**ORBIT CLOSURES OF ZARISKI DENSE SUBGROUPS IN HOMOGENEOUS SPACES**

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**Abstract.** We present a new proof of Benoist-Quint’s finite or dense dichotomy for orbits of Zariski dense subgroups acting on the quotient space of $\text{SO}^0(d, 1)$ by a cocompact lattice.

Our proof is topological. We use ideas from the study of dynamics of unipotent flows on homogeneous spaces of infinite volume. We also use Ratner’s orbit closure theorem for a one-parameter unipotent semigroup action on homogeneous spaces of finite volume.

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1. **Introduction**

Let $G = \text{SO}^0(d, 1)$ for $d \geq 2$, and let $\Delta$ be a cocompact lattice in $G$. Let $\Gamma$ be any Zariski dense subgroup of $G$, acting on the space $\Delta \backslash G$ by right translations.

The aim of this paper is to present a new proof of the following theorem of Benoist-Quint in [2], which was a question of Margulis [6] and Shah [13]:

**Theorem 1.1.** Any $\Gamma$-invariant subset of $\Delta \backslash G$ is either finite or dense.

The proof of Benoist-Quint is based on their classification of stationary measures for random walks on $\Gamma$ on the space $\Delta \backslash G$. Our proof is topological and can be easily modified to all rank one simple Lie groups; for the sake of concreteness, we opted to write it only for $\text{SO}^0(d, 1)$. In the case when $G = \text{SO}^0(2, 1)$, $\Delta$ is a cocompact lattice and $\Gamma$ is a finitely generated Zariski...
dense subgroup with no parabolic elements, Benoist-Oh gave a topological proof of Theorem 1.1 when the $\Gamma$-invariant subset is a single $\Gamma$-orbit [3].

A Zariski dense subgroup of $G$ is either discrete or dense in $G$. Hence it suffices to consider the case when $\Gamma$ is discrete and Zariski dense. Our starting point is then the observation that Theorem 1.1 can be translated into a problem on the orbit closure of unipotent flows on a homogeneous space of infinite volume. If we set $H = \{(g, g) : g \in G\}$ to be the diagonal embedding of $G$ into $G \times G$, a basic simple observation is that Theorem 1.1 is equivalent to the following statement about the $H$-action on the product space $(\Gamma \times \Delta) \setminus (G \times G)$, which has infinite volume unless $\Gamma$ is a lattice.

**Theorem 1.2.** Any $H$-invariant closed subset of $(\Gamma \times \Delta) \setminus (G \times G)$ is either a union of finitely many closed $H$-orbits or dense. In particular, any $H$-orbit is either closed or dense.

When $\Gamma$ is a lattice in $G$, i.e., when the homogeneous space $(\Gamma \times \Delta) \setminus (G \times G)$ has finite volume, Theorem 1.2 is a special case of Ratner’s orbit closure theorem and Mozes-Shah theorem [9].

**On the proofs.** Any Zariski dense discrete subgroup of $G$ contains a Zariski dense Schottky subgroup (Lemma 7.3). Hence in proving Theorem 1.1, we may assume without loss of generality that $\Gamma$ is a convex cocompact Zariski dense subgroup. Set $Z := (\Gamma \times \Delta) \setminus (G \times G)$. Let $A$ denote the diagonal subgroup of $G$, and $U$ denote the horospherical subgroup of $H = \{(g, g) : g \in G\}$. Our proof is based on the study of the action of $U$ on $Z$. Let $\Omega$ denote the subset of $Z$ consisting of all bounded $A \times A$-orbits, which is a compact subset. For $x \in \Omega$, set $T(x) := \{u \in U : xu \in \Omega\}$. Then for any sequence $\lambda_i \to \infty$, we show that the renormalization $T_\infty := \limsup_i \lambda_i^{-1} T(x)$ is locally Zariski dense at $e$, i.e., for any neighborhood $O$ of $e$ in $U$, $T_\infty \cap O$ is Zariski dense in $U$ (Lemma 3.2). This is the key recurrence property we use in carrying out the unipotent dynamics for $U$-action on $Z$. We remark that this recurrence property is much weaker than the notion of thickness used in [8] and [5], where the thick return property was required for any one-parameter subgroup of $U$; the latter strong property holds for hyperbolic manifolds with Fuchsian ends but not for a general convex cocompact manifold such as Schottky manifolds.

The reasons why we can work with this type of weak recurrence property are the following:

1. $U$ is large; its projection to each factor is a horospherical subgroup of $G$.
2. one factor of $Z$, which is $\Delta \setminus G$, is a compact space.

We prove that any closed $H$-invariant subset $X$ contains a $U$-minimal subset $Y$ with respect to $\Omega$ such that

$$Y \mathcal{C} \subset X$$

for some smooth non-degenerate curve $\mathcal{C}$ inside the horospherical subgroup of $\{e\} \times G$. Once we do that, using the invariance of $X$ under the diagonal
embedding of $A$, we can finish the proof; we mention that we use Ratner’s theorem for a one-parameter unipotent semigroup action on the compact homogeneous space $\Delta \setminus G$ in this final step (see sections 6-7).

