TOPOLOGICAL PROOF OF BENOIST-QUINT'S ORBIT CLOSURE THEOREM FOR $\mathrm{SO}(d,1)$

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Abstract. We present a new proof of the following theorem of Benoist-Quint: Let $G := \mathrm{SO}^\circ(d,1)$, $d \geq 2$ and $\Delta < G$ a cocompact lattice. Any orbit of a Zariski dense subgroup Γ of G is either finite or dense in $\Delta\backslash G$. While Benoist and Quint's proof is based on the classification of stationary measures, our proof is topological, using ideas from the study of dynamics of unipotent flows on the infinite volume homogeneous space $\Gamma\backslash G$.

Contents

1. Introduction

Let $G = \mathrm{SO}^\circ(d,1)$ for $d \geq 2$, and Δ a cocompact lattice in G. Let Γ be a Zariski dense subgroup of G, acting on the space $\Delta\backslash G$ by right translations.

The aim of this paper is to present a new proof of the following theorem of Benoist-Quint in [2], which was originally a question of Margulis [9] and Shah [17]:

Theorem 1.1. *Any Γ-invariant subset of $\Delta\backslash G$ is either finite or dense.*

The proof of Benoist-Quint is based on their classification of stationary measures for random walks on Γ on the space $\Delta\backslash G$. Our proof is topological and can be easily modified to all rank one simple Lie groups; for the sake of concreteness, we opted to write it only for $G = \mathrm{SO}^\circ(d,1)$. In the case when $G = \mathrm{SO}^\circ(2,1)$ and $\Gamma < G$ is a convex cocompact Zariski dense subgroup,

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Benoist-Oh gave a topological proof of Theorem 1.1 when the Γ-invariant subset is a single Γ-orbit [3].

Since a Zariski dense subgroup of G is either discrete or dense, it suffices to consider the case when Γ is discrete. Our starting point is then the observation that Theorem 1.1 can be translated into a problem on the orbit closure of unipotent flows on a homogeneous space of *infinite volume.* If we set $H = \{(g,g) : g \in G\}$ to be the diagonal embedding of G into $G \times G$, Theorem 1.1 is equivalent to the following statement about the H-action on the product space $\Gamma\backslash G \times \Delta\backslash G$, which has infinite volume unless Γ is a lattice.

Theorem 1.2. *Any H-invariant closed subset of $(\Gamma\times\Delta)\backslash(G\times G)$ is either a union of finitely many closed H-orbits or dense. In particular, any H-orbit is either closed or dense.*

When Γ is a lattice in G, i.e., when the homogeneous space $(\Gamma\times\Delta)\backslash(G\times G)$ has finite volume, Theorem 1.2 is a special case of Ratner's orbit closure theorem [15] and Mozes-Shah theorem [13].

On the proofs. Any Zariski dense discrete subgroup of G contains a Zariski dense Schottky subgroup (Lemma 7.3). Hence in proving Theorem 1.1, we may assume without loss of generality that Γ is a convex cocompact Zariski dense subgroup.

Set $Z := (\Gamma \times \Delta)\backslash(G \times G)$. Let $\mathcal{A} = \{a_t\}$ be a one-parameter subgroup of diagonalizable elements of G, and $\mathcal{U}$ the contracting horospherical subgroup of G with respect to the choice of $\mathcal{A}$. Let $U < H$ denote the diagonal embedding of $\mathcal{U}$ into $G\times G$. Our proof is based on the study of the action of U on Z. Let Ω denote the subset of Z consisting of all bounded $\mathcal{A}\times\mathcal{A}$-orbits, which is a compact subset. For $x \in \Omega$, consider the return of xU to Ω:

$$\mathsf{T}(x) := \{u \in U : xu \in \Omega\}.$$

For any sequence $\lambda_i \to \infty$, we show that the renormalization

$$\mathsf{T}_\infty := \limsup_i \lambda_i^{-1}\mathsf{T}(x)$$

is locally Zariski dense at e, i.e., for any neighborhood $\mathcal{O}$ of e in U, $\mathsf{T}_\infty \cap \mathcal{O}$ is Zariski dense in U (Lemma 3.2). This is the key recurrence property we use in carrying out the unipotent dynamics for the U-action on Z. We remark that this recurrence property is much weaker than the notion of thickness used in ([11], [12], [4], [8]), where the thick return property was required for *any* one-parameter subgroup of U; the latter strong property does not hold for a general convex cocompact subgroup.

We prove that any closed H-invariant subset X of Z, which is not a union of finitely many closed H-orbits, contains a U-minimal subset Y with respect to Ω such that

$$Y\mathcal{C} \subset X$$

for some non-constant analytic curve $\mathcal{C}$ contained in $\{e\} \times \mathcal{U}$. We then conclude $X = Z$ using the density of translates $x\mathcal{C}a_{-t} \subset \Delta\backslash G$ as $t \to$

$+\infty$ (Theorem 6.1); this last ingredient was proved by Shah [18], using Ratner's measure classification theorem [14] and the linearization techniques ([5], [16]).

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2. Notations and background

Let $G = \mathrm{SO}^\circ(d,1)$, $d \geq 2$. Let $\mathbb{H}^d$ denote the real hyperbolic space of dimension d with boundary $\partial\mathbb{H}^d = \mathbb{S}^{d-1}$. Then G can be identified with the group $\mathrm{Isom}^+(\mathbb{H}^d)$ of orientation preserving isometries of $\mathbb{H}^d$. The isometric action of G on $\mathbb{H}^d$ extends to a transitive action of G on the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^d)$. We identify $\mathbb{H}^d = G/\mathcal{K}$ and $\mathrm{T}^1(\mathbb{H}^d) = G/\mathcal{M}$ where $\mathcal{K}$ and $\mathcal{M}$ are respectively the stabilizers of a point $o \in \mathbb{H}^d$ and a vector $v_o \in \mathrm{T}^1_o(\mathbb{H}^d)$. The group G itself can be understood as the oriented frame bundle $\mathrm{F}(\mathbb{H}^d)$. Let $\mathcal{A} = \{a_t : t \in \mathbb{R}\}$ be the one-parameter subgroup of diagonalizable elements such that $\mathcal{A}$ centralizes $\mathcal{M}$ and the right translation action of a_t on $G/\mathcal{M}$ corresponds to the geodesic flow on $\mathrm{T}^1(\mathbb{H}^d)$. For a tangent vector $v \in \mathrm{T}^1(\mathbb{H}^d)$, we write v^+ for the forward end point of the associated geodesic in the boundary $\mathbb{S}^{d-1}$ and v^- for the backward end point. For $g \in G$, we define

$$g^+ := (gv_o)^+ = gv_o^+ \quad \text{and} \quad g^- := (gv_o)^- = gv_o^-.$$

We denote by $\mathcal{U}$ the contracting horospherical subgroup of G:

$$\mathcal{U} = \{u \in G : a_{-t}ua_t \to e, \quad \text{as } t \to +\infty\}.$$

The group $\mathcal{U}$ is isomorphic to $\mathbb{R}^{d-1}$; we write $\mathcal{U} = \{u_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^{d-1}\}$.

