e^{-Tv_i}| \le \varepsilon e^{-Tv_i}.$$

Set

$$\rho^*(v) = \sum_{i=1}^{n} (v_1 + \cdots + v_i) = \sum_{i=1}^{n} (n+1-i) v_i.$$

The following theorem is a direct combination of Lemma 5.3, Corollary 5.4 and Lemma 5.5:

Theorem 5.2. *Let $\varepsilon > 0$. As $T \to \infty$,*

$$\#\mathcal{Q}_T^*(v, m; \varepsilon) \asymp_\varepsilon e^{\rho^*(v)T};$$

more precisely,

$$\left(\tfrac{2\varepsilon}{(2n-1)3^{2n}}\right)^n \le \liminf_{T\to\infty} \frac{\#\mathcal{Q}_T^*(v,m;\varepsilon)}{e^{\rho^*(v)T}} \le \limsup_{T\to\infty} \frac{\#\mathcal{Q}_T^*(v,m;\varepsilon)}{e^{\rho^*(v)T}} \le (2\varepsilon)^n (n+1)!.$$

Moreover,

$$\#\mathcal{Q}_T^{*,\mathrm{irr}}(v, m; \varepsilon) = \#\mathcal{Q}_T^*(v, m; \varepsilon)(1 + O(e^{-\eta T}))$$

for some $\eta > 0$ depending only on v.

For $T > 1$, define the model polynomial

$$q_{Tv}(x) = \prod_{i=1}^n (x - m_i e^{Tv_i})(x - m_i e^{-Tv_i}) = \sum_{k=0}^{2n} (-1)^{2n-k} b_{2n-k} x^k.$$

Then for any $\varepsilon > 0$, and sufficiently large $T \gg 1$, we have that $b_0 = 1 = b_{2n}$, $b_i = b_{2n-i}$ and

$$(1-\varepsilon) e^{T(v_1+\cdots+v_i)} \le b_i M_i \le (1+\varepsilon) e^{T(v_1+\cdots+v_i)} \qquad \text{for all } 1 \le i \le n$$

where $M_i = \prod_{j=1}^i m_i$.

Let $\mathcal{P}_T^*(v, m; \varepsilon)$ be the set of all monic reciprocal polynomials

$$p(x) = \sum_{k=0}^{2n} (-1)^{2n-k} a_{2n-k} x^k \in \mathbb{Z}[x]$$

such that

$$(1-\varepsilon) e^{T(v_1+\cdots+v_i)} \le a_i M_i \le (1+\varepsilon) e^{T(v_1+\cdots+v_i)} \qquad \text{for all } 1 \le i \le n.$$

Let $\mathcal{P}_T^{**}(v, m; \varepsilon)$ be defined by the condition that

$$(1-(i+1)\varepsilon) e^{T(v_1+\cdots+v_i)} \le a_i M_i \le (1+(i+1)\varepsilon) e^{T(v_1+\cdots+v_i)} \qquad \text{for all } 1 \le i \le n.$$

Clearly we have:

Lemma 5.3. *For all sufficiently small $\varepsilon > 0$, we have, as $T \to \infty$,*

$$\#\mathcal{P}_T^*(v, m; \varepsilon) \sim (2\varepsilon)^n e^{\rho^*(v)T} \quad \text{and} \quad \#\mathcal{P}_T^*(v, m; \varepsilon) \sim (2\varepsilon)^n (n+1)! e^{\rho^*(v)T}.$$

The following follows from Lemma 2.5:

Corollary 5.4 (Root approximation)**.** *For all sufficiently small $\varepsilon > 0$, there exists $T_0 = T_0(v, \varepsilon) > 1$ such that for all $T \ge T_0$, we have*

$$\mathcal{P}_T^*(v, m; \frac{\varepsilon}{c_{2n}}) \subset \mathcal{Q}_T^*(v, m; \varepsilon) \subset \mathcal{P}_T^{**}(v, m; \varepsilon)$$

where $c_{2n} = (2n-1)3^{2n}$.