2. Notations and background

Let $G = \text{SO}^0(d, 1)$, $d \geq 2$. Let $\mathbb{H}^d$ be the real hyperbolic space of dimension $d$. Then $G$ can be identified with $\text{Isom}^+(\mathbb{H}^d)$. The isometric action of $G$ on $\mathbb{H}^d$ extends to a transitive action of $G$ on the unit tangent bundle $T^1(\mathbb{H}^d)$. We identify $\mathbb{H}^d = G/K$ and $T^1(\mathbb{H}^d) = G/\mathcal{M}$ where $K$ is the stabilizer of a point $o \in \mathbb{H}^d$ and $\mathcal{M} < K$ is the stabilizer of a tangent vector $v_o \in T^1_o(\mathbb{H}^d)$. The group $G$ itself can be understood as the oriented frame bundle over $\mathbb{H}^d$. Let $A = \{a_t : t \in \mathbb{R}\}$ be the one-parameter subgroup of semisimple elements such that $A$ centralizes $\mathcal{M}$ and the right translation action of $a_t$ on $G/\mathcal{M}$ corresponds to the geodesic flow on $T^1(\mathbb{H}^d)$. For a tangent vector $v \in T^1(\mathbb{H}^d)$, we write $v^+$ for the forward end point of the associated geodesic in the boundary $\partial \mathbb{H}^d$ and $v^-$ for the backward end point. For $g \in G$, we define

$$g^+ = (gv_o)^+ \quad \text{and} \quad g^- = (gv_o)^-.$$ 

We denote by $\mathcal{U}$ the contracting horospherical subgroup of $G$:

$$\mathcal{U} = \{u \in G : a_{-t}ua_t \rightarrow e, \quad \text{as} \quad t \rightarrow +\infty\}.$$ 

The group $\mathcal{U}$ is isomorphic to $\mathbb{R}^{d-1}$; we use notation $\mathcal{U} = \{u_t : t \in \mathbb{R}^{d-1}\}$.

We use the following notation in the sequel:

- $H = \{(h, h) : h \in G\}$
- $H_1 = G \times \{e\}$ and $H_2 = \{e\} \times G$
- $A = \{(a_t, a_t) : t \in \mathbb{R}\}$
- $A_1 = \mathcal{A} \times \{e\}$, $A_2 = \{e\} \times \mathcal{A}$
- $U = \{(u, u) : u \in \mathcal{U}\}$
- $U_1 = \mathcal{U} \times \{e\}$, $U_2 = \{e\} \times \mathcal{U}$
- $M = \{(m, m) : m \in \mathcal{M}\}$
- $M_1 = \mathcal{M} \times \{e\}$, and $M_2 = \{e\} \times \mathcal{M}$.

Let $\Gamma_1 < H_1$ be a Zariski-dense convex cocompact subgroup and $\Gamma_2 < H_2$ be a cocompact lattice. We assume that both are torsion-free. For each $i = 1, 2$, let

$$S_i := \Gamma_i \setminus \mathbb{H}^d$$

be the associated real hyperbolic manifold, and let $\Lambda_i \subset \partial \mathbb{H}^d = S^{d-1}$ denote the limit set of $\Gamma_i$. By the assumption on $\Gamma_1$, the convex core of $S_1$, which is given by $\Gamma_1 \setminus \text{hull}(\Lambda_1)$ is compact. As $\Gamma_2$ is a lattice in $H_2$, we have $\Lambda_2 = S^{d-1}$.

Set

$$Z_1 = \Gamma_1 \setminus H_1, \quad Z_2 = \Gamma_2 \setminus H_2, \quad Z = Z_1 \times Z_2.$$ 

We also define

$$\text{RF} S_1 = \{x_1 \in Z_1 : x_1 A_1 \text{ bounded}\} = \{[g] \in Z_1 : g^\pm \in \Lambda_1\};$$
Proof. Note that $UO$ the first claim follows. The second claim follows from Lemma 3.1. □

Define

$\Omega = RF S_1 \times Z_2$ and $\Omega_+ = RF_+ S_1 \times Z_2$.

As $\Gamma_1$ is convex cocompact, $RF S_1$ is a compact $A_1 M_1$-invariant subset. Hence $\Omega$ is a compact subset of $Z$ invariant under $\prod_{i=1}^2 A_i M_i$.

3. Local Zariski density of the renormalization of the $U$-recurrence

We often identify $U$ with $\mathbb{R}^{d-1}$ via the map $(u, u) \mapsto u$, and the notation $\|u\|$ means the Euclidean norm of $u$. To ease the notation, we oftentimes write $u \in U$, identifying $u$ with $(u, u)$.

For $x \in \Omega$, we define the following recurrence time of $x$ to $\Omega$ under $U$:

$$T(x) := \{u \in U : x u \in \Omega\}.$$ 

Let $x = (x_1, x_2) \in \Omega \subset Z_1 \times Z_2$. Since $\Omega = RF S_1 \times Z_2$, $(u, u) \in T(x)$ if and only if $x_1 u \in RF S_1$. If we choose $g_1 \in H_1$ so that $x_1 = [g_1]$, then $g_1^+ \in A_1$ since $x_1 \in RF S_1$. Since $(g_1 u)^+ = g_1^+$, and $(g_1 u)^- = g_1 u^-$, we have

$$(3.1) \quad T(x) = \{u \in U : (g_1 u)^- \in A_1\}.$$ 

Since $g_1^+ \in A_1$ and $A_1$ has no isolated point, it follows that $T(x)$ is unbounded.

Lemma 3.1. For $x \in \Omega$, any non-empty open subset of $T(x)$ is Zariski dense in $U$.