We use the following notation in the rest of the paper:

- $H = \{(h,h) : h \in G\}$;
- $H_1 = G \times \{e\}$, $H_2 = \{e\} \times G$, and $\mathcal{H} = H_1 \times H_2$;
- $A = \{(a_t, a_t) : t \in \mathbb{R}\}$;
- $A_1 = \mathcal{A} \times \{e\}$, $A_2 = \{e\} \times \mathcal{A}$;
- $U = \{(u,u) : u \in \mathcal{U}\}$;
- $U_1 = \mathcal{U} \times \{e\}$, $U_2 = \{e\} \times \mathcal{U}$;
- $M = \{(m,m) : m \in \mathcal{M}\}$;
- $M_1 = \mathcal{M} \times \{e\}$, and $M_2 = \{e\} \times \mathcal{M}$.

Let $\Gamma_1 < H_1$ be a Zariski-dense discrete subgroup and $\Gamma_2 < H_2$ be a cocompact lattice. We assume that both Γ_1 and Γ_2 are torsion-free. For each $i = 1, 2$, let

$$\mathcal{S}_i := \Gamma_i \backslash \mathbb{H}^d$$

denote the associated real hyperbolic manifold, and $\Lambda_i \subset \mathbb{S}^{d-1}$ the limit set of Γ_i. As Γ_2 is a lattice in H_2, we have $\Lambda_2 = \mathbb{S}^{d-1}$. We assume that Γ_1 is convex cocompact, that is, $\Gamma_1 \backslash \mathrm{hull}(\Lambda_1)$ is compact where $\mathrm{hull}(\Lambda_1) \subset \mathbb{H}^d$ denotes the convex hull of Λ_1.

Set

$$Z_1 = \Gamma_1\backslash H_1, \quad Z_2 = \Gamma_2\backslash H_2, \text{ and } Z = Z_1 \times Z_2.$$

We define

$$\mathrm{RF}\,\mathcal{S}_1 = \{x_1 \in Z_1 : x_1 A_1 \text{ bounded}\} = \{[g] \in Z_1 : g^{\pm} \in \Lambda_1\};$$

and

$$\mathrm{RF}_+\,\mathcal{S}_1 = \{x_1 \in Z_1 : x_1 A_1^+ \text{ bounded}\} = \{[g] \in Z_1 : g^+ \in \Lambda_1\}$$

where $A_1^+ = \{(a_s, e) : s \geq 0\}$.

Define

$$\Omega = \mathrm{RF}\,\mathcal{S}_1 \times Z_2 \text{ and } \Omega_+ = \mathrm{RF}_+\,\mathcal{S}_1 \times Z_2.$$

As Γ_1 is convex cocompact, $\mathrm{RF}\,\mathcal{S}_1$ is a compact A_1M_1-invariant subset. Hence Ω is a compact subset of Z which is invariant under $\prod_{i=1}^2 A_iM_i$. The set $\mathrm{RF}_+\,\mathcal{S}_1$ is equal to $\mathrm{RF}\,\mathcal{S}_1 \cdot U_1$, and hence Ω_+ is a closed subset of Z invariant under $\prod_{i=1}^2 A_iM_iU_i$.

3. Local Zariski density of renormalization of U-recurrence

We often identify U with $\mathbb{R}^{d-1}$ via the map $(u_{\mathbf{t}}, u_{\mathbf{t}}) \mapsto \mathbf{t}$, and the notation $\|\mathbf{t}\|$ means the Euclidean norm of $\mathbf{t} \in \mathbb{R}^{d-1}$. To ease the notation, we sometimes write $u \in U$, identifying u with (u, u). Similarly we will write $a \in A$, identifying a with (a, a).

For $x \in \Omega$, we define the following recurrence time of x to Ω under U:

$$\mathsf{T}(x) := \{\mathbf{t} \in \mathbb{R}^{d-1} : xu_{\mathbf{t}} \in \Omega\}.$$

For $x = (x_1, x_2) \in \Omega$, note that $\mathbf{t} \in \mathsf{T}(x)$ if and only if $x_1 u_{\mathbf{t}} \in \mathrm{RF}\,\mathcal{S}_1$. If we choose $g_1 \in H_1$ so that $x_1 = [g_1]$, then $g_1^{\pm} \in \Lambda_1$ since $x_1 \in \mathrm{RF}\,\mathcal{S}_1$. Since $(g_1 u_{\mathbf{t}})^+ = g_1^+$, we have

$$\mathsf{T}(x) = \{\mathbf{t} \in \mathbb{R}^{d-1} : (g_1 u_{\mathbf{t}})^- \in \Lambda_1\}. \tag{3.1}$$

Since $(g_1 u_{\mathbf{t}})^- \to g_1^+ \in \Lambda_1$ as $\mathbf{t} \to \infty$ and Λ_1 has no isolated point, it follows that $\mathsf{T}(x)$ is unbounded.

Lemma 3.1. *For $x \in \Omega$, any non-empty open subset of $\mathsf{T}(x)$ is Zariski dense in U.*

Proof. The visual map $U \to \mathbb{S}^{d-1} - \{g_1^+\}$ defined by $u \mapsto (g_1 u)^-$ is a diffeomorphism. Hence by (3.1), the claim follows the well-known fact that no non-empty open subset of Λ_1 is contained in a smooth submanifold in $\mathbb{S}^{d-1}$ of positive co-dimension (cf. [19, Corollary 3.10]). $\square$

Lemma 3.2. *Let $x \in \Omega$. For any sequence $\lambda_i \to +\infty$, there exists $z \in \Omega$ such that*

$$\mathsf{T}_\infty := \limsup_{i\to\infty} \lambda_i^{-1} \mathsf{T}(x) \supset \mathsf{T}(z).$$

In particular, for any neighborhood $\mathcal{O} \subset U$ of e, $\mathsf{T}_\infty \cap \mathcal{O}$ is Zariski dense in U.