Lemma 5.5. *For all sufficiently small $\varepsilon > 0$, there is $\eta > 0$ depending only on v such that for all T large enough, we have*

$$\frac{\#\{p \in \mathcal{P}_T^*(v, m; \varepsilon) \text{ irreducible}\}}{\#\mathcal{P}_T^*(v, m; \varepsilon)} = 1 + O(e^{-\eta T}).$$

Proof. Suppose that $p \in \mathscr{P}_T^*(v,m;\varepsilon)$ is *reducible*. Because p is *reciprocal*, every irreducible factor f of p forces its reciprocal $f^*(x) = x^{\deg f} f(x^{-1})$ to be a factor as well.

Consider first those p that factor as $p = f\cdot f^*$ with f irreducible of degree n. Write the roots of p as $x_i = m_i e^{Tv_i}\big(1+O(\varepsilon)\big)$ and $x_i^{-1} = m_i e^{-Tv_i}\big(1+O(\varepsilon)\big)$, $1 \le i \le n$. Since the constant term of f must be ± 1, if we set $P = \{i : f(x_i) = 0\} = \{j_1 < \cdots < j_\ell\}$, then $1 \le \ell < n$. By Vieta's formula, the k-th coefficient of f is bounded by $\exp\left((v_{j_1} + \cdots + v_{j_k})T\right)$, up to a multiplicative constant. Taking the product over $k = 1, \cdots, \ell$, the coefficient box for f has volume at most a constant multiple of $\exp \sum_{k=1}^{\ell}(v_{j_1} + \cdots + v_{j_k})$. Because $\ell < n$ and $v_1 > \cdots > v_n > 0$, we have $\sum_{k=1}^{\ell}(v_{j_1} + \cdots + v_{j_k}) \le \sum_{k=1}^{n}(v_1 + \cdots + v_k) - v_n = \rho^*(v) - v_n$. Hence

$$\#\{p \text{ with the factorization } p = f\, f^*\} \ll e^{(\rho^*(v)-v_n)T}. \tag{5.1}$$

All remaining reducible polynomials split as

$$p(x) = f_1(x)\, f_2(x), \qquad \deg f_j = 2s_j,\ 1 \le s_j \le n-1,\ s_1 + s_2 = n,$$

with each f_j itself a monic *reciprocal* polynomial in $\mathbb{Z}[x]$.

Let $S \subset \{1,\ldots,n\}$ record which conjugate-pairs $\{x_i, x_i^{-1}\}$ of roots of f_1 in decreasing modulus. Writing $S = \{i_1 > \cdots > i_{s_1}\}$, we obtain

$$f_1 \in \mathscr{P}_T^*\big(u_1, M_1'; \varepsilon\big), \quad f_2 \in \mathscr{P}_T^*\big(u_2, M_2'; \varepsilon\big),$$

where u_j collects the v-coordinates indexed by S and its complement, and M_j' the corresponding sign patterns.

By Lemma 5.3, we get

$$\#\Big\{p \in \mathscr{P}_T^*(v,m;\varepsilon) \text{ encoded by } S\Big\} \ll e^{\big(\rho^*(u_1)+\rho^*(u_2)\big)T}. \tag{5.2}$$

We claim that

$$\Delta(S) := \rho^*(v) - \big(\rho^*(u_1) + \rho^*(u_2)\big) > 0.$$

Write $w = (n, n-1, \ldots, 1)$ so that $\rho^*(v) = w\cdot v$. Inside the factors f_j the largest coefficient weight drops from n to at most $n-1$, while no weight increases. Because $v_1 > \cdots > v_n$, we get $w\cdot v > w'\cdot v$, where w' is the modified weight-vector attached to (u_1, u_2), implying the claim.