Proof. The visual map $U \rightarrow \mathbb{R}^d - \{g_1^+\}$ defined by $u \mapsto g_1 u^-$ is a diffeomorphism. Hence by (3.1), the claim follows the well-known fact that no non-empty open subset of $A_1$ is contained in a smooth submanifold in $\mathbb{R}^{d-1}$ of positive co-dimension (cf. [15, Corollary 3.10]). □

Lemma 3.2. For any $x \in \Omega$ and for any sequence numbers $\lambda_i \rightarrow +\infty$, there exists $z \in \Omega$ such that

$$T_\infty := \limsup_{i \rightarrow \infty} \frac{T(x)}{\lambda_i} \supset T(z).$$

In particular, for any neighborhood $O$ of $e$ in $U$, $T_\infty \cap O$ is Zariski dense in $U$.

Proof. Note that $\lambda_i^{-1} T(x) = \{u_t \in U : x u_{\lambda_i t} \in \Omega\}$. Let $s_i = \frac{1}{2} \log \lambda_i$ so that $a_s u_t a_s^{-1} = u_{\lambda_i t}$. Since $\Omega$ is $A$-invariant,

$$\lambda_i^{-1} T(x) = \{u_t \in U : x a_s u_t \in \Omega\} = T(x a_s).$$

Since $\Omega$ is a compact $A$-invariant subset, passing to a subsequence, $x a_s$ converges to some $z \in \Omega$ as $i \rightarrow \infty$. It follows that $\limsup T(x a_s) \supset T(z)$; the first claim follows. The second claim follows from Lemma 3.1. □
4. Unipotent blowup

For simplicity, set \( H := H_1 \times H_2 \). For a subgroup \( S < H \), we denote by \( N(S) \) the normalizer of \( S \). For a subgroup \( S_i \subset H_i \), \( C_{H_i}(S_i) \) denotes the centralizer of \( S_i \) in \( H_i \).

**Lemma 4.1.** \( N(U) = AMU_1U_2 \).

*Proof.* The inclusion \( AMU_1U_2 \subset N(U) \) is clear. To show the reverse inclusion, let \((g_1, g_2) \in N(U)\). Then for all \((u, u) \in U\), \((g_1ug_1^{-1}, g_2ug_2^{-1}) \in U\) and hence \( g_2^{-1}g_1ug_1^{-1}g_2 = u \). This implies \((g_2^{-1}g_1, e) \in C_{H_1}(U_1)\). Since \( C_{H_1}(U_1) \subset N(U)\),

\[
(g_1, g_2) = (g_2, g_2) \cdot (g_2^{-1}g_1, e) \in N(U),
\]

it follows \((g_2, g_2) \in N(U) \cap H = AMU\). As both \((g_2, g_2)\) and \((g_2^{-1}g_1, e)\) belong to \( AMU_1U_2\), so is \((g_1, g_2)\) and the lemma is proved. \( \square \)

**Lemma 4.2.** Let \( g_i \to e \) in \( H - N(U) \) as \( i \to \infty \). Then for any \( x \in \Omega \) and for any neighborhood \( \mathcal{O} \) of \( e \) in \( H \), there exists sequences \( u'_i \in U \) and \( u_i \in T(x) \) such that

\[
u_i u_i g_i u_i \to \alpha
\]

for some \( \alpha \in (AMU_2 - M) \cap \mathcal{O} \).

*Proof.* Set \( L := A_1M_1U_1^+ \times A_2M_2U_2^+ \) where \( U_i^+ \) is the expanding horospherical subgroup of \( H_i \) for \( i = 1, 2 \). Note that

\[
N(U) \cap L = AMU_2
\]

and that the product map from \( U \times L \) to \( H \) is a diffeomorphism onto a Zariski open neighborhood of \( e \) in \( G \). Following [7], we will construct a quasi-regular map

\[
\psi : U \to N(U) \cap L
\]

associated to the sequence \( g_i \). Except for a Zariski closed subset of \( U \), the product \( g_iu \) can be written as an element of \( UL \) in a unique way. We denote by \( \psi_i(u) \in L \) its \( L \)-component so that \( g_iu \in U\psi_i(u) \). Since \( U \) is an algebraic subgroup, by Chevalley’s theorem, there exists an \( \mathbb{R} \)-regular representation \( H \to GL(W) \) with a distinguished point \( p \in W \) such that \( U = Stab(p) \). Then \( pH \) is locally closed, and

\[
N(U) = \{ g \in H : pgu = pg \text{ for all } u \in U \}.
\]

The map \( \bar{\psi}_i : U \to W \) defined by \( \bar{\psi}_i(u) = pg_iu \) is a polynomial map in \( d - 1 \)-variables of degree uniformly bounded for all \( i \), and \( \bar{\psi}_i(e) \) converges to \( p \) as \( i \to \infty \). As \( g_i \notin N(U) \), \( \bar{\psi}_i \) is non-constant. Denote by \( B(p, r) \) the ball of radius \( r \) centered at \( p \), fixing an \( M \)-invariant norm \( \| \cdot \| \) on \( W \). Since \( pH \) is open in its closure, we can find \( \lambda_0 > 0 \) such that

\[
B(p, \lambda_0) \cap pH \subset pH.
\]
Without loss of generality, we may assume that $\lambda_0 = 1$ by renormalizing the norm. Now define

$$\lambda_i := \sup\{ \lambda \geq 0 : \tilde{\phi}_i(B_U(\lambda)) \subset B(p, 1) \}.$$ 

Note that $\lambda_i < \infty$ as $\phi_i$ is nonconstant, and $\lambda_i \to \infty$ as $i \to \infty$, as $g_i \to e$. We define $\phi_i : U \to W$ by

$$\phi_i(u) := \tilde{\phi}_i(\lambda_i u).$$

This forms an equi-continuous family of polynomials on $U = \mathbb{R}^{d-1}$. Therefore, after passing to a subsequence, $\phi_i$ converges to a non-constant polynomial $\phi$ uniformly on every compact subset of $U$. Moreover $\sup\{ \|\phi(u) - p\| : u \in B_U(1) \} = 1$, $\phi(B_U(1)) \subset pL$, and $\phi(0) = p$. Now the following map $\psi$ defines a non-constant rational map defined on a Zariski open dense neighborhood $U$ of $e$ in $U$:

$$\psi := \rho_L^{-1} \circ \phi$$

where $\rho_L$ is the restriction to $L$ of the orbit map $g \mapsto pg$.