Proof. Note that $\lambda_i^{-1}\mathsf{T}(x) = \{\mathbf{t} \in \mathbb{R}^{d-1} : xu_{\lambda_i \mathbf{t}} \in \Omega\}$. Let $s_i = \frac{1}{2}\log \lambda_i$ so that $a_{s_i} u_{\mathbf{t}} a_{s_i}^{-1} = u_{\lambda_i \mathbf{t}}$. Since Ω is A-invariant,

$$\lambda_i^{-1}\mathsf{T}(x) = \{\mathbf{t} \in \mathbb{R}^{d-1} : xa_{s_i} u_{\mathbf{t}} \in \Omega\} = \mathsf{T}(xa_{s_i}).$$

Since Ω is a compact A-invariant subset, passing to a subsequence, xa_{s_i} converges to some $z \in \Omega$ as $i \to \infty$.

We claim that

$$\limsup_{i\to\infty} \lambda_i^{-1}\mathsf{T}(x) \supset \mathsf{T}(z).$$

Let $x = (x_1, x_2)$, $z = (z_1, z_2)$, and choose $g_1, g_1' \in G$ so that $x_1 = [g_1]$ and $z_1 = [g_1']$. Since $x_1 a_{s_i} \to z_1$ as $i \to \infty$, there exists $\gamma_i \in \Gamma_1$ such that $\gamma_i g_1 a_{s_i} \to g_1'$ as $i \to \infty$. Let $\mathbf{t} \in \mathsf{T}(z)$. For each i, choose $\mathbf{r}_i \in \mathbb{R}^{d-1}$ of minimal Euclidean norm in the set $\{\mathbf{r} \in \mathbb{R}^{d-1} : (\gamma_i g_1 a_{s_i} u_{\mathbf{t}+\mathbf{r}})^- \in \Lambda_1\}$. We claim that $\mathbf{r}_i \to 0$ as $i \to \infty$. Suppose not. Then there exists $c > 0$ such that $B_i(c) \cap \Lambda_1 = \emptyset$ for infinitely many i, where $\mathrm{B}(\mathbf{t}, c) \subset \mathbb{R}^{d-1}$ denotes the closed ball of radius c centered at $\mathbf{t}$, $u_{\mathrm{B}(\mathbf{t},c)} = \{u_{\mathbf{s}} : \mathbf{s} \in \mathrm{B}(\mathbf{t}, c)\}$ and $B_i(c) := (\gamma_i g_1 a_{s_i} u_{\mathrm{B}(\mathbf{t},c)})^-$. Since $B_i(c)$ converges to $B(c) := (g_1' u_{\mathrm{B}(\mathbf{t},c)})^-$ in the Hausdorff topology of closed subsets of $\mathbb{S}^{d-1}$ and $B(c)$ contains a neighborhood of $(g_1' u_{\mathbf{t}})^-$, $B_i(c)$ must contain $(g_1' u_{\mathbf{t}})^-$ for all sufficiently large i. Since $(g_1' u_{\mathbf{t}})^- \in \Lambda_1$ (because $\mathbf{t} \in \mathsf{T}(z)$), we get a contradiction to the hypothesis that $B_i(c) \cap \Lambda_1 = \emptyset$ for infinitely many i's. Therefore $\mathbf{t}_i := \mathbf{t} + \mathbf{r}_i \to \mathbf{t}$ as $i \to \infty$.

On the other hand, $\mathbf{t}_i \in \lambda_i^{-1}\mathsf{T}(x)$ since

$$(\gamma_i g_1 u_{\lambda_i \mathbf{t}_i})^- = (\gamma_i g_1 a_{s_i} u_{\mathbf{t}_i} a_{-s_i})^- = (\gamma_i g_1 a_{s_i} u_{\mathbf{t}_i})^- \in \Lambda_1.$$

This shows that $\mathbf{t} \in \limsup \lambda_i^{-1}\mathsf{T}(x)$, proving the first claim. The second claim follows from the first claim together with Lemma 3.1. $\square$

4. Unipotent blowup

For a subgroup $S < \mathcal{H}$, we denote by $\mathrm{N}(S)$ the normalizer of S in $\mathcal{H}$. For a subgroup $S_i \subset H_i$, $\mathrm{C}_{H_i}(S_i)$ denotes the centralizer of S_i in H_i.

Lemma 4.1. *We have* $\mathrm{N}(U) = AMU_1U_2$.

Proof. The inclusion $\supset$ is clear. To show the reverse inclusion $\subset$, let $(g_1, g_2) \in \mathrm{N}(U)$. Then for all $(u, u) \in U$, $(g_1 u g_1^{-1}, g_2 u g_2^{-1}) \in U$ and hence $g_2^{-1} g_1 u g_1^{-1} g_2 = u$. This implies $(g_2^{-1} g_1, e) \in \mathrm{C}_{H_1}(U_1)$. Since $\mathrm{C}_{H_1}(U_1) \subset \mathrm{N}(U)$, and

$$(g_1, g_2) = (g_2, g_2) \cdot (g_2^{-1} g_1, e) \in \mathrm{N}(U),$$

it follows $(g_2, g_2) \in \mathrm{N}(U) \cap H = AMU$. As both (g_2, g_2) and $(g_2^{-1} g_1, e)$ belong to AMU_1U_2, so does (g_1, g_2). $\square$

Lemma 4.2. *Let $g_i \to e$ in $\mathcal{H} - \mathrm{N}(U)$ as $i \to \infty$, and $x \in \Omega$. Then for any neighborhood $\mathcal{O} \subset \mathcal{H}$ of e, there exist sequences $u_i' \in U$ and $u_i \in \mathsf{T}(x)$ such that, as $i \to \infty$, the sequence $u_i' g_i u_i$ converges to an element in $(AMU_2 - M) \cap \mathcal{O}$.*

Proof. Following [10], we will construct a quasi-regular map

$$\psi : \mathbb{R}^{d-1} \to AMU_2$$

associated to the given sequence g_i. Since U is a real algebraic subgroup of $\mathcal{H}$, by Chevalley's theorem, there exists an $\mathbb{R}$-regular representation $\mathcal{H} \to \mathrm{GL}(W)$ with a distinguished point $p \in W$ such that $U = \mathrm{Stab}(p)$. Then $p\mathcal{H}$ is locally closed, and $\mathrm{N}(U)$ is equal to the set

$$\{g \in \mathcal{H} : pgu = pg \text{ for all } u \in U\}. \tag{4.1}$$

Set $L := A_1M_1U_1^+ \times A_2M_2U_2^+U_2$ where U_i^+ is the expanding horospherical subgroup of H_i for $i = 1, 2$. Note that

$$\mathrm{N}(U) \cap L = AMU_2$$

and that the product map from $U \times L$ to $\mathcal{H}$ is a diffeomorphism onto a Zariski open neighborhood of e.

