Geodesic planes in the convex core of an acylindrical 3-manifold

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Abstract

Let $M$ be a convex cocompact, acylindrical hyperbolic 3-manifold of infinite volume, and let $M^*$ denote the interior of the convex core of $M$. In this paper we show that any geodesic plane in $M^*$ is either closed or dense. We also show that only countably many planes are closed. These are the first rigidity theorems for planes in convex cocompact 3-manifolds of infinite volume that depend only on the topology of $M$.

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1 Introduction

In this paper we establish a new rigidity theorem for geodesic planes in acylindrical hyperbolic 3-manifolds.

Hyperbolic 3-manifolds. Let \( M = \Gamma \backslash \mathbb{H}^3 \) be a complete, oriented hyperbolic 3-manifold, presented as a quotient of hyperbolic space by the action of a discrete group

\[
\Gamma \subset G = \text{Isom}^+ (\mathbb{H}^3).
\]

Let \( \Lambda \subset S^2 = \partial \mathbb{H}^3 \) denote the limit set of \( \Gamma \), and let \( \Omega = S^2 - \Lambda \) denote the domain of discontinuity. The convex core of \( M \) is the smallest closed, convex subset of \( M \) containing all closed geodesics; equivalently,

\[
\text{core}(M) = \Gamma \backslash \text{hull}(\Lambda) \subset M
\]

is the quotient of the convex hull of the limit set \( \Lambda \) of \( \Gamma \). Let \( M^* \) denote the interior of the convex core of \( M \).

Geodesic planes in \( M^* \). Let

\[
f : \mathbb{H}^2 \to M
\]

be a geodesic plane, i.e. a totally geodesic immersion of the hyperbolic plane into \( M \). We often identify a geodesic plane with its image, \( P = f(\mathbb{H}^2) \).

By a geodesic plane \( P^* \subset M^* \), we mean the nontrivial intersection

\[
P^* = P \cap M^* \neq \emptyset
\]

of a geodesic plane in \( M \) with the interior of the convex core. A plane \( P^* \) in \( M^* \) is always connected, and \( P^* \) is closed in \( M^* \) if and only if \( P^* \) is properly immersed in \( M^* \) (§2).

Acylindrical manifolds and rigidity. In this work, we study geodesic planes in \( M^* \) under the assumption that \( M \) is a convex cocompact, acylindrical hyperbolic 3-manifold. The acylindrical condition is a topological one; it means that the compact Kleinian manifold

\[
\overline{M} = \Gamma \backslash (\mathbb{H}^3 \cup \Omega)
\]

has incompressible boundary, and every essential cylinder in \( \overline{M} \) is boundary parallel (§2). We will be primarily interested in the case where \( M \) is a convex cocompact manifold of infinite volume. Under this assumption, \( M \) is acylindrical if and only if \( \Lambda \) is a Sierpiński curve.\(^1\)

Our main goal is to establish:

\(^1\)A compact set \( \Lambda \subset S^2 \) is a **Sierpiński curve** if \( S^2 - \Lambda = \bigcup D_i \) is a dense union of Jordan disks with disjoint closures, and \( \text{diam}(D_i) \to 0 \). Any two Sierpiński curves are homeomorphic [Wy].
**Theorem 1.1** Let \( M \) be a convex cocompact, acylindrical, hyperbolic 3-manifold. Then any geodesic plane \( P^* \) in \( M^* \) is either closed or dense.

As a complement, we will show:

**Theorem 1.2** There are only countably many closed geodesic planes \( P^* \subset M^* \).

We also establish the following topological equidistribution result:

**Theorem 1.3** If \( P^*_i \subset M^* \) is an infinite sequence of distinct closed geodesic planes, then

\[
\lim_{i \to \infty} P^*_i = M^*
\]

in the Hausdorff topology on closed subsets of \( M^* \).

**Remarks.**

1. We do not know of any instance of Theorem 1.1 where \( P^* \) is closed in \( M^* \) but \( P \) is not closed in \( M \).

   *Added in proof.* An example of such an exotic plane in an acylindrical manifold has recently been constructed by Zhang. In his example, the closure of \( P \) is not even locally connected near \( \partial M^* \) [Zh].

   Thus the rigidity of planes described in Theorem 1.1 does not extend beyond the convex core of \( M \).

2. In the special case where \( M \) is compact (so \( M = M^* \)), Theorem 1.1 is due independently to Shah and Ratner (see [Sh], [Rn]).

3. For a general convex cocompact manifold \( M \), there can be uncountably many distinct closed planes in \( M^* \); see the end of §2.

4. Examples of acylindrical manifolds such that \( M^* \) contains infinitely many closed geodesic planes are given in [MMO, Cor.11.5]

5. The study of planes \( P \) that do not meet \( M^* \) can be reduced to the case where \( M \) is a quasifuchsian manifold. This case can be analyzed via the bending lamination (cf. §6).

**Comparison to the case of geodesic boundary.** A convex cocompact hyperbolic 3-manifold \( M \) such that \( \partial \text{core}(M) \) is totally geodesic is automatically acylindrical. For these rigid acylindrical manifolds, the results above
were obtained in our previous work [MMO]. While one would ultimately like to analyze planes in a large class of geometrically finite groups, our previous results covered only countably many examples (by Mostow rigidity).

The present paper makes a major step forward in this program, by developing a new argument for unipotent recurrence which works without geodesic boundary, which is robust enough to be invariant under quasi-isometry, and which is powerful enough to apply to the class of all convex cocompact acylindrical manifolds. The key insight is that one should work with a proper subset of the renormalized frame bundle, defined in terms of thickness of Cantor sets, where we show sufficient recurrence takes place in the acylindrical case.

Figure 1. Limit set of a cylindrical 3-manifold.

The cylindrical case. The acylindrical setting is also close to optimal, since Theorem 1.1 is generally false for cylindrical manifolds.

For example, consider a quasifuchsian group \( \Gamma \) containing a Fuchsian subgroup \( \Gamma' \) of the second kind with limit set \( \Lambda' \subset S^1 \). Given \((a, b) \in \Lambda' \times \Lambda'\), let \( C_{ab} \) denote the unique circle orthogonal to \( S^1 \) such that \( C_{ab} \cap S^1 = \{a, b\} \). It is possible to choose \( \Gamma \) such that \( C_{ab} \cap \Lambda = \{a, b\} \) for uncountably many \((a, b)\); and further, to arrange that the corresponding hyperbolic planes \( P \subset M \) and \( P^* \subset M^* \) have wild closures, violating Theorem 1.1 (cf. [MMO, App. A]).

The same type of example can be embedded in more complicated 3-manifolds with nontrivial characteristic submanifold; an example is shown
in Figure 1.

\[
G = \text{PSL}_2(\mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)
\]

\[
H = \text{PSL}_2(\mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2)
\]

\[
K = \text{SU}(2) / (\pm I)
\]

\[
A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}
\]

\[
N = \left\{ n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{C} \right\}
\]

\[
U = \left\{ n_s : s \in \mathbb{R} \right\}
\]

\[
V = \left\{ n_s : s \in i\mathbb{R} \right\}
\]

\[
F\mathbb{H}^3 = G = \{\text{the frame bundle of } \mathbb{H}^3\}
\]

\[
\mathbb{H}^3 = G/K
\]

\[
S^2 = G/AN = \partial \mathbb{H}^3
\]

\[
\mathcal{C} = G/H = \{\text{the space of oriented circles } C \subset S^2\}
\]

Table 2. Notation for $G$ and some of its subgroups and homogeneous spaces.

**Homogeneous dynamics.** Next we formulate a result in the language of Lie groups and homogeneous spaces, Theorem 1.4, that strengthens both Theorems 1.1 and 1.3.

To set the stage, we have summarized our notation for $G$ and its subgroups in Table 2. We have similarly summarized the spaces attached to an arbitrary hyperbolic 3–manifold $M = \Gamma \backslash \mathbb{H}^3$ in Table 3. (In the definition of $\mathcal{C}^*$, a circle $C \subset S^2$ separates $\Lambda$ if the limit set meets both components of $S^2 - C$.)

**Circles, frames and planes.** Circles, frame and planes are closely related. In fact, if $\mathcal{P}$ denotes the set of all (oriented) planes in $M$, then we have the natural identifications:

\[
\mathcal{P} = \Gamma \backslash \mathcal{C} = FM/H.
\]

Indeed, all three spaces can be identified with $\Gamma \backslash G/H$. We will frequently use these identifications to go back and forth between circles, frames and planes.

When $M^*$ is nonempty (equivalently, when $\Gamma$ is Zariski dense in $G$), the spaces $\mathcal{C}^*$ and $F^*$ correspond to the set of planes $\mathcal{P}^*$ that meet $M^*$. In other words, we have

\[
\mathcal{P}^* = \Gamma \backslash \mathcal{C}^* = F^*/H.
\]
To go from a circle to a plane, let $P$ be the image of $\text{hull}(C) \subset \mathbb{H}^3$ under the covering map from $\mathbb{H}^3$ to $M$. To go from a frame $x \in FM$ to a plane, take the image of $xH$ under the natural projection $FM \to M$.