We have $\psi(e) = e$ and

$$\psi(u) = \lim_{i} \psi_i(\lambda_i u)$$

where the convergence is uniform on compact subsets of $U$. It is easy to check that $\text{Im } \psi \subset N(U) \cap L = AMU_2$ using (4.1). Set

$$T_\infty := \lim_{i \to \infty} \lambda_i^{-1} T(x).$$

Let $O'$ be a neighborhood of $0$ in $U$ such that $\phi(O') \subset pO$. Since $\phi(t)$ is a nonconstant polynomial, it follows from Lemma 3.2 that there exists $t \in O' \cap T_\infty$ such that $\|\phi(t)\|^2 \neq \|p\|^2$.

Let $t_i \in T(x)$ be a sequence such that $\lambda_i^{-1} t_i \to t$ as $i \to \infty$ (by passing to a subsequence). Since $\psi_i \circ \lambda_i \to \psi$ uniformly on compact subsets,

$$\psi(t) = \lim_{i \to \infty} (\psi_i \circ \lambda_i)(t_i/\lambda_i) = \lim_{i \to \infty} \psi_i(t_i) = \lim_{i \to \infty} u_i g_i u_{t_i}.$$

for some sequence $u_i \in U$. Note that $\phi(t) = \rho_p(t)$ for some $\psi(t) \in AMU_2 \cap O$. Since $\|\phi(t)\|^2 \neq \|p\|^2$ and $\|\cdot\|$ is $M$-invariant, we have $\psi(t) \notin M$. Hence this finishes the proof. \hfill $\square$

**Lemma 4.3.** Let $r_i \to e$ in $H_2 - N(U)$. For any $x \in \Omega$, there exists a sequence $u_i \in T(x)$ such that

$$u_i^{-1} r_i u_i \to v$$

for some non-trivial $v \in U_2$.

**Proof.** The Lie algebra of $H_2$ is given by $\mathfrak{h}$. Write $r_i = \exp q_i$ for $q_i \in \mathfrak{h}$. Define a polynomial map $\psi_i : U_2 \to \mathfrak{h}$ by

$$\psi_i(t) = u_t^{-1} q_i u_t \quad \text{for all } t \in U_2.$$
Since $H_2 \cap N(U) = U_2 = C_{H_2}(U_2)$, it follows that $r_i \in H_2 - C_{H_2}(U_2)$. Hence $\psi_i$ is a nonconstant polynomial. Let $\lambda_i > 0$ be the supremum of $\lambda > 0$ such that $\sup_{t \in B_{U_2}(\lambda)} \|\psi_i(t)\| \leq 1$. Then $0 < \lambda_i < \infty$ and $\lambda_i \to \infty$.

Now the rescaled polynomials $\phi_i = \psi_i \circ \lambda_i : U_2 \to \mathfrak{h}$ form an equicontinuous family of polynomials of bounded degree and $\lim_{i \to \infty} \phi_i(0) = 0$. Therefore $\phi_i$ converges to a polynomial $\phi : U_2 \to \mathfrak{h}$ uniformly on compact subsets. Since $\phi(0) = 0$ and $\sup_{t \in B_{U_2}(\lambda)} \|\phi(t)\| = 1$, $\phi$ is a non-constant polynomial.

We claim that $\text{Im}(\phi) \subset \text{Lie}(U_2)$. For any fixed $s \in U_2$, we have $\lambda_i^{-1}s \to 0$, and hence for any $t \in U_2$, 

$$u_s^{-1}\phi(t)u_s = \lim_{i \to \infty} u_{-\lambda_i t - s} q_i u_{\lambda_i t + s} = \lim_{i \to \infty} u_{-\lambda_i (t + \lambda_i^{-1}s)} q_i u_{\lambda_i (t + \lambda_i^{-1}s)} = \lim_{i \to \infty} u_{-\lambda_i t} q_i u_{\lambda_i t} = \phi(t).$$

Hence $\phi(t)$ belongs to the centralizer of $U_2$. Since the centralizer of $U_2$ in $\mathfrak{h}$ is equal to $\text{Lie} U_2$, the claim follows.

Set 

$$T_{\infty} := \lim_{i \to \infty} \lambda_i^{-1} T(x).$$

Fix $t \in T_{\infty}$ such that $\phi(t) \neq 0$; this exists by Lemma 3.2. Let $t_i \in T(x)$ be a sequence such that $\lambda_i^{-1} t_i \to t$ as $i \to \infty$. As $\phi_i \to \phi$ uniformly on compact subsets, it follows that 

$$\phi(t) = \lim_{i \to \infty} \psi_i(\lambda_i \cdot t_i / \lambda_i) = \lim_{i \to \infty} u_{t_i}^{-1} q_i u_{t_i}.$$

Hence, by exponentiating, we obtain that $u_{t_i}^{-1} r_i u_{t_i}$ converges to a non-trivial element of $U_2$. \hfill \Box

5. Relative minimal subsets and additional invariance

Let $X$ be a closed $H$-invariant subset of $Z$. A closed $U$-invariant subset $Y$ of $X$ is called $U$-minimal with respect to $\Omega$ if $Y \cap \Omega \neq \emptyset$ and for any $y \in Y \cap \Omega$, $yU$ is dense in $Y$. Since every $H$-orbit in $Z$ intersects $\Omega$, $X \cap \Omega \neq \emptyset$. By Zorn’s lemma, there exists a $U$-minimal subset $Y$ of $X$ with respect $\Omega$, which we fix in the following.