It now follows that

$$\#\big\{p \in \mathscr{P}_T^*(v,m;\varepsilon) \text{ reducible}\big\} \ll e^{(\rho^*(v)-\eta)T}$$

where $\eta := \min_S \Delta(S) > 0$. Combined with Lemma 5.3, this completes the proof. □

Jordan projections of $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let

$$G = \mathrm{Sp}_{2n}(\mathbb{R}) = \{g \in \mathrm{SL}_{2n}(\mathbb{R}) : g^t J_n g = J_n\} \quad J_n = \begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix} \tag{5.3}$$

where $\bar{I}_n$ is the anti-diagonal identity matrix.

Then $\mathfrak{a}^+$ is a positive Weyl chamber. For a loxodromic element $g \in G$, its Jordan projection is given by

$$\lambda(g) = (\lambda_1(g), \cdots, \lambda_n(g), -\lambda_n(g), \cdots, -\lambda_1(g)) \in \operatorname{int}\mathfrak{a}^+$$

and its eigenvalue datum is

$$\mathcal{E}(g) = (m_1(g)e^{\lambda_1(g)}, \cdots, m_n(g)e^{\lambda_n(g)}, m_n(g)e^{-\lambda_n(g)}, \cdots, m_1(g)e^{-\lambda_1(g)})$$

where $m_i(g) \in \{\pm 1\}$, $1 \le i \le n$.

Theorem 5.6. *([33], [20], [1]) Every integral monic reciprocal polynomial is the characteristic polynomial of some element of* $\mathrm{Sp}_{2n}(\mathbb{Z})$.

We define $\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m)$ and $\mathsf{E}^\star_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m)$ exactly as in Definition 1.8, replacing $\mathrm{SL}_n(\mathbb{Z})$ by $\mathrm{Sp}_{2n}(\mathbb{Z})$ throughout.

Observe that

$$\rho^*(v) = \rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^n (n+1-i)v_i$$

where $\rho_{\mathrm{Sp}_{2n}}$ is the half-sum of all positive roots of $(\mathfrak{sp}_{2n}(\mathbb{R}), \mathfrak{a})$.

Theorem 5.7. *Let* $v \in \operatorname{int}\mathfrak{a}^+$ *and* $m = (m_1, \dots, m_n) \in \{\pm 1\}^n$. *For* $\varepsilon > 0$, *set*

$$J_T^{\mathrm{Sp}}(v,m;\varepsilon) := \Big\{(\lambda(\gamma), m(\gamma)) : \gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}),\ \|\lambda(\gamma) - Tv\|_{\max} \le \varepsilon,\ m(\gamma) = m\Big\}.$$

For all sufficiently small $\varepsilon > 0$, *we have*

$$\left(\frac{2\varepsilon}{(2n-1)3^{2n}}\right)^n \le \liminf_{T\to\infty} \frac{\#J_T^{\mathrm{Sp}}(v,m;\varepsilon)}{e^{\rho_{\mathrm{Sp}_{2n}}(v)T}} \le \limsup_{T\to\infty} \frac{\#J_T^{\mathrm{Sp}}(v,m;\varepsilon)}{e^{\rho_{\mathrm{Sp}_{2n}}(v)T}} \le (2\varepsilon)^n (n+1)!.$$

Consequently

$$\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) \;=\; \rho_{\mathrm{Sp}_{2n}}(v).$$

Proof. For $(\lambda(\gamma), m(\gamma)) \in J_T^{\mathrm{Sp}}(v,m;\varepsilon)$, set

$$p(x) \;=\; \prod_{i=1}^n \big(x - m_i e^{\lambda_i(\gamma)}\big)\big(x - m_i e^{-\lambda_i(\gamma)}\big) \;\in\; \mathfrak{Q}_T^*(v,m;\varepsilon).$$

The assignment $(\lambda(\gamma), m(\gamma)) \mapsto p(x)$ is injective, so

$$\#J_T^{\mathrm{Sp}}(v,m;\varepsilon) \;\le\; \#\mathfrak{Q}_T^*(v,m;\varepsilon).$$