When $\Lambda$ is connected and consists of more than one point (e.g. when $M$ is acylindrical), it is easy to see that:

$$\overline{C^*} = \{ C \in C : C \text{ meets } \Lambda \}.$$  

Thus the closures of the dense sets arising in Theorem 1.4 below are quite explicit.

$$M = \Gamma \backslash \mathbb{H}^3 = (\text{the quotient hyperbolic 3-manifold})$$

$$\mathcal{M} = \Gamma \backslash (\mathbb{H}^3 \cup \Omega)$$

$$\text{core}(M) = \Gamma \backslash \text{hull}(\Lambda)$$

$$M^* = \text{int}(\text{core}(M))$$

$$FM = \Gamma \backslash G = (\text{the frame bundle of } M)$$

$$F^* = \{ x \in FM : x \text{ is tangent to a plane } P \text{ that meets } M^* \}$$

$$C^* = \{ C \in C : C \text{ separates } \Lambda \}$$

**Table 3. Spaces associated to $M = \Gamma \backslash \mathbb{H}^3$.**

The closed or dense dichotomy. We can now state our main result from the perspective of homogeneous dynamics.

**Theorem 1.4** Let $M = \Gamma \backslash \mathbb{H}^3$ be a convex cocompact, acylindrical 3-manifold. Then any $\Gamma$-invariant subset of $C^*$ is either closed or dense in $C^*$. Equivalently, any $H$-invariant subset of $F^*$ is either closed or dense in $F^*$.

(The equivalence is immediate from equation (1.2).)

This result sharpens Theorem 1.1 to give the following dichotomy on the level of the tangent bundles:

**Corollary 1.5** The normal bundle to a geodesic plane $P^* \subset M^*$ is either closed or dense in the tangent bundle $TM^*$.

Beyond the acylindrical case. This paper also establishes several results that apply outside the acylindrical setting. For example, Theorems 2.1, 4.1, 5.1 and 6.1 only require the assumption that $M$ has incompressible
boundary. In fact, the main argument pivots on a result relating Cantor sets and Sierpiński curves, Theorem 3.4, that involves no groups at all.

**Discussion of the proofs.** We conclude with a sketch of the proofs of Theorems 1.1 through Theorem 1.4.

Let $M = \Gamma \backslash \mathbb{H}^3$ be a convex cocompact acylindrical 3–manifold of infinite volume, with limit set $\Lambda$ and domain of discontinuity $\Omega$. The horocycle and geodesic flows on the frame bundle $FM = \Gamma \backslash G$ are given by the right actions of $U$ and $A$ respectively. The renormalized frame bundle of $M$ is the compact set defined by

$$RFM = \{ x \in FM : xA \text{ is bounded} \}.$$  \hspace{1cm} (1.3)

In §2 we prove Theorem 1.2 by showing that the fundamental group of any closed plane $P^* \subset M^*$ contains a free group on two generators. We also show that Theorems 1.1 and 1.3 follow from Theorem 1.4. The remaining sections develop the proof of Theorem 1.4.

In §3 we show that $\Lambda$ is a Sierpiński curve of positive modulus. This means there exists a $\delta > 0$ such that the modulus of the annulus between any two components $D_1, D_2$ of $S^2 - \Lambda$ satisfies

$$\text{mod}(S^2 - (D_1 \cup D_2)) \geq \delta > 0.$$  

We also show that if $\Lambda$ is a Sierpiński curve of positive modulus, then there exists a $\delta > 0$ such that $C \cap \Lambda$ contains a Cantor set $K$ of modulus $\delta$, whenever $C$ separates $\Lambda$. This means that for any disjoint components $I_1$ and $I_2$ of $C - K$, we have

$$\text{mod}(S^2 - (I_1 \cup I_2)) \geq \delta > 0.$$  

This result does not involve Kleinian groups and may be of interest in its own right.

In §4 we use this uniform bound on the modulus of a Cantor set to construct a compact, $A$–invariant set

$$RF_k M \subset RFM$$  

with good recurrence properties for the horocycle flow on $FM$. We also show that when $k$ is sufficiently large, $RF_k M$ meets every $H$–orbit in $F^*$.

The introduction of $RF_k M$ is one of the central innovations of this paper that allows us to handle acylindrical manifolds with quasifuchsian boundary. When $M$ is a rigid acylindrical manifold, $RF_k M = RFM$ for all $k$ sufficiently large, so in some sense $RF_k M$ is a substitute for the renormalized frame bundle. For a more detailed discussion, see the end of §4.
In §5 we shift our focus to the boundary of the convex core. Using the theory of the bending lamination, we give a precise description of $C \cap \Lambda$ in the case where $C$ comes from a supporting hyperplane for the limit set.

In §§6 and 7, we formulate two density theorems for hyperbolic 3-manifolds $M$ with incompressible boundary. These results do not require that $M$ is acylindrical. Each section gives a criterion for a sequence of circles $C_n \in \mathcal{C}$ to have the property that $\bigcup \Gamma C_n$ is dense in $\mathcal{C}$.

In §6 we show that density holds if $C_n \to C \not\in \mathcal{C}$ and $\lim (C_n \cap \Lambda)$ is uncountable. The proof relies on the analysis of the convex hull given in §5.

In §7 we show that density holds if $C_n \to C \in \mathcal{C}$ and $C \not\in \bigcup \Gamma C_n$, provided $C \cap \Lambda$ contains a Cantor set of positive modulus. The proof uses recurrence, minimal sets and homogeneous dynamics on the frame bundle, and follows a similar argument in [MMO]. It also relies on the density result of §6.

When $M$ is acylindrical, the Cantor set condition is automatic by §3. Thus Theorem 1.4 follows immediately from the density theorem of §7.

**Question.** We conclude by mentioning an open problem that goes beyond the acylindrical case. Let $P^* \subset M^*$ be a plane in a quasifuchsian manifold, and suppose the corresponding circle satisfies $|C \cap \Lambda| > 2$. Does it follow that $P^*$ is closed or dense in $M^*$?

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## 2 Planes in acylindrical manifolds

In this section we will prove Theorem 1.2, and show that our other main results, Theorems 1.1 and 1.3, follow from Theorem 1.4 on the homogeneous dynamics of $H$ acting on $F^*$.

Let $M = \Gamma \backslash \mathbb{H}^3$ be a convex cocompact hyperbolic 3-manifold. We first describe how the topology of $M$ influences the shape of planes in $M^*$. Here are the two main results.

**Theorem 2.1** If $M$ has incompressible boundary, then the fundamental group of any closed plane $P^* \subset M^*$ is nontrivial.

**Theorem 2.2** If $M$ is acylindrical, then the fundamental group of any closed plane $P^* \subset M^*$ contains a free group on two generators.

The second result immediately implies Theorem 1.2, which we restate as follows:
Corollary 2.3 If $\overline{M}$ is acylindrical, then there are at most countably many closed planes $P^* \subset M^*$.

Proof. In this case $P^*$ corresponds to a circle $C$ whose stabilizer $\Gamma^C$ (as discussed below) is isomorphic to the fundamental group of $P^*$, and contains a free group on two generators $\langle a, b \rangle$. Since $C$ is the unique circle containing the limit set of $\langle a, b \rangle \subset \Gamma$, and there are only countably many possibilities for $(a, b)$, there are only countable possibilities for $P^*$.

In the remainder of this section, we first develop general results about planes in 3-manifolds, and prove Theorems 2.1 and 2.2. Then we derive Theorems 1.1 and 1.3 from Theorem 1.4. Finally we show by example that a cylindrical manifold can have uncountably many closed planes $P^* \subset M^*$.

Topology of 3-manifolds. We begin with some topological definitions.

Let $D^2$ denote a closed 2-disk, and let $C^2 \cong S^1 \times [0, 1]$ denote a closed cylinder. Let $N$ be a compact 3-manifold with boundary. We say $N$ has incompressible boundary if every continuous map $f : (D^2, \partial D^2) \to (N, \partial N)$ can be deformed, as a map of pairs, so its image lies in $\partial N$. (This property is automatic if $\partial N = \emptyset$.)

Similarly, $N$ is acylindrical if it has incompressible boundary and every continuous map $f : (C^2, \partial C^2) \to (N, \partial N)$, injective on $\pi_1$, can be deformed into $\partial N$. That is, every incompressible disk or cylinder in $N$ is boundary parallel.

When $N = \overline{M} = \Gamma \backslash (\mathbb{H}^3 \cup \Omega)$ is a compact Kleinian manifold, these properties are visible on the sphere at infinity: the limit set $\Lambda$ of $\Gamma$ is connected iff $\overline{M}$ has incompressible boundary, and $\overline{M}$ is acylindrical iff $\Lambda$ is a Sierpiński curve or $\Lambda = S^2$.

For more on the topology of hyperbolic 3-manifolds, see e.g. [Th2], [Mor], and [Md].

Topology of planes. Next we discuss the fundamental group of a plane $P \subset M$, and the corresponding plane $P^* \subset M^*$. These definitions apply to an arbitrary hyperbolic 3-manifold.