**Lemma 5.1.** If $\pi_i : Z \to Z_i$ denotes the canonical projection, we have 

$$\pi_1(Y) = RF^+_S 1 \text{ and } \pi_2(Y) = Z_2.$$

**Proof.** The claim follows since $U_1$ and $U_2$ act minimally on $RF^+_S 1$ and $Z_2$ respectively [15]. \hfill \Box

**Lemma 5.2.** Let $S$ be a closed subgroup of $N(U)$ containing $U$. For any $y \in Y \cap \Omega$, the orbit $yS$ is not locally closed.

**Proof.** Suppose that $yS$ is locally closed for some $y \in Y \cap \Omega$. We claim that there exists $u_i \to \infty$ in $U$ such that $yu_i \to y$. Let 

$$Q := \{ z \in Y : \exists i \to \infty \text{ such that } yu_i \to y \}.$$
Since $T(y)$ is unbounded, there exists $u_i \to \infty$ in $U$ such that $yu_i \in Y \cap \Omega$. Since any limit of the sequence $yu_i$ belongs to $Q \cap \Omega$, we have $Q \cap \Omega \neq \emptyset$. Since $Q$ is a closed $U$-invariant set, $Q = Y$ by the relative $U$-minimality of $Y$. In particular, $y \in Q$, proving the claim. We may assume that $y = [e]$ without loss of generality. Let $\Gamma = \Gamma_1 \times \Gamma_2$. Since $yS$ is locally closed, $yS$ is homeomorphic to $(S \cap \Gamma) \setminus S$. Therefore there exists $\delta_i \in S \cap \Gamma$ such that $\delta_i u_i \to e$ as $i \to \infty$.

Since $N(U) = AMU_1 U_2$, writing $\delta_i = a_i r_i$ for $a_i \in A$ and $r_i \in MU_1 U_2$, it follows that $a_i \to e$ as $i \to \infty$. On the other hand, note that $a_i$ is non-trivial as $\Gamma$ does not contain any elliptic or parabolic element. This is a contradiction, as there exists a positive lower bound for the translation lengths of elements of $\Gamma_1 \cup \Gamma_2$. \hfill \qed

**Lemma 5.3.** For any $y \in Y \cap \Omega$, there exist $y \in Y$ and $g_i \to e$ in $\mathcal{H} - N(U)$ such that $yg_i \in Y$.

**Proof.** Suppose not. Then there is an open neighborhood $\mathcal{O}$ of $e$ such that
\begin{equation}
(5.1) \quad y\mathcal{O} \cap Y \subset yN(U).
\end{equation}
We may assume the map $g \mapsto yg \in X$ is injective on $\mathcal{O}$. Set
\[
S := \{ g \in N(U) : Yg = Y \}
\]
which is a closed subgroup of $N(U)$ containing $U$. We will show that $yS$ is locally closed; this contradicts Lemma 5.2. We first claim that
\begin{equation}
(5.2) \quad y\mathcal{O} \cap Y \subset yS.
\end{equation}
If $g \in \mathcal{O}$ such that $yg \in Y$, then $g \in N(U)$. Therefore $Yg = \overline{ygU} = \overline{ygU} \subset Y$.
Moreover, since $Yg \subset Y \subset \Omega_+$ and $\Omega_+ = \Omega U$, we have $Yg \cap \Omega \neq \emptyset$. Hence $Yg = Y$, proving that $g \in S$. Now, (5.2) implies that $yS$ is open in $Y$. On the other hand, since $U \subset S$, we get $Y = \overline{yS}$. Therefore, $yS$ is locally closed, finishing the proof. \hfill \qed

**Proposition 5.4** (Translate of $Y$ inside of $Y$). There exists a one-parameter subsemigroup $S < AMU_2$ such that $S \not\subset M$ and
\[
YS \subset Y.
\]

**Proof.** It suffices to prove that there exists a sequence $\beta_k \to e$ in $AMU_2 - M$ such that $Y \beta_k \subset Y$ (cf. [5, Lemma 10.5]). Choose $y \in Y \cap \Omega$. By Lemma 5.3, there exists $g_i \to e$ in $\mathcal{H} - N(U)$ such that $yg_i \in Y$. Set
\[
T(y) := \{ u \in U : yu \in Y \cap \Omega \}.
\]
Let $\mathcal{O}_k$ be a decreasing sequence of neighborhoods of $e$ in $G$ so that $\cap_k \mathcal{O}_k = \{ e \}$. By Lemma 4.2 to $g_i^{-1}$, there exist $u_i' \in U$ and $u_i \in T(y)$ such that $(u_i')^{-1} g_i^{-1} u_i \to \alpha_k$ for some $\alpha_k \in (AMU_2 - M) \cap \mathcal{O}_k$.