Conversely, if $p \in \mathfrak{Q}_T^*(v,m;\varepsilon)$, then Theorem 5.6 and Corollary 5.4 produce a $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ with $(\lambda(\gamma), m(\gamma)) \in J_T^{\mathrm{Sp}}(v,m;\varepsilon)$. The map $p \mapsto (\lambda(\gamma), m(\gamma))$ is injective, hence

$$\#\mathfrak{Q}_T^*(v,m;\varepsilon) \;\le\; \#J_T^{\mathrm{Sp}}(v,m;\varepsilon).$$

Hence the claim follows from Theorem 5.2. □

The lower bound below follows directly from the above theorem and the upper bound will be proved in Theorem 6.2.

Theorem 5.8. *Let $v \in \operatorname{int}\mathfrak{a}^+$ and $m = (m_1, \dots, m_n) \in \{\pm 1\}^n$. For every $0 < \varepsilon < 1$, there exist $C_1, C_2 > 0$ such that*

$$C_1 e^{\rho_{\mathrm{Sp}_{2n}}(v)T} \leq \#\Big\{[\gamma] \in [\mathrm{Sp}_{2n}(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \leq \varepsilon, m(\gamma) = m\Big\} \leq C_2 e^{2\rho_{\mathrm{Sp}_{2n}}(v)T}.$$

In particular,

$$\rho_{\mathrm{Sp}_{2n}}(v) \leq \underline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m) \ \leq \ \overline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m) \leq 2\rho_{\mathrm{Sp}_{2n}}(v). \tag{5.4}$$

In [32], Yang establishes a bijection between the set of $\mathrm{Sp}_{2n}(\mathbb{Z})$–conjugacy classes and a distinguished subset of units of degree $2n$. Through this correspondence, Theorem 5.8 can be viewed as a result about the growth of that collection of algebraic units.

6. Upper bound for $\mathsf{E}^{\star}_{\Gamma}$ for a general lattice

Let G be a connected semisimple real algebraic group. Fix a Cartan involution so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition into ± 1 eigenspaces. Let $K < G$ be the maximal compact subgroup with Lie algebra $\mathfrak{k}$, and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra with closed positive chamber $\mathfrak{a}^+$. Let ρ_G denote the half-sum of all positive roots of $(\mathfrak{g}, \mathfrak{a})$.

Write $A = \exp\mathfrak{a}$, $A^+ = \exp\mathfrak{a}^+$, and let $M = Z_K(A)$. Every $g \in G$ decomposes as a commuting product $g = g_h g_e g_u$ of hyperbolic, elliptic and unipotent elements, and the hyperbolic part g_h is G-conjugate to a unique element $\exp\lambda(g) \in A^+$; we call $\lambda(g)$ the *Jordan projection*. If $\lambda(g) \in \operatorname{int}\mathfrak{a}^+$ we say g is *loxodromic*; then g_u is the identity and g_e is conjugate to an element $m(g) \in M$, unique up to M-conjugacy. We denote by $[\Gamma]_{\mathrm{lox}}$ the set of Γ-conjugacy classes of all loxodromic elements of Γ.

Definition 6.1 (Directional entropy for Γ)**.** Let $\Gamma < G$ be a lattice. Let $\|\cdot\|$ be any norm on $\mathfrak{a}$. For any vector $v \in \operatorname{int}\mathfrak{a}^+$, define the directional *Jordan-entropy* functions by

$$\overline{\mathsf{E}}_{\Gamma}(v) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log N_\varepsilon(T, v)}{T}, \quad \underline{\mathsf{E}}_{\Gamma}(v) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log N_\varepsilon(T, v)}{T}$$

where $N_\varepsilon(T, v) = \#\{\lambda(\gamma) : \gamma \in \Gamma : \|\lambda(\gamma) - \mathbb{R}_+ v\| \leq \varepsilon, \|\lambda(\gamma)\| \leq T\}$.