For precision it is useful to think of a plane $P$ as being specified by an oriented circle $C \subset S^2$, whose convex hull covers $P$. More precisely, the plane attached to $C$ is given by the map

$$\tilde{f} : \text{hull}(C) \cong \mathbb{H}^2 \subset \mathbb{H}^3 \to M = \Gamma \backslash \mathbb{H}^3$$
with image $\tilde{f}(\mathbb{H}^2) = P$. The stabilizer of the circle $C$ in $G$ is a conjugate $xHx^{-1}$ of $H = \text{PSL}_2(\mathbb{R})$; hence its stabilizer in $\Gamma$ is given by

$$\Gamma^C = \Gamma \cap xHx^{-1}.$$ 

Let

$$S = \Gamma^C \setminus \text{hull}(C).$$

Then the map $\tilde{f}$ descends to give an immersion

$$f : S \to M$$

with image $P$. The immersion $f$ is generically injective if $P$ is orientable; otherwise, it is generically two–to–one (and there is an element in $\Gamma$ that reverses the orientation of $C$).

We refer to

$$\pi_1(S) \cong \Gamma^C$$

as the fundamental group of $P$ (keeping in mind caveats about orientability).

**Planes in the convex core.** Now suppose $P^* = P \cap M^*$ is nonempty. In this case

$$S^* = f^{-1}(M^*)$$

is a nonempty convex subsurface of $S$, with $\pi_1(S^*) = \pi_1(S)$. The map

$$f : S^* \to P^* \subset M^*$$

presents $S^*$ as the (orientable) normalization of $P^*$, i.e. as the smooth surface obtained by resolving the self-intersections of $P^*$. Similarly, the frame bundle of $P$ with its branches separated is given by

$$FP = xH \subset FM$$

for some $x \in F^*$. (One should consistently orient $C$ and $P$ to define $FP$.)

To elucidate the connections between these objects, we formulate:

**Proposition 2.4** Let $M$ be an arbitrary hyperbolic 3–manifold. Suppose $C \in \mathcal{C}^*$ and $x \in F^*$ correspond to the same plane $P^* \subset M^*$. Then the following are equivalent:

1. $\Gamma C$ is closed in $\mathcal{C}^*$.
2. The inclusion $\Gamma C \subset \mathcal{C}^*$ is proper.
3. $xH$ is closed in $F^*$.

4. $P^*$ is closed in $M^*$.

5. The normalization map $f : S^* \to P^*$ is proper.

In (2) above, $\Gamma C$ is given the discrete topology.

**Proof.** If $\Gamma C$ is not discrete in $C^*$, then by homogeneity it is perfect (it has no isolated points). But a closed perfect set is uncountable, so $\Gamma C$ is not closed. Thus (1) implies that $\Gamma C \subset C^*$ is closed and discrete, which implies (2); and clearly (2) implies (1). The remaining equivalences are similar, using equation (1.2) to relate $P^*$, $C^*$ and $F^*$.

**Compact deformations.** In the context of proper mappings, the notion of a compact deformation is also useful.

Let $f_0 : X \to Y$ be a continuous map. We say $f_1 : X \to Y$ is a compact deformation of $f_0$ if there is a continuous family of maps $f_t : X \to Y$ interpolating between them, defined for all $t \in [0,1]$, and a compact set $X_0 \subset X$ such that $f_t(x) = f_0(x)$ for all $x \not\in X_0$.

Let $P^* \subset M^*$ be a hyperbolic plane with normalization $f_0 : S^* \to M^*$. We say $Q^* \subset M^*$ is a compact deformation of $P^*$ if it is the image of $S^*$ under a compact deformation $f_1$ of $f_0$.

**Theorem 2.5** Let $M = \Gamma \backslash \mathbb{H}^3$ be an arbitrary 3-manifold, and let $K \subset M^*$ be a submanifold such that the induced map

$$\pi_1(K) \to \pi_1(M)$$

is surjective. Then $K$ meets every geodesic plane $P^* \subset M^*$ and every compact deformation $Q^*$ of $P^*$.

**Corollary 2.6** If $\pi_1(M)$ is finitely generated, then there is a compact submanifold $K \subset M^*$ that meets every plane $P^* \subset M^*$.

**Proof.** Provided $M^*$ is nonempty, $\pi_1(M^*)$ is isomorphic to $\pi_1(M)$; and since the latter group is finitely generated, there is a compact submanifold $K \subset M^*$ (say a neighborhood of a bouquet of circles) whose fundamental group surjects onto $\pi_1(M^*)$. ■
Proof of Theorem 2.5. We will use the fact that $S^0$ and $S^1$ can link in $S^2$.

Let $P^*$ be a plane in $M^*$, arising from a circle $C \subset S^2$ with an associated map $f : S \to P$ as above. Since $P$ meets $M^*$, there are points in the limit set of $\Gamma$ on both sides of $C$. Since the endpoints of closed geodesics are dense in $\Lambda \times \Lambda$ (cf. [Eb]), we can find a hyperbolic element $g \in \Gamma$ such that its two fixed points

$$\text{Fix}(g) = \{a_1, a_2\} \subset S^2$$

are separated by $C$, and the convex hull of $\{a_1, a_2\}$ in $\mathbb{H}^3$ projects to a closed geodesic $\delta \subset M$. Note that $\text{Fix}(g) \cong S^0$ and $C \cong S^1$ are linked in $S^2$.

Since $\pi_1(K)$ maps onto $\pi_1(M)$, the loop $\delta$ is freely homotopic to a loop $\gamma \subset K$.

Let $f_0 = f|S^*$. Suppose $f_0 : S^* \to M^*$ has a compact deformation $f_1$ with image $Q^*$ disjoint from $K$, and hence disjoint from $\gamma$. Extend this deformation trivially to the rest of $S$, to obtain a compact deformation $f_1$ of the geodesic immersion $f : S \to P$. Then $f_1(S)$ is disjoint from $\gamma$. Lifting $f_1$ to the universal cover of $S$, we obtain a continuous map

$$\tilde{f}_1 : \text{hull}(C) \to \mathbb{H}^3$$

that is a bounded distance from the identity map. In particular, its image is a disk $D$ spanning $C$.

Similarly, a suitable lift of $\gamma$ gives a path $\tilde{\gamma} \subset \mathbb{H}^3$, disjoint from $D$, that joins $a_1$ to $a_2$. This contradicts the fact that $C$ separates $a_1$ from $a_2$ in $S^2$. $\blacksquare$

We can now proceed to the:

Proof of Theorem 2.1 (The incompressible case). For the beginning of the argument, we only use the fact that $\overline{M}$ is compact and $M^*$ is nonempty. Using the nearest point projection, it is straightforward to show that $\text{core}(M)$ is homeomorphic to $\overline{M}$. Thus its interior $M^*$ deformation retracts onto a compact submanifold $K \subset M^*$, homeomorphic to $\overline{M}$, such that the inclusion is a homotopy equivalence; in particular, $\pi_1(K) \cong \pi_1(M^*)$.

Consider a closed plane $P^* \subset M^*$, arising as the image of a proper map $f : S^* \to P^*$ as above. We can also arrange that $K$ is transverse to $f$, so its preimage

$$S_0 = f^{-1}(K) \subset S^*$$

is a compact, smoothly bounded region in $S^*$. (However $S_0$ need not be connected.)
We claim that, after changing \( f \) by a compact deformation, we can arrange that the inclusion of each component of \( S_0 \) into \( S^* \) is injective on \( \pi_1 \). This is a standard argument in 3-dimensional topology. If the inclusion is not injective on \( \pi_1 \), then there is a compact disk \( D \subset S^* \) with \( D \cap S_0 = \partial D \). The map \( f \) sends \((D, \partial D)\) into \((M^*, K)\). Since \( K \) is a deformation retract of \( M^* \), \( f|D \) can be deformed until it maps \( D \) into \( K \), while keeping \( f|\partial D \) fixed. Then \( D \) becomes part of \( S_0 \). This deformation is compact because \( D \) is compact. Since \( \partial S_0 \) has only finitely many components, only finitely many disks of this type arise, so after finitely many compact deformations of \( f \), the inclusion \( S_0 \subset S^* \) becomes injective on \( \pi_1 \).

Now we use the assumption that \( K \cong \mathcal{M} \) has incompressible boundary. Suppose that \( \pi_1(S^*) \) is trivial. Then \( \pi_1 \) is trivial for each component of \( S_0 \), and hence each component of \( S_0 \) is a disk. By construction the deformed map \( f \) restricts to give a map of pairs

\[
f : (S_0, \partial S_0) \to (K, \partial K).
\]

Since \( K \) has incompressible boundary, we can further deform \( f|S_0 \) so it sends the whole surface \( S_0 \) into \( \partial K \). Then the image \( Q^* \) of \( f \) gives a compact deformation of \( P^* \) that is disjoint from \( K^* = K - \partial K \). But \( \pi_1(K^*) \) maps onto \( \pi_1(M) \), contradicting Theorem 2.5. Thus \( \pi_1(S^*) \) is nontrivial.

**Proof of Theorem 2.2 (The acylindrical case).** The proof follows the same lines as the incompressible case. If \( \pi_1(S^*) \) does not contain a free group on two generators, then \( S^* \) is a disk or an annulus. After a compact deformation, we can assume that the inclusion \( S_0 = f^{-1}(K) \subset S^* \) is injective on \( \pi_1 \). Thus each component of \( S_0 \) is also a disk or an annulus. Since \( K \) is acylindrical, after a further compact deformation of \( f \) we can arrange that \( f(S_0) \subset \partial K \), leading to a contradiction.