Since $Y \cap \Omega$ is compact, $yu_i$ converges to some $y_k \in Y \cap \Omega$ as $i \to \infty$, by passing to a subsequence. Hence as $i \to \infty$,
\[
yg_i u_i' = (yu_i)((u_i')^{-1} g_i^{-1} u_i)\to y_k \alpha_k^{-1} \in Y.
\]
Since $y_k \in Y \cap \Omega$ and $\alpha_k \in N(U)$, it follows that $Y\alpha_k^{-1} \subset Y$. It remains to set $\beta_k := \alpha_k^{-1}$.

**Proposition 5.5** (Translate of $Y$ inside of $X$). Suppose that there exists $y \in Y \cap \Omega$ such that $X - yH$ is not closed. Then there exists some non-trivial $v \in U_2$ such that

$$Yv \subset X.$$ 

**Proof.** By the hypothesis, there exists a sequence $g_i \to e$ in $H - H$ such that $yg_i \in X$. Since $X$ is $H$-invariant, we may assume $g_i \in H_2$. Note that $N(U) \cap H_2 = U_2$. Hence if $g_i \in N(U)$ for some $i$, then we can simply take $v := g_i$.

Now suppose that $g_i \notin N(U)$ for all $i$. Hence, by Lemma 4.3, there exists $u_i \in T(y)$ such that $u_i^{-1}g_i u_i \to v$ for some non-trivial $v \in U_2$. Observe

$$(yu_i)(u_i^{-1}g_i u_i) = yg_i u_i \in X.$$ 

The sequence $yu_i$ converges to some $y_0 \in Y \cap \Omega$, by passing to a subsequence. Hence $y_0v \in X$. It follows $Yv \subset X$ by the relative minimality of $Y$. 

6. Expansion of a smooth curve inside a horospherical subgroup

We will need the following lemma in the next section: it is in the proof of this lemma for which we use Ratner’s theorem ([10], [11]):

**Lemma 6.1.** For any smooth curve $C : [0,1] \to U_2$ with $C'(0) \neq 0$, there exists $y_2 \in Z_2$ such that for any sequence $t_i \to +\infty$,

$$\limsup_{i \to \infty} y_2 C a_{-t_i} = Z_2.$$ 

**Lemma 6.2.** Let $D : [0,1] \to U_2$ be a smooth curve with $D(0) = 0$ and $D'(0) \neq 0$. Then for any sequence $t_i \to +\infty$, the set

$$\liminf_{i \to \infty} a_{t_i} D a_{-t_i}$$

contains a one-parameter subsemigroup $V^+ < U_2$.

**Proof.** Let us identify $U_2 \simeq \mathbb{R}^{d-1}$ and write $D(t) = tD'(0) + t^2 R(t)$ for some continuous function $R : [0,1] \to U_2$. Let $\gamma \geq 0$ be arbitrary. We claim

$$\liminf a_{t_i} D a_{-t_i}$$

contains $rD'(0)$. We set $s_i := re^{-t_i}$. Then $s_i \to 0$ and

$$a_{t_i} D(s_i) a_{-t_i} = rD'(0) + r^2 e^{-t_i} R(re^{-t_i}) \to rD'(0)$$

as $i \to \infty$.

This proves the lemma.

**Proposition 6.3.** Let $V^+$ be a one-parameter subsemigroup of $U_2$. For any $z \in Z_2$, there exists $q \in M_2$ such that $zqV^+$ is dense in $Z_2$.

**Proof.** Let $V$ be the one-parameter subgroup spanned by $V^+$. By Ratner’s uniform distribution theorem [10], for any $x \in Z_2$, $xV^+$ is dense if and only if $xV$ is dense. Moreover, since $Z_2$ is compact, the set $\{x \in Z_2 : xV \text{ is not dense in } Z_2\}$ is contained in the singular set

$$\mathcal{S}(V) := \bigcup \Gamma_2 \backslash \Gamma_2 X(L, V)$$
where $L$ belongs to some countable collection of proper reductive algebraic subgroups of $H_2$ such that $[e]L$ is closed and $L \cap H_2$ is a Zariski dense lattice in $L$ and $X(L,V) = \{g \in H_2 : gVg^{-1} \subset L\}$ ([10], [12, Proposition 3.10]). Choose $g \in H_2$ so that $z = [g]$. Suppose that $zM_2 \subset \mathcal{J}(V)$. Then there exists a proper reductive algebraic subgroup $L$ such that a positive measurable subset of $gM$ is contained in $X(L,V)$. Since $X(L,V)$ is a real algebraic submanifold, it follows that $gM \subset X(L,V)$. Hence $mV_m^{-1} \subset g^{-1}Lg$ for all $m \in M_2$. Since $\{mV_m^{-1} : m \in M_2\} = U_2$, we get $U_2 \subset g^{-1}Lg$. This is a contradiction, since no proper reductive subgroup of $H_2$ can contain a horospherical subgroup $U_2$.

**Proof of Lemma 6.1.** Define $D(t) = C(t) - C(0)$ (in the additive notation for $U_2 = \mathbb{R}^{d-1}$), and set $D = D[0,1]$. Let $t_i \to +\infty$. Then $\lim \inf a_{t_i}D_{a_{-t_i}}$ contains a one-parameter semigroup $V^+$ of $U_2$ by Lemma 6.2.