Similarly,

$$\overline{\mathsf{E}}^{\star}_{\Gamma}(v) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log M_\varepsilon(T, v)}{T}, \quad \underline{\mathsf{E}}^{\star}_{\Gamma}(v) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log M_\varepsilon(T, v)}{T}$$

where $M_\varepsilon(T, v) := \#\{[\gamma] \in [\Gamma] : \|\lambda(\gamma) - \mathbb{R}_+ v\| \leq \varepsilon, \|\lambda(\gamma)\| \leq T\}$. These definitions are independent of the choice of a norm. When the lower and upper values coincide, we write $\mathsf{E}_{\Gamma}(v)$ and $\mathsf{E}^{\star}_{\Gamma}(v)$, respectively.

Theorem 6.2. *For all $v \in \operatorname{int}\mathfrak{a}^+$ and $\varepsilon > 0$, there exists $C > 0$ such that for all $T \geq 1$,*

$$\#\Big\{[\gamma] \in [\Gamma] : \|\lambda(\gamma) - \mathbb{R}_+ v\| \leq \varepsilon, \|\lambda(\gamma)\| \leq T\Big\} \leq C e^{2\rho_G(v)T}.$$

In particular,

$$\overline{\mathsf{E}}^{\star}_{\Gamma}(v) \le 2\rho_G(v).$$

Cartan counting and Upper bound. Let $\mu(g) \in \mathfrak{a}^+$ denote the Cartan projection of $g \in G$, i.e. the unique element with

$$g \in K e^{\mu(g)} K.$$

If we use the norm on $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$, then for all $g \in G$, we have $\|\mu(g)\| = d(go, o)$ where $o = [K] \in G/K$ and d is the Riemannian distance on the symmetric space G/K. Counting lattice points subject to constraints on the Cartan projection $\mu(g)$ is considerably better understood than the analogous problem for the Jordan projection; see, for example, ([9], [10], [14], [5], [13], etc). In particular, following the method of Eskin-McMullen[10], we can count lattice points whose Cartan projections lie in prescribed tubes or cones by combining the mixing of the A–action on $\Gamma\backslash G$ with the strong wavefront lemma stated below.

Lemma 6.3 (Strong wavefront lemma)**.** [14, Theorem 3.7] *Let $\mathcal{C} \subset \operatorname{int} \mathfrak{a}^+$ be closed and at positive distance from every wall of $\mathfrak{a}^+$. For any neighborhoods $\mathcal{O}_K \subset K$ and $\mathcal{O}_A \subset A$ of e, there exists a neighborhood $U \subset G$ of e such that for any $g = k_1 a k_2 \in K(\exp \mathcal{C})K$, we have*

$$UgU \subset k_1\mathcal{O}_K\, a\mathcal{O}_A\, k_2 O_K.$$

Theorem 6.4. *Let $\Gamma < G$ be a lattice and $v \in \operatorname{int} \mathfrak{a}^+$. For any $\varepsilon > 0$ we have, as $T \to \infty$,*

$$\#\big\{\gamma \in \Gamma : \|\mu(\gamma) - Tv\| \le \varepsilon\big\} \ \sim\ C\, e^{2\rho_G(v)T}$$

for some constant $C = C(\varepsilon) > 0$.