Since $p\mathcal{H}$ is open in its closure and pL is a open neighborhood of p in $p\mathcal{H}$, we can choose an M-invariant norm on W such that

$$B(p, 1) \cap \overline{p\mathcal{H}} \subset pL \tag{4.2}$$

where $B(p, r) \subset W$ denotes the norm ball of radius r centered at p.

For each i, we define $\tilde{\phi}_i : \mathbb{R}^{d-1} \to W$ by

$$\tilde{\phi}_i(\mathbf{t}) = pg_iu_{\mathbf{t}}$$

which is a polynomial map in $d-1$-variables with degree uniformly bounded for all i. Note that $\tilde{\phi}_i(0)$ converges to p as $i \to \infty$. As $g_i \notin \mathrm{N}(U)$, $\tilde{\phi}_i$ is non-constant.

Now define

$$\lambda_i := \sup\{\lambda \geq 0 : \tilde{\phi}_i(B(\lambda)) \subset B(p, 1)\}$$

where $B(\lambda)$ denotes the norm ball of radius λ centered at 0 in $\mathbb{R}^{d-1}$. Note that $\lambda_i < \infty$ as $\tilde{\phi}_i$ is nonconstant, and that $\lambda_i \to \infty$ as $g_i \to e$. Reparametrizing $\tilde{\phi}_i$ by λ_i, we define $\phi_i : \mathbb{R}^{d-1} \to W$:

$$\phi_i(\mathbf{t}) := \tilde{\phi}_i(\lambda_i\mathbf{t}).$$

Note that $\sup\{\|\phi_i(\mathbf{t}) - p\| : \mathbf{t} \in B(1)\} = 1$, and $\lim_{i\to\infty} \phi_i(0) = p$. Since the polynomials ϕ_i have uniformly bounded degree, it follows that after passing to a subsequence, ϕ_i converges to a non-constant polynomial $\phi : \mathbb{R}^{d-1} \to W$ uniformly on every compact subset of $\mathbb{R}^{d-1}$.

Since pL is a Zariski open neighborhood of p in $\overline{p\mathcal{H}}$, the following map ψ defines a non-constant rational map on a Zariski open neighborhood of 0 in $\mathbb{R}^{d-1}$:

$$\psi := \rho_L^{-1} \circ \phi$$

where ρ_L is the restriction to L of the orbit map $g \mapsto pg$.

Since $g_i \to e$, without loss of generality, we may assume that $g_i \in UL$ for all i. Except for a Zariski closed subset of $\mathbb{R}^{d-1}$, the product $g_iu_{\mathbf{t}}$ can be written as an element of UL in a unique way. We denote by $\psi_i(\mathbf{t}) \in L$ its L-component so that $g_iu_{\mathbf{t}} \in U\psi_i(\mathbf{t})$.

We have $\psi(0) = e$ and

$$\psi(\mathbf{t}) = \lim_{i\to\infty} \psi_i(\lambda_i \mathbf{t})$$

where the convergence is uniform on compact subsets of $\mathbb{R}^{d-1}$. It is easy to check that $\operatorname{Im}\psi \subset \mathrm{N}(U) \cap L = AMU_2$ using (4.1). Set

$$\mathsf{T}_\infty := \limsup_{i\to\infty} \lambda_i^{-1}\mathsf{T}(x).$$

Given a neighborhood $\mathcal{O} \subset \mathcal{H}$ of e, let $\mathcal{O}'$ be a neighborhood of 0 in $\mathbb{R}^{d-1}$ such that $\phi(\mathcal{O}') \subset p\mathcal{O}$. Since ϕ is a nonconstant polynomial, it follows from Lemma 3.2 that there exists $\mathbf{t} \in \mathcal{O}' \cap \mathsf{T}_\infty$ such that $\|\phi(\mathbf{t})\|^2 \neq \|p\|^2$.

Let $\mathbf{t}_i \in \mathsf{T}(x)$ be a sequence such that $\lambda_i^{-1}\mathbf{t}_i \to \mathbf{t}$ as $i \to \infty$ (by passing to a subsequence). Since $\psi_i \circ \lambda_i \to \psi$ uniformly on compact subsets,

$$\psi(\mathbf{t}) = \lim_{i\to\infty} (\psi_i \circ \lambda_i)\left(\lambda_i^{-1}\mathbf{t}_i\right) = \lim_{i\to\infty} \psi_i(\mathbf{t}_i) = \lim_{i\to\infty} u_{\mathbf{s}_i} g_i u_{\mathbf{t}_i}$$

for some sequence $\mathbf{s}_i \in \mathbb{R}^{d-1}$. Note that $\phi(\mathbf{t}) = p\psi(\mathbf{t})$ with $\psi(\mathbf{t}) \in AMU_2 \cap \mathcal{O}$. Since $\|\phi(\mathbf{t})\|^2 \neq \|p\|^2$ and $\|\cdot\|$ is M-invariant, we have $\psi(\mathbf{t}) \notin M$. Hence this finishes the proof. $\square$

Lemma 4.3. *Let $r_i \to e$ in $H_2 - \mathrm{N}(U)$. For any $x \in \Omega$, there exists a sequence $\mathbf{t}_i \in \mathsf{T}(x)$ such that the sequence $u_{-\mathbf{t}_i} r_i u_{\mathbf{t}_i}$ converges to a non-trivial element of U_2.*

Proof. Write $r_i = \exp q_i$ for $q_i \in \mathfrak{h} = \mathrm{Lie}(H_2)$. We write $U_2 = \{u_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^{d-1}\}$. Define a polynomial map $\psi_i : \mathbb{R}^{d-1} \to \mathfrak{h}$ by

$$\psi_i(\mathbf{t}) = u_{\mathbf{t}}^{-1} q_i u_{\mathbf{t}} \quad \text{ for all } \mathbf{t} \in \mathbb{R}^{d-1}.$$

Since $H_2 \cap \mathrm{N}(U) = U_2 = \mathrm{C}_{H_2}(U_2)$, it follows that $r_i \in H_2 - \mathrm{C}_{H_2}(U_2)$. Hence ψ_i is a nonconstant polynomial. Let λ_i be the supremum of $\lambda > 0$ such that $\sup_{\mathbf{t}\in B(\lambda)} \|\psi_i(\mathbf{t})\| \leq 1$ where $B(\lambda)$ denotes the ball in $\mathbb{R}^{d-1}$ of radius λ centered at 0. Then $0 < \lambda_i < \infty$ and $\lambda_i \to \infty$.

Now the rescaled polynomials $\phi_i := \psi_i \circ \lambda_i : \mathbb{R}^{d-1} \to \mathfrak{h}$ are uniformly bounded on the unit ball with uniformly bounded degree and $\lim_{i\to\infty} \phi_i(0) = 0$. Therefore, by passing to a subsequence, ϕ_i converges to a polynomial $\phi : \mathbb{R}^{d-1} \to \mathfrak{h}$ uniformly on compact subsets. Since $\sup_{\mathbf{t}\in B(1)} \|\phi(\mathbf{t})\| = 1$, ϕ is not a constant.