Fix any compact orbit $zA_2M_2$. By Proposition 6.3, we may assume that $zV^+$ is dense by modifying $z$ using an element of $M_2$. Note that since $zaV^+ = zV^+a$ for any $a \in A_2$, we have $zaV^+$ dense for any $a \in A_2$. Hence $za_{-t_i}$ converges to some $z_0$ where $z_0 \in zA_2$ and $m \in M_2$, by passing to a subsequence. Then $z_m^{-1}a_{-t_i} \to z_0$. Set $y_2 = z_m^{-1}C(0)^{-1}$. Then $y_2Ca_{-t_i} = z_m^{-1}Da_{-t_i}$, and hence

$$\lim_{i \to \infty} y_2Ca_{-t_i} = \lim_{i \to \infty} z_m^{-1}a_{-t_i}(a_{t_i}Da_{-t_i}) \supset z_0V^+ = Z_2.$$}

7. INVARINCE BY SMOOTH CURVES AND CONCLUSION

**Theorem 7.1.** Let $X$ be a closed $H$-invariant subset of $Z$. Let $Y \subset X$ be a $U$-minimal subset with respect to $\Omega$. Suppose that there exists $y \in Y \cap \Omega$ such that $X - yH$ is not closed. Then there exists a smooth curve $C : [0,1] \to U_2$ such that $C'(0) \neq 0$ and $YC \subset X$.

**Proof.** By Proposition 5.4, there exists a one-parameter subsemigroup $S \subset AMU_2$ such that $S \not\subset M$ and $YS \subset Y$. Now $S$ is either an unbounded subsemigroup of $w^{-1}AMw$ for some $w \in U_2$, or is contained in $MU_2$ but not in $M$.

**Case 1:** $S \subset w^{-1}AMw$ and $S$ is unbounded.

Suppose that $w = e$; so $S = \{(m_ta_t, m_ta_t) : t \geq 0\}$. By Proposition 5.5, there exists a nontrivial $v \in U_2$ such that $Yv \subset X$. Observe $YSvAM \subset YvAM \subset X$. Define $C : [0,1] \to U_2$ by

$$C(t) = (e, m_ta_tv^{-1}a^{-1}_t).$$

Since $C \subset SvAM$, we have $YC \subset X$.

Next, suppose $w \neq e$ and write $S = \{(m_ta_t, w^{-1}m_ta_tw) : t \geq 0\}$. Observe that $YSAM \subset X$, and define $C : [0,1] \to U_2$ by

$$C(t) = (e, w^{-1}m_ta_tw^{-1}a^{-1}_t).$$
Since $C \subset SAM$, we have $YC \subset X$. In this case, it is clear that $C'(0) \neq 0$.

**Case 2:** $S \subset MU_2$. There exists $\xi \in \text{Lie} M$ and $\eta \in \text{Lie} U_2 - \{0\}$ such that $S = \{ \phi(t) := \exp(t(\xi + \eta)) : t \geq 0 \}$. Let $\delta(t) \in M$ be the unique element such that $\phi(t)\delta(t) \in U_2$. Define $C : [0, 1] \to U_2$ by $C(t) := \phi(t)\delta(t)$. Since $\phi(0) = e, \phi'(0) = \xi + \eta,$ and $\delta(0) = e$, we have

$$C'(0) = \phi'(0)\delta(0) + \phi(0)\delta'(0) = \xi + \eta + \delta(0)$$

On the other hand, since $C'(0) \in \text{Lie}(U_2)$ and $\delta'(0) \in \text{Lie} M$, it follows $\delta(0) = -\xi$ and $C'(0) = \eta \neq 0$.

**Proposition 7.2.** Let $E$ be an $H$-invariant subset of $Z$ which is not closed. Then $E$ is dense in $Z$.

**Proof.** Let $X$ denote the closure of $E$. By the assumption, there exists $x \in X - E$. Since any $H$-orbit meets $\Omega$, we may assume $x \in (X - E) \cap \Omega$, by modifying $x$ using an element of $H$. We claim that there exists a $U$-minimal subset $Y$ of $X$ with respect to $\Omega$ such that for some $y \in Y \cap \Omega$, $X - yH$ is not closed.

If $E$ is locally closed, then $X - E$ is a closed subset. Let $Y$ be a $U$-minimal subset of $X - E$ with respect to $\Omega$. Choose $y \in Y \cap \Omega$. Then $X - yH$ is not closed, since $y \in X - E$.

If $E$ is not locally closed, then $X - E$ is not closed. Let $Y$ be a $U$-minimal closed subset of $xH$ with respect to $\Omega$. If $Y \cap \Omega \subset xH$, choose $y \in Y \cap \Omega$. If $Y \cap \Omega \not\subset xH$, then choose $y \in (Y \cap \Omega) - xH$. We can then check that $X - yH$ is not closed.

Therefore, $Y$ satisfies the hypothesis of Theorem 7.1. Therefore there exists a smooth curve $C : [0, 1] \to U_2$ such that $C'(0) \neq 0$ and

$$YC \subset X.$$  

By Lemma 6.1, there exists $y_2 \in Z_2$ such that for any sequence $t_i \to +\infty$,

$$\limsup_{i \to \infty} y_2Ca_{-t_i} = Z_2. \tag{7.1}$$

By Lemma 5.1, we can choose $y_1 \in RF_+S_1$ such that $(y_1, y_2) \in Y$. By replacing $(y_1, y_2)$ with $(y_1u, y_2u)$ for some $(u, u) \in U$, we may assume $y_1 \in RF_{S_1}$, as (7.1) holds for $y_2u$ as well.