Proof. Fix $\varepsilon > 0$ and put

$$b_{T,\varepsilon} = \big\{u \in \mathfrak{a}^+ : \|u - Tv\| < \varepsilon\big\}, \qquad Z_T = K\, \exp(b_{T,\varepsilon})\, K.$$

For $g = k_1(\exp v)k_2 \in K(\exp \mathfrak{a}^+)K$, the Haar measure is

$$dg = \prod_{\alpha} \sinh \alpha(v) dk_1 dv dk_2,$$

where the product runs over all positive roots, counted with multiplicity [17]. We obtain that

$$\operatorname{Vol} Z_T \sim\ C_\varepsilon\, e^{2\rho_G(v)T} \tag{6.1}$$

for some constant $C_\varepsilon > 0$. Since $v \in \operatorname{int} \mathfrak{a}^+$, the set $b_{T,\varepsilon}$ has a positive distance from all walls of $\mathfrak{a}^+$. Lemma 6.3 and (6.1) imply that the family $\{Z_T\}_{T\gg 1}$ is *well-rounded*: for any $\eta > 0$, there exists an open neighborhood U_η of e in G such that

$$Z_{T-\eta} \subset \bigcap_{u_1,u_2 \in U_\eta} u_1 Z_T u_2 \subset \bigcup_{u_1,u_2 \in U_\eta} u_1 Z_T u_2 \subset Z_{T+\eta}$$

and

$$\limsup_{\eta\to 0} \frac{\mathrm{Vol}(Z_{T+\eta})}{\mathrm{Vol}(Z_{T-\eta})} = 1.$$

Define the counting function $F_T = F_{Z_T}$ on $(\Gamma\times\Gamma)\backslash(G\times G)$ by

$$F_T([g_1],[g_2]) = \sum_{\gamma\in\Gamma} \chi_{Z_T}(g_1^{-1}\gamma g_2)$$

so that $F_T([e],[e]) = \#\Gamma\cap Z_T$. If ϕ_η is the approximation of the identity function on G supported on the η-neighborhood of e in G and $\Phi_\eta([g]) = \sum_{\gamma\in\Gamma}\phi_\eta(\gamma g)$, then the standard unfolding argument gives that

$$\langle F_T, \Phi_\eta\otimes\Phi_\eta\rangle := \int F_T(x_1,x_2)\Phi_\eta(x_1)\Phi_\eta(x_2)dx_1dx_2 = \int_{g\in Z_T}\langle\Phi_\eta, g.\Phi_\eta\rangle_{L^2(\Gamma\backslash G)}dg.$$

Using strong mixing of the G-action on $L^2(\Gamma\backslash G)$ [16], we get

$$\langle F_T, \Phi_\eta\otimes\Phi_\eta\rangle \sim \frac{1}{\mathrm{Vol}(\Gamma\backslash G)}\,\mathrm{Vol}\,Z_T.$$

Noting that

$$\langle F_{T-\eta}, \Phi_\eta\otimes\Phi_\eta\rangle \le F_T([e],[e]) \le \langle F_{T+\eta}, \Phi_\eta\otimes\Phi_\eta\rangle,$$

the well-roundness property of the family $\{Z_T\}$ implies that

$$F_T([e],[e]) \;\sim\; \frac{1}{\mathrm{Vol}(\Gamma\backslash G)}\,\mathrm{Vol}\,Z_T.$$

□

The following can be deduced from [31, Theorem 1.2] for arithmetic lattices (see the proof of [24, Theorem 3.1]). For rank one groups, this is a standard fact which follows from the thick-thin decomposition of rank one locally symmetric manifolds of finite volume. Hence by Margulis arithmeticity theorem [22], we get:

Theorem 6.5. *Let $\Gamma < G$ be a lattice. There exists a compact subset $Q\subset G$ such that any compact AM-orbit in $\Gamma\backslash G$ is of the form $\Gamma\backslash\Gamma gAM$ for some $g\in Q$.*

Corollary 6.6. *For any lattice $\Gamma < G$, there is $C > 1$ such that for any conjugacy class $[\gamma]\in[\Gamma]_{\mathrm{lox}}$, there exists $\gamma'\in[\gamma]$ such that*

$$\|\lambda(\gamma) - \mu(\gamma')\| \le C.$$

Proof. Let Q be a compact subset in Theorem 6.5. We claim that there exists a representative $\gamma'\in[\gamma]$ such that

$$\gamma' \;=\; g\,e^{\lambda(\gamma)}m_\gamma\,g^{-1}, \qquad m_\gamma\in M,\; g\in Q.$$