**Rigidity of planes from homogeneous dynamics.** Now suppose \( M = \Gamma \backslash \mathbb{H}^3 \) is a convex cocompact, acylindrical 3-manifold. Assume we know Theorem 1.4, which states that under this hypothesis:

*Any \( \Gamma \)-invariant set \( E \subset \mathbb{C}^* \) is closed or dense in \( \mathbb{C}^* \).*

We can then prove the other two main results stated in the introduction.

**Proof of Theorem 1.1.** Let \( P^* \) be a geodesic plane in \( M^* \), and let \( E = \Gamma C \) be the corresponding set of circles. Then by Theorem 1.4, \( E \) is either closed or dense in \( \mathbb{C}^* \), and hence \( P^* \) is either closed or dense in \( M^* \).
Proof of Theorem 1.3. Let $P^*_i$ be a sequence of distinct closed planes in $M^*$. We wish to show that $\lim P^*_i = M^*$ in the Hausdorff topology on closed subsets of $M^*$. To see this, first pass to a subsequence so that $P^*_i$ converges to $Q^* \subset M^*$. It suffices to show that $Q^* = M^*$ for every such subsequence. Since each $P^*_i$ is nowhere dense, to show that $Q^* = M^*$ and complete the proof, it suffices to show that $\bigcup P^*_i$ is dense in $M^*$.

Let $E_i \subset C^*$ be the $\Gamma$–orbit corresponding to $P_i$, and let $E = \bigcup E_i$. Since the planes $P_i$ are distinct, the sets $E_i$ are disjoint. By Corollary 2.6, there exists a compact set $K \subset M^*$ that meets every $P^*_i$, so there exists a compact set $K' \subset C^*$ meeting every $E_i$. Thus we can choose $C_i \in E_i \cap K'$ and pass to a subsequence such that

$$C_i \to C_\infty \in K' \subset C^*$$

and $C_\infty \notin E$. (If $C_\infty \in E_i = \Gamma C_i$, just drop that term from the sequence.) Since $E$ is not closed in $C^*$, it is dense in $C^*$ by Theorem 1.4. Consequently $\bigcup P^*_i$ is dense in $M^*$, as desired.

Example: uncountably many geodesic cylinders. To conclude, we show that Theorem 2.2 and Corollary 2.3 do not hold for general convex cocompact manifolds with incompressible boundary.

In fact, in such a manifold one can have uncountably many distinct closed planes $P^* \subset M^*$, each with cyclic fundamental group. For a concrete example of this phenomenon, consider a closed geodesic $\gamma$ and the corresponding plane $P$ in the quasifuchsian manifold $M = M_\theta$ discussed in [MMO, Cor. A.2]. In this construction, $\gamma$ is a simple curve in the boundary of the convex core of $M$, and $P \cong \gamma \times \mathbb{R}$ is a hyperbolic cylinder properly embedded in $M$. Consequently $P^* \subset M^*$ is a properly immersed cylinder in $M^*$. By varying the angle that $P$ meets the boundary of $\text{core}(M_\theta)$ along $\gamma$, we obtain a continuous family of properly immersed planes in $M^*$.

3 Moduli of Cantor sets and Sierpiński curves

The rest of the paper is devoted to the proof of Theorem 1.4.

In this section we define the modulus of a Cantor set $K \subset S^1$ (or in any circle $C \subset S^2$), as well as the modulus of a Sierpiński curve $K \subset S^2$. We then prove:

**Theorem 3.1** Let $\Lambda$ be the limit set of $\Gamma$, where $M = \Gamma \backslash \mathbb{H}^3$ is a convex cocompact acylindrical 3–manifold of infinite volume. Then there exists a $\delta > 0$ such that:
1. $\Lambda$ is a Sierpiński curve of modulus $\delta$, and

2. $C \cap \Lambda$ contains a Cantor set of modulus $\delta$, whenever the circle $C \subset S^2$ separates $\Lambda$.

**The modulus of a Sierpiński curve.** For background on conformal invariants and quasiconformal maps, see [LV].

We begin with some definitions. An **annulus** $A \subset S^2$ is an open region whose complement consists of two components. Provided neither component is a single point, $A$ is conformally equivalent to a unique round annulus of the form

$$ A_R = \{ z \in \mathbb{C} : 1 < |z| < R \}, $$

and its **modulus** is defined by

$$ \text{mod}(A) = \frac{\log R}{2\pi}. $$

(More geometrically, $A$ is conformally equivalent to a Euclidean cylinder of radius 1 and height $\text{mod}(A)$.) Since the modulus is a conformal invariant, we have

$$ \text{mod}(A) = \text{mod}(g(A)) \quad \forall g \in G. \quad (3.1) $$

Recall that a compact set $\Lambda \subset S^2$ is a **Sierpiński curve** if its complement

$$ S^2 - \Lambda = \bigcup D_i $$

is a dense union of Jordan disks $D_i$ with disjoint closures, whose diameters tend to zero. We say $\Lambda$ has **modulus** $\delta$ if

$$ \inf_{i \neq j} \text{mod}(S^2 - (\overline{D_i} \cup \overline{D_j})) \geq \delta > 0. $$

**The modulus of an annulus** $A \subset S^1$. Let $C \subset S^2$ be a circle and let $A \subset C$ be an ‘annulus on $C$’, meaning an open set such that $C - A = I_1 \cup I_2$ is the union of two disjoint intervals (circular arcs). We extend the notion of modulus to this 1–dimensional situation by defining

$$ \text{mod}(A, C) = \text{mod}(S^2 - (I_1 \cup I_2)). $$

Clearly $\text{mod}(gA, gC) = \text{mod}(A, C)$ for all $g \in G$, and consequently $\text{mod}(A, C)$ depends only on the cross-ratio of the 4 endpoints of $A$. The cross ratio is controlled by the lengths of the components $A_1, A_2$ of $A$ and the components $I_1, I_2$ of $C - A$. From this observation and monotonicity of the modulus [LV, I.6.6] it is easy to show:
Proposition 3.2 There are increasing continuous functions \( \delta(t), \Delta(t) > 0 \) such that
\[
\delta(t) < \text{mod}(A, C) < \Delta(t),
\]
where \( t \) is the ratio of lengths
\[
t = \frac{\min(|A_1|, |A_2|)}{\min(|I_1|, |I_2|)}.
\]
The same result holds with \( t \) replaced by \( d(\text{hull}(I_1), \text{hull}(I_2)) \).

For later reference we recall the following result due to Teichmüller [LV, Ch II, Thm 1.1]:

Proposition 3.3 Let \( I_1 \) and \( I_2 \) be the two components of \( C - A \). Then
\[
\text{mod}(B) \leq \text{mod}(A, C)
\]
for any annulus \( B \subset S^2 \) separating the endpoints of \( I_1 \) from those of \( I_2 \).

The modulus of a Cantor set. Let \( K \subset C \subset S^2 \) be a compact subset of a circle, such that its complement
\[
C - K = \bigcup I_i
\]
is a union of open intervals with disjoint closures. Note that \( C \) is uniquely determined by \( K \) (and we allow \( K = C \)). We say \( K \) has modulus \( \delta \) if we have
\[
\inf_{i \neq j} \text{mod}(A_{ij}, C) \geq \delta > 0,
\]
where \( A_{ij} = C - T_i \cup T_j \). We will be primarily interested in the case where \( K \) is a Cantor set, meaning \( \bigcup I_i \) is dense in \( C \).

Slices. Next we show that circular slices of a Sierpiński curve inherit positivity of the modulus. This argument makes no reference to 3–manifolds.

Theorem 3.4 Let \( \Lambda \subset S^2 \) be a Sierpiński curve of modulus \( \delta > 0 \). Then there exists a \( \delta' > 0 \) such that \( C \cap \Lambda \) contains a Cantor set \( K \) of modulus \( \delta' \) whenever \( C \) is a circle separating \( \Lambda \).

Proof. Let \( S^2 - \Lambda = \bigcup D_i \) express the complement of \( \Lambda \) as a union of disjoint disks. Each disk \( D_i \) meets the circle \( C \) in a collection of disjoint open intervals (see Figure 4). The proof will be based on a study of the interaction of intervals from different components.
Let $U = C - \Lambda = \bigcup U_i$, where

$$U_i = C \cap D_i.$$  

Note that distinct $U_i$ have disjoint closures, and $\text{diam } U_i \to 0$, since these two properties hold for the disks $D_i$. The open set $U_i$ may be empty.

We may assume $U$ is dense in $C$, since otherwise we can just choose a suitable Cantor set $K \subset C - U$. On the other hand, no $U_i$ is dense in $C$; if it were, we would have $C \subset D_i$, contrary to our assumption that $C$ separates $\Lambda$. It follows that $U_i$ is nonempty for infinitely many values of $i$.

Let us say an open interval $I = (a,b) \subset C$, with distinct endpoints, is a bridge of type $i$ if $a,b \in \partial U_i$. Note that an ascending union of bridges of type $i$ is again a bridge of type $i$, provided its endpoints are distinct.