We claim that $\mathrm{Im}(\phi) \subset \mathrm{Lie}(U_2)$. For any fixed $\mathbf{s} \in \mathbb{R}^{d-1}$, we have $\lambda_i^{-1}\mathbf{s} \to 0$, and hence for any $\mathbf{t} \in \mathbb{R}^{d-1}$,

$$\begin{aligned} u_{\mathbf{s}}^{-1}\phi(\mathbf{t})u_{\mathbf{s}} &= \lim_{i\to\infty} u_{-\lambda_i\mathbf{t}-\mathbf{s}} q_i u_{\lambda_i\mathbf{t}+\mathbf{s}} \\ &= \lim_{i\to\infty} u_{-\lambda_i(\mathbf{t}+\lambda_i^{-1}\mathbf{s})} q_i u_{\lambda_i(\mathbf{t}+\lambda_i^{-1}\mathbf{s})} \\ &= \lim_{i\to\infty} u_{-\lambda_i\mathbf{t}} q_i u_{\lambda_i\mathbf{t}} = \phi(\mathbf{t}). \end{aligned}$$

Hence $\phi(\mathbf{t})$ belongs to the centralizer of U_2. Since the centralizer of U_2 in $\mathfrak{h}$ is equal to $\mathrm{Lie}\, U_2$, the claim follows.

Set

$$\mathsf{T}_\infty := \limsup_{i\to\infty} \lambda_i^{-1}\mathsf{T}(x).$$

Fix $\mathbf{t} \in \mathsf{T}_\infty$ such that $\phi(\mathbf{t}) \neq 0$; this exists by Lemma 3.2. Let $\mathbf{t}_i \in \mathsf{T}(x)$ be a sequence such that $\lambda_i^{-1}\mathbf{t}_i \to \mathbf{t}$ as $i \to \infty$. As $\phi_i \to \phi$ uniformly on compact subsets, it follows that

$$\phi(\mathbf{t}) = \lim_{i\to\infty} (\psi_i \circ \lambda_i)(\lambda_i^{-1}\mathbf{t}_i) = \lim_{i\to\infty} \psi_i(\mathbf{t}_i) = \lim_{i\to\infty} u_{\mathbf{t}_i}^{-1} q_i u_{\mathbf{t}_i}.$$

Hence, by exponentiating, we obtain that $u_{\mathbf{t}_i}^{-1} r_i u_{\mathbf{t}_i}$ converges to a non-trivial element of U_2. □

5. Relative minimal subsets and additional invariance

Let X be a closed H-invariant subset of Z. A closed U-invariant subset Y of X is called U-minimal with respect to Ω if $Y \cap \Omega \neq \emptyset$ and for any $y \in Y \cap \Omega$, yU is dense in Y. Since every H-orbit in Z intersects Ω, $X \cap \Omega \neq \emptyset$. By Zorn's lemma, there exists a U-minimal subset Y of X with respect Ω, which we fix in the following.

Lemma 5.1. *If $\pi_i : Z \to Z_i$ denotes the canonical projection for $i = 1, 2$, we have*

$$\pi_1(Y) = \mathrm{RF}_+ \mathcal{S}_1 \quad \textit{and} \quad \pi_2(Y) = Z_2.$$

Proof. The claim follows since U_1 and U_2 act minimally on $\mathrm{RF}_+ \mathcal{S}_1$ and Z_2 respectively [19]. □

Lemma 5.2. *Let S be a closed subgroup of $\mathrm{N}(U)$ containing U. For any $y \in Y \cap \Omega$, the orbit yS is not locally closed.*

Proof. Suppose that yS is locally closed for some $y \in Y \cap \Omega$. We claim that there exists a sequence $u_i \to \infty$ in U such that $yu_i \to y$ as $i \to \infty$. Let

$$Q := \{z \in Y : z = \lim_{i\to\infty} yu_i \text{ for some } u_i \to \infty \text{ in } U\}.$$

Since $\mathsf{T}(y)$ is unbounded, there exists $u_i \to \infty$ in U such that $yu_i \in Y \cap \Omega$. Since any limit of the sequence yu_i belongs to $Q \cap \Omega$, we have $Q \cap \Omega \neq \emptyset$. Since Q is a closed U-invariant set, $Q = Y$ by the relative U-minimality of Y. In particular, $y \in Q$, proving the claim. We may assume that $y = [e]$ without loss of generality. Let $\Gamma := \Gamma_1 \times \Gamma_2$. Since yS is locally closed, yS is homeomorphic to the quotient $(S \cap \Gamma)\backslash S$. Therefore there exists $\delta_i \in S \cap \Gamma$ such that $\delta_i u_i \to e$ as $i \to \infty$.

Since $\mathrm{N}(U) = AMU_1U_2$, writing $\delta_i = a_i r_i$ for $a_i \in A$ and $r_i \in MU_1U_2$, it follows that $a_i \to e$ as $i \to \infty$. Write $\delta_i = (\delta_i^1, \delta_i^2) \in \Gamma_1 \times \Gamma_2$. In the case when $a_i = e$ for all sufficiently large i, it follows from $u_i \to \infty$ in U that δ_i^1 must be a parabolic element of Γ_1, yielding a contradiction to the convex cocompactness of Γ_1. In the case when $a_i \neq e$ for an infinite subsequence, we again get a contradiction, because there is a uniform positive lower bound for all translation lengths of elements of Γ_1. This finishes the proof. □

Lemma 5.3. *For any $y \in Y \cap \Omega$, there exists a sequence $g_i \to e$ in $\mathcal{H} - \mathrm{N}(U)$ such that $yg_i \in Y$.*

Proof. Suppose not. Then there is an open neighborhood $\mathcal{O} \subset \mathcal{H}$ of e such that

$$y\mathcal{O} \cap Y \subset y\,\mathrm{N}(U). \tag{5.1}$$

We may assume the map $g \mapsto yg \in X$ is injective on $\mathcal{O}$ by shrinking $\mathcal{O}$ if necessary. Set

$$S := \{g \in \mathrm{N}(U) : Yg = Y\}$$

which is a closed subgroup of $\mathrm{N}(U)$ containing U. We will show that yS is locally closed; this contradicts Lemma 5.2. We first claim that

$$y\mathcal{O} \cap Y \subset yS. \tag{5.2}$$

If $g \in \mathcal{O}$ such that $yg \in Y$, then $g \in \mathrm{N}(U)$ by (5.1). Therefore $Yg = \overline{yU}g = \overline{ygU} \subset Y$. Moreover, since $Yg \subset Y \subset \Omega_+$ and $\Omega_+ = \Omega U$, we have $Yg \cap \Omega \neq \emptyset$. By the minimality assumption on Y, $Yg = Y$, proving that $g \in S$, and hence (5.2).