To see this, since γ is loxodromic, its centralizer in G is of the form $hAMh^{-1}$ with $\Gamma\backslash\Gamma hAM$ compact [26]. Since $h=\gamma_0 g a_0 m_0\in\Gamma QAM$ with

$g_0 \in Q$ by Theorem 6.5 and $\gamma = he^{\lambda(\gamma)}mh^{-1}$ for some $m \in M$, it suffices to set

$$\gamma' = ge^{\lambda(\gamma)}(m_0 m m_0^{-1})g^{-1}.$$

Therefore there is $C > 1$ depending only on Q such that $\|\lambda(\gamma) - \mu(\gamma')\| \le C$ by [4, Lemma 4.6]. □

Since $\gamma' \in [\gamma]$, the map $[\gamma] \to \gamma'$ is an injective map to Γ. Hence we get:

Corollary 6.7. *Let $\Gamma < G$ be a lattice. For any bounded subset $B \subset \mathfrak{a}^+$,*

$$\#\{[\gamma] \in [\Gamma] : \lambda(\gamma) \in B\} < \infty.$$

Proof of Theorem 6.2. Let $B_\varepsilon(0) \subset \mathfrak{a}$ be the ball of radius ε about the origin, and fix $v \in \operatorname{int} \mathfrak{a}^+$. Suppose that $\gamma \in \Gamma$ satisfies $\lambda(\gamma) \in Tv + B_\varepsilon(0)$ for all sufficiently large T. Then since $v \in \operatorname{int} \mathfrak{a}^+$ and T is large, γ is loxodromic. Hence, by Corollary 6.6, there is $\gamma' \in [\gamma]$ such that $\|\lambda(\gamma) - \mu(\gamma')\| \le C$.

Thus, by the injectivity of the map $[\gamma] \to \gamma'$,

$$\#\big\{[\gamma] : \lambda(\gamma) \in Tv + B_\varepsilon(0)\big\} \;\le\; \#\big\{\gamma' \in \Gamma : \mu(\gamma') \in Tv + B_C(0)\big\}.$$

Applying Theorem 6.4 proves the claim.

Remark 6.8. In [25], Quint introduced the growth indicator

$$\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$$

of a Zariski dense discrete subgroup $\Gamma < G$. Let $\mathcal{L}_\Gamma$ be the limit cone of Γ, that is, the asymptotic cone of the Cartan projection $\mu(\Gamma)$. For $v \in \operatorname{int} \mathcal{L}_\Gamma$, it is equal to

$$\psi_\Gamma(v) = \|v\| \inf_{\mathcal{C}} \limsup_{T\to\infty} \frac{\log \#\{\gamma \in \Gamma : \|\mu(\gamma)\| \le T, \mu(\gamma) \in \mathcal{C}\}}{T}$$

where the infimum is taken over all open cones $\mathcal{C} \subset \mathfrak{a}^+$ containing v. If $\Gamma < G$ is a lattice, then $\mathcal{L}_\Gamma = \mathfrak{a}^+$ and $\psi_\Gamma = 2\rho_G$.

While $\psi_\Gamma(v) < +\infty$ for all $v \in \mathfrak{a}^+$ and for any discrete subgroup Γ, the directional entropy

$$\overline{E}^\star_\Gamma(v) = \|v\| \cdot \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log \#\{[\gamma] \in \Gamma : \|\lambda(\gamma)\| \le T, \|\lambda(\gamma) - \mathbb{R}_+ v\| \le \varepsilon\}}{T}$$

may take the value $+\infty$; this already occurs for a normal subgroup of a cocompact lattice of $\mathrm{SL}_2(\mathbb{R})$ of infinite index. Theorem 6.2 shows that for Γ lattice, $\overline{E}^\star_\Gamma(v) \le \psi_\Gamma(v) = 2\rho_G(v)$ for all $v \in \operatorname{int} \mathfrak{a}^+$. It is shown in [6] that if Γ is a Zariski dense *Borel Anosov* subgroup of G, then $\overline{E}^\star_\Gamma(v) = \psi_\Gamma(v)$ for all $v \in \operatorname{int} \mathcal{L}_\Gamma$.