Our goal is to construct a sequence of disjoint bridges $I_1, I_2, I_3, \ldots$ such that $|I_1| \geq |I_2| \geq \cdots$ and $K = C - \bigcup I_i$ is a Cantor set of modulus $\delta'$. To start the construction, choose any bridge $I_1 \subset C$. After changing coordinates by a Möbius transformation $g \in G_C$, we can assume that $I_1$ fills at least half the circle; i.e. $|I_1| > |C|/2$. This will ensure that $|I_1| \geq |I_k|$ for all $k > 1$.

Next, let $I_2$ be a bridge of maximal length among all those which are disjoint from $I_1$ and of a different type from $I_1$. Such a bridge exists because $\text{diam}(U_i) \to 0$, so only finitely many types of bridges are competing to be $I_2$. To complete the initial step, enlarge $I_1$ to a maximal interval of the same type, disjoint from $I_2$.

Proceeding inductively, let $I_{k+1} \subset C$ be a bridge of maximum length among all bridges disjoint from $I_1, \ldots, I_k$. Since $I_1$ is a maximal bridge of
its type among those disjoint from \( I_2 \), and vice–versa, the intervals \((I_1, I_2, I_k)\) are of 3 distinct types, for all \( k \geq 3 \). Consequently \(|I_2| \geq |I_k|\) for all \( k > 2 \).

Note that the bridges so constructed have disjoint closures. Indeed, if \( I_i \) and \( I_j \) were to have an endpoint \( a \) in common, with \( i < j \), then \( I_i \cup \{a\} \cup I_j \) would be a longer interval of the same type as \( I_i \), contradicting to stage \( i \) of the construction.

Since \( U \) is dense in \( C \), it follows that at any finite stage there is a bridge disjoint from all those chosen so far, and thus the inductive construction continues indefinitely. By construction, we have

\[ |I_1| \geq |I_2| \geq |I_3| \cdots \]

and by disjointness, \(|I_k| \to 0\). Moreover, \( \bigcup I_k \) is dense in \( C \). Otherwise, by density of \( U \), we would be able to find a bridge \( J \) disjoint from all \( I_k \), and longer than \( I_k \) for all \( k \) sufficiently large, contradicting the construction of \( I_k \).

Let \( K = C - \bigcup_1^\infty I_k \). Since the intervals \( I_k \) have disjoint closures, and their union is dense in \( C \), \( K \) is a Cantor set. We have \( K \subset \Lambda \) since \( \partial I_k \subset \Lambda \) for all \( k \).

Now consider any two indices \( i < j \). Let

\[ A = C - (I_i \cup I_j) = A_1 \cup A_2, \]

where the open intervals \( A_1 \) and \( A_2 \) are disjoint. If the bridges \( I_i \) and \( I_j \) have types \( s \neq t \) respectively, then the annulus

\[ B = S^2 - (D_s \cup D_t) \]

separates \( \partial I_i \) from \( \partial I_j \), and hence

\[ \text{mod}(A, C) \geq \text{mod}(B) \geq \delta > 0 \]

by Proposition 3.3.

On the other hand, if \( I_i \) and \( I_j \) have the same type \( s \), then \( i, j > 2 \), and there must be a bridge \( I_k \), \( k < i \), such that \( I_1 \cup I_k \) separates \( I_i \) from \( I_j \). Otherwise, we could have combined \( I_i \) and \( I_j \) to obtain a longer bridge at step \( i \).

It follows that

\[ t = \frac{\min(|A_1|, |A_2|)}{\min(|I_i|, |I_j|)} \geq \frac{\min(|I_k|, |I_k|)}{\min(|I_i|, |I_j|)} = \frac{|I_k|}{|I_j|} \geq 1, \]

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since \( k < i < j \). By Proposition 3.2, this implies that
\[
\text{mod}(A,C) > \delta_0 > 0
\]
where \( \delta_0 \) is a universal constant. Thus the Theorem holds with \( \delta' = \min(\delta_0, \delta) \).

**Limit sets.** We can now complete the proof of Theorem 3.1.

**Theorem 3.5** Let \( M = \Gamma \setminus \mathbb{H}^3 \) be a convex cocompact acylindrical 3–manifold of infinite volume. Then its limit set \( \Lambda \) is a Sierpiński curve of modulus \( \delta \) for some \( \delta > 0 \).

**Proof.** First suppose that every component of \( \Omega = S^2 - \Lambda = \bigcup D_i \) is a round disk, i.e. suppose that \( M \) is a rigid acylindrical manifold. By compactness, there exists an \( L > 0 \) such that the hyperbolic length of any geodesic arc \( \gamma \subset \text{core}(M) \) orthogonal to the boundary at its endpoints is greater than \( L \). Consequently \( d_{ij} = d(\text{hull}(D_i), \text{hull}(D_j)) \geq L \) for any \( i \neq j \). Since the modulus of \( S^2 - (\overline{D_i} \cup \overline{D_j}) \) is given by \( d_{ij}/(2\pi) \), \( \Lambda \) is a Sierpiński curve of modulus \( \delta = L/(2\pi) > 0 \).

To treat the general case, recall that for any convex cocompact acylindrical manifold \( M \), there exists a rigid acylindrical manifold \( M' = \Gamma' \setminus \mathbb{H}^3 \) such that \( \Gamma' \) is \( K \)-quasiconformally conjugate to \( \Gamma \). Since a \( K \)-quasiconformal map distorts the modulus of an annulus by at most a factor of \( K \), and the limit set \( \Lambda' \) of \( \Gamma' \) is a Sierpiński curve with modulus \( \delta' > 0 \), \( \Lambda \) itself is a Sierpiński curve of modulus \( \delta = \delta'/K > 0 \).

**Proof of Theorem 3.1.** Combine Theorems 3.4 and 3.5.

### 4 Recurrence of horocycles

Let \( M = \Gamma \setminus \mathbb{H}^3 \) be an arbitrary 3–manifold. In this section we will define, for each \( k > 1 \), a closed, \( A \)-invariant set
\[
\text{RF}_k M \subset \text{RF} M
\]
consisting of points with good recurrence properties under the horocycle flow generated by \( U \) (for terminology see Tables 2 and 3). We will then show:
Theorem 4.1 Let $M = \Gamma \backslash \mathbb{H}^3$ be a convex cocompact acylindrical 3–manifold. We then have

$$F^* \subset (\text{RF}_k M) H$$

for all $k$ sufficiently large. More precisely, every plane $P^* \subset M^*$ is tangent to a frame in $\text{RF}_k M$.

We conclude by comparing the general result above to results that hold only when $\partial M^*$ is totally geodesic.

We remark that $(\text{RF}_k M) H$ is usually not closed, even when $M$ is acylindrical, because there can be circles $C \in \mathbb{C}^*$ such that $|C \cap \Lambda| = 1$.

**Thick sets.** We begin by defining $\text{RF}_k M$. Let us say a closed set $T \subset \mathbb{R}$ is $k$–thick if

$$[1, k] \cdot |T| = [0, \infty).$$

In other words, given $x \geq 0$ there exists a $t \in T$ with $|t| \in [x, kx]$. Note that if $T$ is $k$–thick, so is $\lambda T$ for all $\lambda \in \mathbb{R}^*$.

If the translate $T - x$ is $k$–thick for every $x \in T$, we say $T$ is globally $k$–thick. A set $K \subset U$ is (globally) $k$–thick if its image under an isomorphism $U \cong \mathbb{R}$ is (globally) $k$–thick.

**Unipotent recurrence.** For $x \in \text{RF}_M$, the unipotent orbit $xU$ almost never remains in $\text{RF}_M$. Provided, however, there is a thick set $K \subset U$ such that $xK \subset \text{RF}_M$, we have sufficient recurrence to carry through many arguments that would be automatic if $xU$ were bounded. The key point is to combine thickness with the polynomial behavior of unipotent flows. This theme is developed in detail in [MMO, §8], and it motivates the definition of $\text{RF}_k M$ below.

Let

$$U(z) = \{ u \in U : zu \in \text{RF}_M \}$$

(4.1)

denote the return times of $z \in \text{FM}$ to the renormalized frame bundle under the horocycle flow. We define $\text{RF}_k M$ for each $k > 1$ by

$$\text{RF}_k M = \left\{ z \in \text{RF}_M : \text{there exists a globally } k \text{–thick set } K \text{ with } 0 \in K \subset U(z) \right\}.$$ 

Let

$$U(z, k) = \{ u \in U : zu \in \text{RF}_k M \}.$$ 

**Proposition 4.2** Suppose the convex core of $M$ is compact. Then for any $k > 1$, the set $\text{RF}_k M$ is a compact, $\mathcal{A}$–invariant subset of $\text{RF}_M$. Moreover, $U(z, k)$ is $k$–thick for each $z \in \text{RF}_k M$. 

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Proof. Using compactness of $RFM$, it is easily verified that if $z_n \to z$ in $FM$ then $\limsup U(z_n) \subset U(z)$. One can also check that if $K_n \subset U$ is a sequence of globally $k$–thick sets with $0 \in K_n$, then $\limsup K_n$ is also globally $k$–thick. Consequently $RF_k M \subset RFM$ is closed, and hence compact.