Therefore yS is an open U-invariant subset of Y. Since $Y = \overline{yS}$, it follows that yS is locally closed. □

By a one-parameter semigroup of $\mathcal{H}$, we mean a subset of the form $\{\exp(t\xi) : t \geq 0\}$ for some non-zero ξ in the Lie algebra of $\mathcal{H}$.

Proposition 5.4 (Translate of Y inside of Y)**.** *There exists a one-parameter subsemigroup $S < AMU_2$ such that $S \not\subset M$ and*

$$YS \subset Y.$$

Proof. It suffices to prove that there exists a sequence $\beta_k \to e$ in $AMU_2 - M$ such that $Y\beta_k \subset Y$ (cf. [8, Lemma 10.5]). Choose $y \in Y \cap \Omega$. By Lemma 5.3, there exists $g_i \to e$ in $\mathcal{H} - \mathrm{N}(U)$ such that $yg_i \in Y$. Let $\mathcal{O}_k$ be a decreasing sequence of neighborhoods of e in G so that $\bigcap_k \mathcal{O}_k = \{e\}$. Fix k. Applying Lemma 4.2 to the sequence g_i^{-1}, we get $u_i' \in U$ and $u_i \in \mathsf{T}(y) = \{u \in U : yu \in \Omega\}$ such that $u_i' g_i^{-1} u_i$ converges to some element $\alpha_k \in (AMU_2 - M) \cap \mathcal{O}_k$.

Since $Y \cap \Omega$ is compact, by passing to a subsequence, yu_i converges to some $y_k \in Y \cap \Omega$ as $i \to \infty$. Hence as $i \to \infty$,

$$yg_i(u_i')^{-1} = (yu_i)(u_i' g_i^{-1} u_i)^{-1} \to y_k \alpha_k^{-1} \in Y.$$

Since $y_k \in Y \cap \Omega$ and $\alpha_k \in \mathrm{N}(U)$, it follows that $Y\alpha_k^{-1} \subset Y$. It remains to set $\beta_k := \alpha_k^{-1}$. □

Proposition 5.5 (Translate of Y inside of X)**.** *Suppose that there exists $y \in Y \cap \Omega$ such that $X - yH$ is not closed. Then there exists a non-trivial element $v \in U_2$ such that*

$$Yv \subset X.$$

Proof. By the hypothesis, there exists a sequence $g_i \to e$ in $\mathcal{H} - H$ such that $yg_i \in X$. Since X is H-invariant, we may assume $g_i \in H_2$. Note that $\mathrm{N}(U) \cap H_2 = U_2$. Hence if $g_i \in \mathrm{N}(U)$ for some i, then we can simply take $v := g_i$.

Now suppose that $g_i \notin \mathrm{N}(U)$ for all i. By Lemma 4.3, there exists $u_i \in \mathsf{T}(y)$ such that $u_i^{-1} g_i u_i \to v$ for some non-trivial $v \in U_2$. Observe

$$(yu_i)(u_i^{-1} g_i u_i) = y g_i u_i \in X.$$

By passing to a subsequence, yu_i converges to some $y_0 \in Y \cap \Omega$. Since $y_0 v \in X$, it follows that $Yv \subset X$ by the relative minimality of Y. □

6. Expansion of an analytic curve inside a horospherical subgroup

For $1 \le k \le d-2$, let $\mathscr{S}_k$ denote the collection of all k-dimensional spheres $S \subset \mathbb{S}^{d-1}$ such that $\Gamma_2 S$ is closed in the space of all k-dimensional spheres of $\mathbb{S}^{d-1}$, and set

$$\mathscr{S} := \bigcup_{1 \le k \le d-2} \mathscr{S}_k.$$

For each $1 \le k \le d-2$, there exists a connected reductive subgroup $L_k \simeq \mathrm{SO}^\circ(k+1, 1)$ such that the convex hull of any sphere $S \subset \mathbb{S}^{d-1}$ of dimension k is equal to $\pi(g_S L_k) = \pi(g_S \,\mathrm{N}(L_k))$ for some $g_S \in H_2$, where $\pi : H_2 = \mathrm{F}(\mathbb{H}^d) \to \mathbb{H}^d$ is the base-point projection. Moreover the space of k-dimensional spheres of $\mathbb{S}^{d-1}$ is homeomorphic to the quotient space $H_2/\mathrm{N}(L_k)$. It follows that $S \in \mathscr{S}_k$ if and only if $[g_S]\,\mathrm{N}(L_k)$ is closed. We note that $\mathscr{S}$ consists of countably many spheres (cf. [8, Coro. 5.8]).

We deduce the following density statement from the equidistribution result [18, Theorem 1.5]:

Theorem 6.1. *Let $\mathcal{C} : [0,1] \to U_2$ be a non-constant analytic curve. Let $g_2 \in H_2$ be such that $(g_2 \mathcal{C}(0))^- \in \mathbb{S}^{d-1}$ is not contained in any sphere in $\mathscr{S}$. Then for any sequence $t_i \to +\infty$,*

$$\limsup_{i \to \infty} [g_2] \mathcal{C} a_{-t_i} = Z_2.$$

Proof. Let S be the smallest sphere of $\mathbb{S}^{d-1}$ which contains the subset $(g_2 \operatorname{Im}(\mathcal{C}))^- = \{(g_2 \mathcal{C}(s))^- : s \in [0,1]\}$. As $\mathcal{C}$ is non-constant, the dimension k of S is at least 1. Since $(g_2 \mathcal{C}(0))^- \in S$, S is not contained in any sphere in $\mathscr{S}$ by the hypothesis on $(g_2 \mathcal{C}(0))^-$. Since $\mathcal{C}$ is non-constant analytic, $\{s \in [0,1] : \mathcal{C}'(s) = 0\}$ is a finite set. Similarly, it follows from the hypothesis on $\mathcal{C}$ that for any $S_0 \in \mathscr{S}$, the set $\{s \in [0,1] : (g_2 \mathcal{C}(s))^- \in S_0\}$ is finite. Now the claim follows from the equidistribution theorem [18, Theorem 1.5]. □

7. Invariance by analytic curves and conclusion

Theorem 7.1. *Let X be a closed H-invariant subset of Z. Let $Y \subset X$ be a U-minimal subset with respect to Ω. Suppose that there exists $y \in Y \cap \Omega$ such that $X - yH$ is not closed. Then there exists an analytic curve $\mathcal{C} : [0,1] \to U_2$ such that $\mathcal{C}'(0) \neq 0$ and*

$$Y\mathcal{C} \subset X.$$

Proof. By Proposition 5.4, there exists a one-parameter subsemigroup $S \subset MAU_2$ such that $S \not\subset M$ and $YS \subset Y$. Now S is either an unbounded subsemigroup of $w^{-1}MAw$ for some $w \in U_2$, or contained in MU_2 but not in M.