Upper bound without directional restriction. We will use the following for the upper bound:

Theorem 6.9. *Let $\Gamma < G$ be a lattice in G. If $\mathcal{C}$ is a convex cone in $\mathfrak{a}^+$ with non-empty interior and $\mathcal{C}_T = \{v \in \mathcal{C} : \|v\| < T\}$, then*

$$\#\Gamma \cap K\exp(\mathcal{C}_T)K \sim C \cdot e^{2\rho_G(u_{\mathcal{C}})T} T^{(\operatorname{rank} G-1)/2}$$

where $\|\cdot\|$ is the norm on $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$ and $u_{\mathcal{C}}$ is the unique unit vector such that $2\rho_G(u_{\mathcal{C}}) = \max_{\|u\|=1, u\in\mathcal{C}} 2\rho_G(u)$.

Proof. In [14, Lemma 5.4], it is shown that for $\mathcal{C} = \mathfrak{a}^+$,

$$\operatorname{Vol}(K\exp(\mathcal{C}_T)K) \sim C \cdot e^{2\rho_G(u_{\mathcal{C}})T} T^{(\operatorname{rank} G-1)/2}.$$

The same proof works for any convex cone $\mathcal{C}$ with non-empty interior.

By Theorem 6.3, the family $Z_T = K\exp(\mathcal{C}_T)K$, $T \geq 1$ is well-rounded, as in the proof of Theorem 6.4. Consequently, by the same argument used there, we get

$$\#\Gamma \cap K\exp(\mathcal{C}_T)K \sim \operatorname{Vol}(Z_T).$$

□

Corollary 6.10. *Let $\Gamma < G$ be a lattice. There exist $C > 0$ such that for all $T > 1$,*

$$\#\{[\gamma] \in [\Gamma]_{\text{lox}} : \|\lambda(\gamma)\| < T\} \leq Ce^{2\|\rho_G\|T} T^{(\operatorname{rank} G-1)/2}$$

where $\|\rho_G\| = \max_{u\in\mathfrak{a}^+, \|u\|=1} \rho_G(u)$.

Proof. Let $[\gamma] \to \gamma'$ be the injective map from the conjugacy classes of loxodromic elements to Γ given in Corollary 6.6. Therefore

$$\#\{[\gamma] \in [\Gamma]_{\text{lox}} : \|\lambda(\gamma)\| < T\} \leq \#\{\gamma' \in \Gamma, \|\mu(\gamma')\| < T + C\}$$

where $C > 1$ is as in Corollary 6.6. Therefore the upper bound follows from Theorem 6.9. □

We remark that in [7], some upper bound for cocompact lattices of G was obtained. We record the following for $\operatorname{SL}_n(\mathbb{Z})$:

Corollary 6.11. *There exist $C_1, C_2 > 0$ such that for all $T > 1$,*

$$C_1 e^{d_n T/2} \leq \#\{[\gamma] \in [\operatorname{SL}_n(\mathbb{Z})]_{\text{lox}} : \|\lambda(\gamma)\|_{\text{Euc}} < T\} \leq C_2 T^{(n-2)/2} e^{d_n T}$$

where $d_n = \sqrt{\frac{n(n^2-1)}{3}}$.

Proof. The lower bound follows from Corollary 4.3. Since the norm on $\mathfrak{a}$ induced by the Killing form on $\mathfrak{sl}_n(\mathbb{R})$ is a constant multiple of the Euclidean norm on $\mathfrak{a}$, the upper bound follows from Corollary 6.10 and (4.3). □

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