Since $U(za)$ is a rescaling of $U(z)$ for any $a \in A$, and the notion of thickness is scale–invariant, $RF_k M$ is $A$–invariant. For the final assertion, observe that $U(z, k)$ contains the thick set $K \subset U(z)$ posited in the definition of $RF_k M$.

**Thickness and moduli.** To complete the proof Theorem 4.1, we just need to relate thickness to the results of §3. For the next statement, we regard $\hat{R} = R \cup \{\infty\}$ as a circle on $S^2 \cong \hat{C}$.

**Proposition 4.3** Let $K \subset \hat{R}$ be a Cantor set of modulus $\delta > 0$ containing $\infty$. Then $T = K \cap R$ is a globally $k$–thick subset of $R$, where $k > 1$ depends only on $\delta$.

**Proof.** Use Proposition 3.2 to relate the modulus of $K$ to the relative sizes of gaps in $R - K$.

**Proof of Theorem 4.1.** Since $M$ is acylindrical, by Theorem 3.1 there exists a $\delta > 0$ such that for any $C \in C^*$, there exists a Cantor set $K$ of modulus $\delta$ with

$$K \subset C \cap \Lambda \subset S^2.$$ 

By Proposition 4.3, there exists a $k_0$ such that $T \subset R$ is globally $k_0$–thick whenever $T \cup \infty$ is a Cantor set of modulus $\delta$.

Let $P^*$ be a plane in $M^*$. Choose $C \in C^*$ such that the image of hull$(C)$ in $M^*$ contains $P^*$. Let $K \subset C \cap \Lambda$ be the Cantor set of modulus $\delta$ provided by Theorem 3.1.

By a change of coordinates, we can arrange that $0, \infty \in K \subset \hat{R}$. Let $\bar{z} \in F\hat{H}^3$ be any frame tangent to hull$(\hat{R})$ along the geodesic $\gamma$ joining zero to infinity, and let $z$ denote its projection to $FM$. Then $z$ is tangent to $P^*$. It is readily verified that there exists an isomorphism $U \cong R$ sending $U(z)$ to $R \cap \Lambda$. Since $0 \in K \subset R \cap \Lambda$ and $K$ is globally $k_0$–thick, we have $z \in RF_{k_0} M$ as well. Thus the Theorem holds for all $k \geq k_0$.

**Comparison with the rigid case.** We conclude by comparing the case of a general convex cocompact acylindrical 3-manifold $M$, treated by Theorems 3.1 and 4.1, with the rigid case, studied in [MMO].

In the rigid case, every component $D_i$ of $S^2 - \Lambda$ is a round disk; hence $C \cap D_i$ is connected for all $C \in C^*$, and one can show:
$K = C \cap \Lambda$ is a compact set of definite modulus $\forall C \in \mathcal{C}^\ast$.

See [MMO, Lemma 9.2]. Similarly, all horocycles passing through $RF_M$ are recurrent, and $RF_kM = RF_M$ for all $k$ sufficiently large.

On the other hand, when $M$ is not rigid, there are cases where both these properties fail. For example, suppose the bending measure of $hull(\Lambda)$ has an atom of mass $\theta$ along the geodesic $\gamma$ joining $p, q \in \Lambda$. Then we can change coordinates on $S^2 \cong \hat{C}$ so that $p = 0$, $q = \infty$, and $\Lambda$ is contained in the wedge defined by $| \arg(z) | < \pi - \theta/2$. Then the circle $C \in \mathcal{C}^\ast$ defined by $\text{Re}(z) = 1$ cannot meet the limit set in a set of positive modulus, since $\infty$ is an isolated point of $C \cap \Lambda$.

Similarly, the horocycle in $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$ defined by $\eta(t) = (it, 1)$ crosses $\gamma$ when $t = 0$, and satisfies $d(\eta(t), hull(\Lambda)) \to \infty$ as $|t| \to \infty$. Projecting to $M$, we obtain a divergent horocycle orbit $xU$ with $x \in RF_M$. In particular, $x \in RF_M - RF_kM$ for all $k$.

Nevertheless $C \cap \Lambda$ can contain a Cantor set of positive modulus, consistent with Theorem 3.1.

5 The boundary of the convex core

In this short section we analyze the behavior of $C \cap \Lambda$ for circles that meet the limit set but do not separate it. The result we need does not require that $M$ is acylindrical, only that its convex core is compact.

**Theorem 5.1** Let $M = \Gamma \backslash \mathbb{H}^3$ be a convex cocompact 3-manifold with limit set $\Lambda$. Let $C$ be the boundary of a supporting hyperplane for $hull(\Lambda)$. Then:

1. $\Gamma^C$ is a convex cocompact Fuchsian group; and
2. There is a finite set $\Lambda_0$ such that

$$C \cap \Lambda = \Lambda(\Gamma^C) \cup \Gamma^C \Lambda_0.$$ 

Here $\Lambda(\Gamma^C)$ denotes the limit set of $\Gamma^C = \{ g \in \Gamma : g(C) = C \}$.

**Corollary 5.2** If the projection of $hull(C)$ to $M$ gives a plane $P$ disjoint from $M^\ast$ but tangent to a frame in $RF_kM$, then $\Gamma^C$ is nonelementary.

**Proof.** The hypotheses guarantee that $C$ does not separate $\Lambda$, and $C \cap \Lambda$ contains an (uncountable) Cantor set of positive modulus. Then by the preceding result, $\Lambda(\Gamma^C)$ is uncountable, so $\Gamma^C$ is nonelementary.  

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Proof of Theorem 5.1. We will use the theory of the bending lamination, developed in [Th1], [EpM], [KaT] and elsewhere.

If $M^*$ is empty, then $\Lambda$ is contained in a circle and the result is immediate. The desired result is also immediate if $C \cap \Lambda$ is finite, because $\Lambda(\Gamma^C) \subset C \cap \Lambda$.

Now assume $C \cap \Lambda$ is infinite and $M^*$ is nonempty. Then $K = \partial \text{core}(M)$ is a finite union of disjoint compact pleated surfaces with bending lamination $\beta$. Let

$$f : S = \Gamma^C \setminus \text{hull}(C \cap \Lambda) \to M$$

be the natural projection. Since $|C \cap \Lambda| > 2$, $S$ is a metrically complete hyperbolic surface with geodesic boundary, with nonempty interior $S_0$. The map $f$ sends $S_0$ isometrically to a component of $K - \beta$; in particular, $S_0$ has finite area. It follows that the ends of $S_0$ consist of the regions between finitely many pairs of geodesics which are tangent at infinity; for an example, see Figure 5. Consequently, we can find a finite set $\Lambda_0 \subset \Lambda$ (corresponding to the finitely many ends of $S_0$) such that

$$C \cap \Lambda = \Lambda(\Gamma^C) \cup \Gamma^C \Lambda_0.$$ 

The group $\Gamma^C$ is convex cocompact because $S$ has finite area and $\Gamma$ contains no parabolic elements.

Figure 5. A surface with a crown.

6 Planes near the boundary of the convex core

In this section we take a step towards the proof of Theorem 1.4 by establishing two density results.

Theorem 6.1 Let $M = \Gamma \setminus \mathbb{H}^3$ be a convex cocompact 3–manifold with incompressible boundary. Consider a sequence of circles $C_n \to C$ with $C_n \in \mathcal{C}^*$
but \( C \not\in \mathcal{C}^* \). Suppose that \( L = \liminf(C_n \cap \Lambda) \) is uncountable. Then \( \bigcup \Gamma C_n \) is dense in \( \mathcal{C}^* \).

Under the same assumptions on \( M \) we obtain:

**Corollary 6.2** Consider an \( H \)-invariant set \( E \subset F^* \) and fix \( k > 1 \). If the closure of \( E \cap RF_k M \) contains a point outside \( F^* \), then \( E \) is dense in \( F^* \).

**Proof.** Consider a sequence \( x_n \in E \cap RF_k M \) such that \( x_n \rightarrow x \in RF_k M - F^* \). We then have a corresponding sequence of circles \( C_n \in \mathcal{C}^* \) such that \( C_n \rightarrow C \not\in \mathcal{C}^* \). (The circles are chosen so that \( x_n \) is tangent to the image of \( \text{hull}(C_n) \) in \( M \)).

Pass to a subsequence such that \( U(x_n) \) (defined using equation (4.1)) converges, in the Hausdorff topology, to a closed set \( K \subset U(x) \). Then \( C_n \cap \Lambda \) also converges, to a compact set \( L \subset C \) homeomorphic to the 1-point compactification of \( K \). The fact that \( x_n \in RF_k M \) implies that \( K \) contains a globally \( k \)-thick set; hence \( K \) is uncountable, so \( L \) is as well. Then by the result above, \( \bigcup \Gamma C_n \) is dense in \( \mathcal{C}^* \), so \( E \) is dense in \( F^* \).

Roughly speaking, these results show that planes \( P^* \) that are nearly tangent to \( \partial M^* \) are also nearly dense in \( M^* \), subject to a condition on \( RF_k M \) that is automatic in the acylindrical case by Theorem 4.1.