Case 1: $S \subset w^{-1}MAw$ for some $w \in U_2$ and S is unbounded.
Case 1.a: $w = e$. In this case, we have $S = \{(m_t a_t, m_t a_t) : t \geq 0\} \subset MA$. By Proposition 5.5, there exists a nontrivial $v \in U_2$ such that $Yv \subset X$. Observe $YSvAM \subset YvAM \subset X$. Define $\mathcal{C} : [0,1] \to U_2$ by

$$\mathcal{C}(t) = (e, m_t a_t v a_t^{-1} m_t^{-1}).$$

Since $\mathcal{C} \subset SvAM$, we have $Y\mathcal{C} \subset X$. If $\xi \in \mathrm{Lie}(\mathcal{AM})$ such that $m_t a_t = \exp t\xi$, then $\mathcal{C}(t)$ is given by $\mathrm{Ad}_{\exp t\xi} v$ in the additive notation. Hence $\mathcal{C}$ is analytic and $\mathcal{C}'(0) = \mathrm{ad}_\xi(v) \neq 0$, since $\xi \notin \mathrm{Lie}(\mathcal{M})$.
Case 1.b: $w \neq e$. We write $S = \{(m_t a_t, w^{-1} m_t a_t w) : t \geq 0\}$. Observe that $YSAM \subset X$, and define $\mathcal{C} : [0,1] \to U_2$ by

$$\mathcal{C}(t) = (e, w^{-1} m_t a_t w a_t^{-1} m_t^{-1}).$$

Since $\mathcal{C} \subset SAM$, we have $Y\mathcal{C} \subset X$. If $\xi \in \mathrm{Lie}(\mathcal{AM})$ such that $m_t a_t = \exp t\xi$, then $\mathcal{C}(t)$ is given by $(\mathrm{Ad}_{e^{t\xi}} w) - w$ in the additive notation. Hence $\mathcal{C}$ is analytic and $\mathcal{C}'(0) = \mathrm{ad}_\xi(w) \neq 0$, since $\xi \notin \mathrm{Lie}(\mathcal{M})$.

Case 2: $S \subset MU_2$. Write $S = \{\exp(t(\xi + \eta)) : t \geq 0\}$ where $\xi \in \mathrm{Lie}\, M$ and $\eta \in \mathrm{Lie}\, U_2 - \{0\}$. Define $\mathcal{C} : [0,1] \to U_2$ so that $\mathcal{C}(t)$ is the U_2-component of $\exp(t(\xi + \eta))$, which is explicitly given by $\sum_{n=1}^{\infty} \frac{(-\xi)^{n-1}\eta}{n!} t^n$ in the additive notation. So $\mathcal{C}(t)$ is analytic and $\mathcal{C}'(0) = \eta \neq 0$. Since $\mathcal{C} \subset SM$, we have $Y\mathcal{C} \subset X$. □

Proposition 7.2. *Let E be an H-invariant subset of Z which is not closed. Then E is dense in Z.*

Proof. Let X denote the closure of E. By the assumption that E is not closed, there exists $x \in X - E$. Since any H-orbit meets Ω, we may assume $x \in (X - E) \cap \Omega$, by modifying x using an element of H.

We claim that there exists a U-minimal subset Y of X with respect to Ω such that for some $y \in Y \cap \Omega$, $X - yH$ is not closed.

If E is locally closed, then $X - E$ is a closed subset. Let Y be a U-minimal subset of $X - E$ with respect to Ω. Choose $y \in Y \cap \Omega$. Then $X - yH$ is not closed, since $y \in X - E$.

If E is not locally closed, then $X - E$ is not closed. Let Y be a U-minimal closed subset of $\overline{xH}$ with respect to Ω. If $Y \cap \Omega \subset xH$, choose $y \in Y \cap \Omega$.

If $Y \cap \Omega \not\subset xH$, then choose $y \in (Y \cap \Omega) - xH$. We can then check that $X - yH$ is not closed.

Therefore, by Theorem 7.1, there exists a non-constant analytic curve $\mathcal{C} : [0,1] \to U_2$ such that

$$Y\mathcal{C} \subset X.$$

By Theorem 6.1, there exists $y_2 \in Z_2$ such that for any sequence $t_i \to +\infty$,

$$\limsup_{i \to \infty} y_2 \mathcal{C} a_{-t_i} = Z_2. \tag{7.1}$$

By Lemma 5.1, we can choose $y_1 \in \mathrm{RF}_+ \mathcal{S}_1$ such that $(y_1, y_2) \in Y$. Choose $g_i \in H_i$ so that $y_i = [g_i]$.

We claim that we can choose $\mathbf{t} \in \mathbb{R}^{d-1}$ so that $(g_1 u_{\mathbf{t}})^- \in \Lambda_1$ and $(g_2 u_{\mathbf{t}} \mathcal{C}(0))^-$ does not belong to any sphere contained in $\mathscr{S}$. It is convenient to use the upper-half space model of $\mathbb{H}^d$ in which we have $\partial \mathbb{H}^d = \mathbb{R}^{d-1} \cup \{\infty\}$, and can take $v_o \in \mathrm{T}^1(\mathbb{H}^d)$ to be the upward normal vector at $o = (0, \cdots, 0, 1)$ so that $v_o^+ = \infty$ and $v_o^- = 0$. Then for all $\mathbf{t} \in \mathbb{R}^{d-1}$, we have $(u_{\mathbf{t}})^+ = \{\infty\}$ and $(u_{\mathbf{t}})^- = u_{\mathbf{t}}(0) = \mathbf{t}$. Suppose that the claim does not hold. Then for any $\mathbf{t} \in \mathbb{R}^{d-1}$ such that $\mathbf{t} \in g_1^{-1} \Lambda_1 \cap \mathbb{R}^{d-1}$, we have $\mathbf{t} + \mathcal{C}(0) \in g_2^{-1} S \cap \mathbb{R}^{d-1}$ for some sphere $S \in \mathscr{S}$. Therefore, $\Lambda_1 - g_1(\infty) \subset \bigcup_{S \in \mathscr{S}} g_1 \mathcal{C}(0)^{-1} g_2^{-1} S$. This is a contradiction, since Λ_1, being the limit set of a Zariski dense subgroup of G, cannot be contained in the union of countably many proper sub-spheres of $\mathbb{S}^{d-1}$ by [7, Coro. 1.4].