Since $y_1$ belongs to the compact $A_1$-invariant subset $RF_{S_1}$, there exists $t_i \to +\infty$ such that $y_1a_{-t_i}$ converges to some $z_1 \in RF_{S_1}$. As $(y_1, y_2) \in Y$ and $X$ is $A$-invariant, it follows

$$(y_1a_{-t_i}, y_2Ca_{-t_i}) \subset X.$$  

By (7.1), we obtain $\{z_1\} \times Z_2 \subset X$. Since $X$ is $H$-invariant, this implies $X = Z$.

A collection of elements $g_1, \cdots, g_k \in SO^o(d, 1), k \geq 2$, is called a Schottky generating set if there exist mutually disjoint closed round balls $B_1, \cdots, B_k$ and $B_1', \cdots, B_k'$ in $S^{d-1}$ such that $g_i$ maps the exterior of $B_i$ onto the interior of $B_i'$ for each $i = 1, \cdots, k$. A subgroup of $SO^o(d, 1)$ is called a (classical)
Schottky subgroup if it is generated by some Schottky generating set. It is easy to see that a Schottky subgroup is a convex cocompact subgroup.

The following lemma is well-known (e.g., [1, Proposition 4.3]). We give a short elementary proof.

**Lemma 7.3.** Any Zariski dense discrete subgroup $\Gamma$ of $SO^d(\mathbb{R})$ contains a Zariski dense Schottky subgroup.

*Proof.* Let $\Lambda$ denote the limit set of $\Gamma$. For each loxodromic element $\gamma \in \Gamma$, $\gamma^+$ and $\gamma^-$ are respectively the attracting and repelling fixed points of $\gamma$. As $\Gamma$ is non-elementary, it follows from [4, Proposition 2.7] that the set \[(\gamma^+, \gamma^-) : \gamma \text{ is a loxodromic element of } \Gamma\] is a dense subset of $\Lambda \times \Lambda$.

Choose two loxodromic elements $\gamma_1, \gamma_2 \in \Gamma$ such that $\{\gamma_1^+\}$ and $\{\gamma_2^+\}$ are disjoint from each other. Let $S_1$ be the smallest sub-sphere of $S^{d-1}$ which contains $\{\gamma_i^+ : i = 1, 2\}$. If $S_1 \neq S^{d-1}$, then we choose a loxodromic element $\gamma_3 \in \Gamma$ so that $\{\gamma_3^+\} \cap S_1 = \emptyset$. Let $S_2$ be the smallest sub-sphere of $S^{d-1}$ which contains $\{\gamma_i^+ : i = 1, 2, 3\}$. Then $\dim S_2 > \dim S_1$. Continuing in this fashion, we can find a sequence of loxodromic elements $\gamma_1, \cdots, \gamma_m$ of $\Gamma$ with $m \leq d-1$ such that the sets $\{\gamma_i^\pm\}$ are all mutually disjoint and their union is not contained in any proper sub-sphere of $S^{d-1}$.

Now we can find a sufficiently large $k$ such that $\gamma_i^k$, $i = 1, \cdots, m$ form a Schottky generating set. Let $\Gamma_0$ be the subgroup generated by them. Since the limit set of $\Gamma_0$ contains the set of all fixed points of $\gamma_i^k$, which is equal to $\{\gamma_i^\pm : i = 1, \cdots, m\}$, it is not contained in any proper sub-sphere of $S^{d-1}$. Hence $\Gamma_0$ is Zariski dense.

□

**Proof of Theorems 1.1 and 1.2.** In order to use the notations introduced in sections 2-6, let $\Gamma_1$ be a Zariski dense discrete subgroup and $\Gamma_2$ be a cocompact lattice in $G$. The equivalence of Theorems 1.1 and 1.2 follows since the following are all equivalent to each other for any $(g_1, g_2) \in H_1 \times H_2$:

1. The orbit $[(g_1, g_2)]H$ is closed (resp. dense) in $(\Gamma_1 \times \Gamma_2) \setminus (H_1 \times H_2)$;
2. The orbit $(\Gamma_1 \times \Gamma_2)(g_1, g_2)$ is closed (resp. dense) in $(H_1 \times H_2)/H$;
3. The product $\Gamma_2 g_2 g_1^{-1} \Gamma_1$ is closed (resp. dense) in $G$;
4. The orbit $[g_2 g_1^{-1}] \Gamma_1$ is closed (resp. dense) in $\Gamma_2 \setminus H_2$.

By Lemma 7.3, $\Gamma_1$ contains a Zariski dense convex cocompact subgroup, say, $\Gamma_0$. Since any $\Gamma_1$-invariant subset of $\Gamma_2 \setminus G$ is $\Gamma_0$-invariant, it suffices to prove Theorem 1.1 (or Theorem 1.2) for $\Gamma_0$. Therefore we assume that $\Gamma_1$ is convex cocompact without loss of generality. Suppose that $X$ is a closed $H$-invariant subset of $Z = \Gamma_1 \setminus H_1 \times \Gamma_2 \setminus H_2$, and suppose that $X \neq Z$. If $X$ consists of finitely many $H$-orbits, then each of them must be closed by Proposition 7.2. Now suppose that $X$ contains infinitely many $H$-orbits, say $x_i H$. Each $x_i H$ should be closed again by Proposition 7.2. Consider the set $E := \bigcup x_i H$. Recalling that every $H$-orbit meets $\Omega$, we may assume that $x_i \in \Omega$ and it converges to some $x \in \Omega - E$; if $x \in x_j H$, then we replace...
$E$ by $\bigcup_{i>j} x_i H$. Since $E$ is not closed, by Proposition 7.2, $E$ is dense in $Z$. This proves Theorem 1.2.

References


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