**Fuchsian dynamics.** The proof of Theorem 6.1 exploits the dynamics of the Fuchsian group \( \Gamma C \). Given an open round disk \( D \subset S^2 \) and a closed subset \( E \subset \partial D \), we let \( \text{hull}(E,D) \subset D \) denote the convex hull of \( E \) in the hyperbolic metric on \( D \).

The principle we will use is [MMO, Cor. 3.2], which we restate as follows.

**Theorem 6.3** Let \( M = \Gamma \backslash \mathbb{H}^3 \) be a convex cocompact hyperbolic 3–manifold. Let \( D \subset S^2 \) be a round open disk that meets \( \Lambda \), and let \( C = \partial D \). Suppose \( \Gamma C \) is a nonelementary, finitely generated group, and let \( C_n \rightarrow C \) be a sequence of circles such that

\[
C_n \cap \text{hull}(\Lambda(\Gamma C), D) \neq \emptyset.
\]

Then the closure of \( \bigcup \Gamma C_n \) in \( \mathcal{C} \) contains every circle that meets \( \Lambda \).

**Proof of Theorem 6.1.** Let \( D \) and \( D' \) denote the two components of \( S^2 - C \). Since \( C \not\in \mathcal{C}^* \), at least one of the components, say \( D' \), is contained in \( \Omega \). Since \( L \subset C \cap \Lambda \) is uncountable, \( \Gamma C \) is nonelementary and finitely generated by Theorem 5.1. Consider an ideal pentagon

\[
X = \text{hull}(V, D) \subset \text{hull}(\Lambda(\Gamma C), D) \tag{6.1}
\]
whose five vertices \( V \) lie in \( L \). Since \( L = \liminf C_n \cap \Lambda \), we can find ‘vertices’
\[
V_n \subset C_n \cap \Lambda, \quad |V_n| = 5,
\]
such that \( V_n \to V \). In particular, \( |C_n \cap \Lambda| \geq 3 \) for all \( n \).

Note that \( C_n \) is the unique circle passing through any three points of \( V_n \). If three of these points were to lie in \( D' \), then we would have \( C_n \subset D' \), and hence \( |C_n \cap \Lambda| \leq 1 \), since \( C_n \neq C = \partial D' \) and \( D' \subset \Omega \). Hence \( |V_n \cap D| \geq 3 \). Since \( |C_n \cap C| \leq 2 \), at least two adjacent components of \( C_n - V_n \) are contained in \( D \). It follows easily that \( C_n \) meets \( \mathrm{hull}(V,D) \) for all \( n \) sufficiently large.

Using equation (6.1) we can then apply Theorem 6.3 to conclude that \( \bigcup \Gamma C_n \) is dense in \( \mathcal{C}^* \), since every \( C \in \mathcal{C}^* \) meets \( \Lambda \).

\[\text{\bf 7 Planes far from the boundary}\]

In this section we finally prove Theorem 1.4, which we restate as Corollary 7.2. The proof rests on the following more general density theorem.

**Theorem 7.1** Let \( M = \Gamma \setminus \mathbb{H}^3 \) be a convex cocompact 3-manifold with incompressible boundary. Let \( C_i \to C \) be a convergent sequence in \( \mathcal{C}^* \), with \( C \not\in \bigcup \Gamma C_i \).

Suppose that \( C \cap \Lambda \) contains a Cantor set of positive modulus. Then \( \bigcup \Gamma C_i \) is dense in \( \mathcal{C}^* \).

**Corollary 7.2** If \( M = \Gamma \setminus \mathbb{H}^3 \) is a convex cocompact acylindrical 3-manifold, then any \( \Gamma \)-invariant set \( E \subset \mathcal{C}^* \) is either closed or dense in \( \mathcal{C}^* \).

**Proof.** Suppose \( E \) is not closed in \( \mathcal{C}^* \). Then we can find a sequence \( C_i \in E \) converging to \( C \in \mathcal{C}^* - E \). Since \( M \) is acylindrical, \( C \) meets \( \Lambda \) in a Cantor set of positive modulus, by Theorem 3.1. Since \( E \) is \( \Gamma \)-invariant, the preceding result shows that \( \bigcup \Gamma C_i \) is dense in \( \mathcal{C}^* \), so the same is true for \( E \).

The proof of Theorem 7.1 follows the same lines as the proof of Theorem 7.3 in [MMO, §9]. We will freely quote results from [MMO] in the course of the proof. The notation from Table 2 for the subgroups \( U,V,A,N \) of \( G \) and other objects will also be in play. A generalization of Theorem 7.1 to manifolds with compressible boundary is stated at the end of this section.

**Setup in the frame bundle.** To prepare for the proof, we first reformulate it in terms of the frame bundle.
Let $C_i \to C$ as in the statement of Theorem 7.1. Since $C \cap \Lambda$ contains a Cantor set of positive modulus, by Proposition 4.3 we can choose $k > 1$ and $x_\infty \in \text{RF}_k M \cap F^*$ such that $x_\infty H$ corresponds to $\Gamma C$. Let us also choose $x_i \to x_\infty$ in $F^*$ such that $x_i H$ corresponds to $\Gamma C_i$. Since $C \not\in \bigcup \Gamma C_i$, we also have

$$x_\infty \notin E = \bigcup x_i H.$$ 

To prove Theorem 7.1 we need to show:

$E$ is dense in $F^*$.

We may also assume that:

$$\text{The set } \overline{E} \cap \text{RF}_k M \cap F^* \text{ is compact.} \quad (7.1)$$

Otherwise $\overline{E} \cap F^* = F^*$ by Corollary 6.2, and hence $E$ is dense in $F^*$.

**Dynamics of semigroups.** We say that $L \subset G$ is a $1$–parameter semigroup if there exists a nonzero $\xi \in \text{Lie}(G)$ such that

$L = \{ \exp(t\xi) : t \geq 0 \}$.

To show a set is dense in $F^*$, we will use the following fact.

**Proposition 7.3** Let $L \subset V$ be a $1$–parameter semigroup. Then $x LH$ contains $F^*$ for all $x \in F^*$.

**Proof.** Let $C \in C^*$ be a circle corresponding to $xH$. Then $x LH$ corresponds to a family of circles $C_\alpha$ such that $\bigcup C_\alpha$ contains one of the components of $S^2 - C$. Since $C \in C^*$, both components meet the limit set. Hence $\overline{\Gamma C_\alpha} \supset C^*$ for some $\alpha$ by [MMO, Cor. 4.2].

**The staccato horocycle flow.** Recall that the compact set $\text{RF}_k M$ is invariant under the geodesic flow $A$. Moreover, Proposition 4.2 states that

$U(z, k) = \{ u \in U : zu \in \text{RF}_k M \}$

is a thick subset of $U$, for all $z \in \text{RF}_k M$. In other words, $\text{RF}_k M$ is also invariant under the *staccato horocycle flow*, which is interrupted outside of $U(z, k)$.

**Recurrence.** Next we define a compact set $W$ with

$$x_\infty \in W \subset \overline{E} \cap F^*.$$
with good recurrence properties for the horocycle flow. Namely, we let
\[ W = \begin{cases} (E - E) \cap RF_k M \cap F^* & \text{if this set is compact, and} \\ E \cap RF_k M \cap F^* & \text{otherwise.} \end{cases} \] (7.2)
(This definition is motivated by the proof of Lemma 7.6.)

In either case, \( W \) is compact by assumption (7.1). Since \( \overline{E} \cap F^* \) is \( H \)-invariant, we have
\[ WA = W \quad \text{and} \quad WU \cap RF_k M \subset W. \]
The second inclusion gives good recurrence; namely, we have
\[ xU(x, k) \subset W \] (7.3)
for all \( x \in W \); and \( U(x, k) \) is thick, because \( W \subset RF_k M \).

**The horocycle flow.** We now exploit the fact that \( E \) is invariant under the horocycle flow. The 1-parameter horocycle subgroup \( U \subset H \) is distinguished by the fact that its normalizer contains (with finite index) the large subgroup \( AN \subset G \). If \( X \) is \( U \)-invariant, then so is \( Xg \) for any \( g \in AN \).

**Minimal sets.** A closed set \( Y \) is a \( U \)-minimal set for \( E \) with respect to \( W \) if \( Y \subset \overline{E} \), \( Y \) meets \( W \), \( YU = Y \), and
\[ \overline{yU} = Y \quad \text{for all} \quad y \in Y \cap W. \]
Note that \( \overline{E} \) itself has all these properties except for the last. The existence of a minimal set \( Y \) follows from the Axiom of Choice and compactness of \( W \). From now on we will assume that:
\[ Y \text{ is a } U \text{-minimal set for } \overline{E} \text{ with respect to } W. \]

To show that \( \overline{E} \) is large, our strategy is to show it contains \( Yg \) for many \( g \in AN \). To this end, we remark that for \( g \in AN \):
\[ \text{If } (Y \cap W)g \text{ meets } \overline{E}, \text{ then } Yg \subset \overline{E}. \]
Indeed, in this case by minimality we have:
\[ \overline{E} \supset ygU = \overline{yU}g = Yg, \] (7.4)
where \( yg \in (Y \cap W)g \cap \overline{E} \).