By replacing (y_1, y_2) with $(y_1 u_{\mathbf{t}}, y_2 u_{\mathbf{t}})$, we may now assume that $y_1 \in \mathrm{RF}\, \mathcal{S}_1$, as (7.1) holds for $y_2 u_{\mathbf{t}}$ as well by Theorem 6.1.

Since y_1 belongs to the compact A_1-invariant subset $\mathrm{RF}\, \mathcal{S}_1$, there exists $t_i \to +\infty$ such that $y_1 a_{-t_i}$ converges to some $z_1 \in \mathrm{RF}\, \mathcal{S}_1$. As $(y_1, y_2) \in Y$ and X is A-invariant, it follows

$$(y_1 a_{-t_i}, y_2 \mathcal{C} a_{-t_i}) \subset X.$$

By (7.1), we obtain $\{z_1\} \times Z_2 \subset X$. Since X is H-invariant, this implies $X = Z$. $\square$

A collection of elements $g_1, \cdots, g_k \in \mathrm{SO}^\circ(d,1)$, $k \geq 2$, is called a Schottky generating set if there exist mutually disjoint closed round balls $B_1, \cdots, B_k$ and $B_1', \cdots, B_k'$ in $\mathbb{S}^{d-1}$ such that g_i maps the exterior of B_i onto the interior of B_i' for each $i = 1, \cdots, k$. A subgroup of $\mathrm{SO}^\circ(d,1)$ is called a (classical) Schottky subgroup if it is generated by some Schottky generating set. It is easy to see that a Schottky subgroup is a convex cocompact subgroup.

The following lemma is well-known (e.g., [1, Proposition 4.3]). We give a short elementary proof.

Lemma 7.3. *Any Zariski dense discrete subgroup Γ of $\mathrm{SO}^\circ(d,1)$ contains a Zariski dense Schottky subgroup.*

Proof. Let Λ denote the limit set of Γ. For each hyperbolic element $\gamma \in \Gamma$, γ^+ and γ^- are respectively the attracting and repelling fixed points of γ.

As Γ is non-elementary, it follows from [6, Proposition 2.7] that the set $\{(\gamma^+, \gamma^-) : \gamma \text{ is a hyperbolic element of } \Gamma\}$ is a dense subset of $\Lambda \times \Lambda$.

Choose two hyperbolic elements $\gamma_1, \gamma_2 \in \Gamma$ such that $\{\gamma_1^\pm\}$ and $\{\gamma_2^\pm\}$ are disjoint from each other. Let S_1 be the smallest sub-sphere of $\mathbb{S}^{d-1}$ which contains $\{\gamma_i^\pm : i = 1, 2\}$. If $S_1 \neq \mathbb{S}^{d-1}$, then we choose a hyperbolic element $\gamma_3 \in \Gamma$ so that $\{\gamma_3^\pm\} \cap S_1 = \emptyset$. Let S_2 be the smallest sub-sphere of $\mathbb{S}^{d-1}$ which contains $\{\gamma_i^\pm : i = 1, 2, 3\}$. Then $\dim S_2 > \dim S_1$. Continuing in this fashion, we can find a sequence of hyperbolic elements $\gamma_1, \cdots, \gamma_m$ of Γ with $m \leq d - 1$ such that the sets $\{\gamma_i^\pm\}$ are all mutually disjoint and their union is not contained in any proper sub-sphere of $\mathbb{S}^{d-1}$. Now for a sufficiently large k, we can find pairwise disjoint round balls $B_i^\pm$ in $\mathbb{S}^{d-1}$ such that γ_i^k maps the exterior of B_i^- to the interior of B_i^+ for each i; this is possible as $\gamma_i^\pm$ are all distinct and for each i, $B_i^\pm$ can be chosen arbitrarily close to $\gamma_i^\pm$ as we make k large. Hence they form a Schottky generating set. Let Γ_0 be the subgroup generated by them. Since the limit set of Γ_0 contains all fixed points of γ_i^k, that is, $\{\gamma_i^\pm : i = 1, \cdots, m\}$, it is not contained in any proper sub-sphere of $\mathbb{S}^{d-1}$. Hence Γ_0 is Zariski dense. $\square$

Proof of Theorems 1.1 and 1.2. In order to use the notations introduced in sections 2-6, let $\Gamma_1 < H_1$ be a Zariski dense discrete subgroup and Γ_2 be a cocompact lattice in H_2. Since Γ_1 is countable and $\Gamma_2 \backslash H_2$ is compact, a closed Γ_1-orbit in $\Gamma_2 \backslash H_2$ is necessarily finite.

For any $(g_1, g_2) \in H_1 \times H_2$, observe that the following are all equivalent to each other:

(1) The orbit $[(g_1, g_2)]H$ is closed (resp. dense) in $(\Gamma_1 \times \Gamma_2) \backslash (H_1 \times H_2)$;
(2) The orbit $(\Gamma_1 \times \Gamma_2)[(g_1, g_2)]$ is closed (resp. dense) in $(H_1 \times H_2)/H$;
(3) The product $\Gamma_2 g_2 g_1^{-1} \Gamma_1$ is closed (resp. dense) in G;
(4) The orbit $[g_2 g_1^{-1}]\Gamma_1$ is finite (resp. dense) in $\Gamma_2 \backslash H_2$.

We first claim Theorem 1.2 when Γ_1 is convex cocompact. Suppose that X is a closed H-invariant subset of $Z = \Gamma_1 \backslash H_1 \times \Gamma_2 \backslash H_2$, and suppose that $X \neq Z$. If X consists of finitely many H-orbits, then each of them must be closed by Proposition 7.2. Now suppose that X contains infinitely many H-orbits, say $x_i H$. Each $x_i H$ should be closed again by Proposition 7.2. Consider the set $E := \bigcup x_i H$. Recalling that every H-orbit meets Ω, we may assume that $x_i \in \Omega$ and it converges to some $x \in \Omega - E$; if $x \in x_j H$, then we replace E by $\bigcup_{i>j} x_i H$. Since E is not closed, by Proposition 7.2, E is dense in Z. This proves the claim. In view of the above equivalence, Theorem 1.1 follows when Γ_1 is convex cocompact.

Since any Zariski dense discrete subgroup of H_1 contains a Zariski dense convex cocompact subgroup by Lemma 7.3, Theorem 1.1 follows. This implies Theorem 1.2 for a general Zariski dense discrete subgroup again in view of the above equivalence.

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