**Translation of \( Y \) inside of \( Y \).** The fact that horocycles in \( Y \) return frequently to \( W \) allows one to deduce additional invariance properties for \( Y \) itself. Note that the orbits of \( AV \) are orthogonal to the orbits of \( U \) in the Riemannian metric on \( FM \).
Lemma 7.4 There exists a 1-parameter semigroup \( L \subset AV \) such that 
\( YL \subset Y \).

**Proof.** In the rigid acylindrical case, this is Theorem 9.4 in [MMO] for \( W = RFM \). The only property of \( RFM \) used in the proof is the \( k \)-thickness of \( \{ u \in U : xu \in RFM \} \) for any \( x \in RFM \). Hence the proof works verbatim with \( W \) replacing \( RFM \), in view of equation (7.3). In fact \( YL = Y \) since \( id \in L \).

Translation of \( Y \) inside of \( E \). Our next goal is to find more elements \( g \in G \) that satisfy \( Yg \subset E \). Consider the closed set 
\[
S(Y) = \{ g \in G : (Y \cap W)g \cap E \neq \emptyset \}.
\]
Since \( E \) is \( H \)-invariant, we have \( S(Y)H = S(Y) \).

Lemma 7.5 If \( S(Y) \) contains a sequence \( g_n \to \text{id} \) in \( G \setminus H \), then there exists \( v_n \in V \setminus \{ \text{id} \} \) tending to \( \text{id} \) such that 
\( Yv_n \subset E \).

**Proof.** Let \( g_n \in S(Y) \) be a sequence tending to \( \text{id} \) in \( G \setminus H \). First suppose that there is a subsequence, which we continue to denote by \( \{ g_n \} \), of the form \( g_n = v_nh_n \in VH \). Since \( g_n \notin H \), we have \( v_n \neq \text{id} \) for all \( n \). The claim then follows from the \( H \)-invariance of \( S(Y) \) and the \( U \)-minimality of \( Y \), see (7.4).

Therefore, assume that \( g_n \notin VH \) for all large \( n \). Since \( g_n \in S(Y) \), there exist \( y_n \in Y \cap W \) such that \( y_nh_n \in E \).

Since \( Y \) is \( U \)-invariant and \( WU \cap RF_kM \subset RF_kM \), we have \( yU(y,k) \subset Y \) for all \( y \in Y \), and \( U(y,k) \) is a \( k \)-thick subset of \( U \).

Therefore, by [MMO, Thm. 8.1], for any neighborhood \( G_0 \) of the identity in \( G \) we can choose \( u_n \in U(y_n,k) \) and \( h_n \in H \) such that 
\[
u_n^{-1}g_nh_n \to v \in V \cap G_0 - \{ \text{id} \}.
\]
After passing to a subsequence, we have \( y_nu_n \to y_0 \in Y \cap W \). Hence 
\[
y_nh_n = (y_nu_n)(u_n^{-1}g_nh_n) \in E
\]
converges to \( y_0v \in E \).

Since \( Y \) is \( U \)-minimal with respect to \( W \) and \( y_0 \in Y \cap W \), we have 
\[
\overline{y_0vU} = \overline{y_0Uv} = Yv \subset E.
\]
Since \( G_0 \) was an arbitrary neighborhood of the identity, the claim follows. 

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Choosing $Y$. In general there are many possibilities for the minimal set $Y$, and it may be hard to describe a particular one, since the existence of a minimal set is proved using the Axiom of Choice. The next result shows that, nevertheless, we can choose $Y$ so it remains inside $\mathcal{E}$ under suitable translations transverse to $H$ but still in $AN$.

**Lemma 7.6** There exists a $U$-minimal set $Y$ for $\mathcal{E}$ with respect to $W$, and a sequence $v_n \to \text{id}$ in $V - \{\text{id}\}$, such that

$$Yv_n \subset \mathcal{E}$$

for all $n$.

**Proof.** By Lemma 7.5, it suffices to show that $Y$ can be chosen so that $S(Y)$ contains a sequence $g_n \to \text{id}$ in $G - H$. We break the analysis into two cases, depending on whether or not $E$ meets the compact set $W$.

First consider the case where $E$ is disjoint from $W$. Let $Y$ be a $U$-minimal set for $\mathcal{E}$ with respect to $W$. Choose $y \in Y \cap W$. Since $Y \subset \mathcal{E}$, there exist $g_n \to \text{id}$ such that $yg_n \in E$. Then $y \notin E$, and hence $g_n \in G - H$, so we are done.

Now suppose $E$ meets $W$. Then $W - E$ is not closed, by equation (7.2). So in this case there exists a sequence $x_n \in W - E$ with $x_n \to x \in E \cap W$. In particular, $\overline{xH} \cap W \neq \emptyset$. Thus there exists a $U$-minimal set $Y$ for $\overline{xH}$ with respect to $W$.

We now consider two cases. Assume first that $Y \cap W \subset xH$. Pick $y \in Y \cap W$; then $y = xh$ for some $h \in H$. Since $x_n \to x$ we have $x_nh \to y$. Now writing $yg_n = x_nh$, we have $g_n \to \text{id}$. As $y \in xH \subset E$ and $x_n \notin E$, we have $g_n \in G - H$, and we are done.

Now suppose that $W \cap Y \not\subset xH$. Choose $y \in (W \cap Y) - xH$. Since we have $Y \subset \overline{xH}$, there exist $g_n \to \text{id}$ with $yg_n \in xH$. Moreover, $g_n \in G - H$ since $y \notin xH$, and the proof is complete in this case as well. 

**Semigroups.** We are now ready to complete the proof of Theorem 7.1. We will exploit the 1-parameter semigroup $L \subset AV$ guaranteed by Lemma 7.4. To discuss the possibilities for $L$, let us write the elements of $V$ and $A$ as

$$v(s) = \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$ 

We then have two semigroups in $V$, defined by $V_{\pm} = \{v(s) : \pm s \geq 0\}$, and two similar semigroups in $A_{\pm}$ in $A$. It will also be useful to introduce the
interval
\[ V_{[a,b]} = \{ v(s) : s \in [a, b] \}. \]

In the notation above, if \( L \subset AV \) is a 1-parameter semigroup, then either

(i) \( L = V_\pm \);
(ii) \( L = A_\pm \); or
(iii) \( L = v^{-1}A_\pm v \), for some \( v \in V, v \neq \text{id} \).

**Proof of Theorem 7.1.** To complete the proof, it only remains to show we have \( F^* \subset \overline{E} \).

Choose \( Y \) and \( v_n \in V \) so that \( Yv_n \subset \overline{E} \) as in Lemma 7.6. Write \( v_n = v(s_n) \); then \( s_n \to 0 \) and \( s_n \neq 0 \). Passing to a subsequence, we can assume \( s_n \) has a definite sign, say \( s_n > 0 \).

By Lemma 7.4, there is a 1–parameter semigroup \( L \subset AV \) such that \( YL \subset Y \).

The rest of the argument breaks into 3 cases, depending on whether \( L \) is of type (i), (ii) or (iii) in the list above.

(i). If \( L = V_\pm \), then we have \( F^* \subset YLH \subset \overline{EH} = \overline{E} \) by Proposition 7.3, and we are done.

(ii). Now suppose \( L = A_\pm \). Let
\[ B = \{ \text{id} \} \cup \bigcup A_\pm v_n A. \]
Since \( YL \subset Y \) and \( Yv_n A \subset \overline{EA} = \overline{E} \) for all \( n \), we have
\[ YB \subset \overline{E}. \]

Note that \( a(t)v(s)a(-t) = v(e^{2t}s) \). Consequently we have
\[ v(e^{2t}s_n) \in B \]
for all \( n \) and all \( t \) with \( a(t) \in L = A_\pm \).

Suppose \( L = A_\pm \). Since \( s_n \to 0 \) and \( s_n > 0 \), in this case we have \( V_+ \subset B \); hence \( YV_+ H \subset \overline{E} \) and we are done as in case (i).

Now suppose \( L = A_- \). In this case at least we obtain an interval
\[ V_{[0,s_1]} \subset B. \]
Choose a sequence $a_n \in A$ such that $V_+ = \bigcup a_n V_{[0,s_1]} a_n^{-1}$. Consider $y \in Y \cap W$. Since $ya_n^{-1} \in W$, and $W$ is compact, after passing to a subsequence we can assume that

$$ya_n^{-1} \to y_0 \in W \subset F^*.$$  

We then have

$$y_0 V_+ = \bigcup ya_n^{-1}(a_n V_{[0,s_1]} a_n^{-1}) \subset \mathcal{E},$$

which again implies that $F^* \subset \mathcal{E}$, by Proposition 7.3.

(iii). Finally, consider the case $L = v^{-1} A \pm v$ for some $v \in V, v \neq \text{id}$. We then have $YB \subset \mathcal{E}$ where

$$B = v^{-1} A \pm vA.$$ 

By an easy computation, $B$ contains $V_{[0,\pm s]}$ for some $s > 0$, and the argument is completed as in case (ii).

The compressible case. In conclusion, we remark that Theorems 6.1 and 7.1 remain true without the hypothesis that $M$ has incompressible boundary, provided we replace $C^*$ with

$$C^\# = \{ C \in C^* : C \text{ meets } \Lambda \}$$

and require that $M^*$ is nonempty. The proofs are simple variants of those just presented.

References


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