ADELIC VERSION OF MARGULIS ARITHMETICITY THEOREM

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Abstract. In this paper, we generalize Margulis's S-arithmeticity theorem to the case when S can be taken as an infinite set of primes. Let R be the set of all primes including infinite one ∞ and set $\mathbb{Q}_\infty = \mathbb{R}$. Let S be any subset of R. For each $p \in S$, let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors and $D_p \subset G_p(\mathbb{Q}_p)$ be a compact open subgroup for almost all finite prime $p \in S$. Let (G_S, D_p) denote the restricted topological product of $G_p(\mathbb{Q}_p)$'s, $p \in S$ with respect to D_p's. Note that if S is finite, $(G_S, D_p) = \prod_{p \in S} G_p(\mathbb{Q}_p)$. We show that if $\sum_{p \in S} \operatorname{rank}_{\mathbb{Q}_p}(G_p) \geq 2$, any irreducible lattice in (G_S, D_p) is a *rational* lattice. We also present a criterion on the collections G_p and D_p for (G_S, D_p) to admit an irreducible lattice. In addition, we describe discrete subgroups of $(G_{\mathbb{A}}, D_p)$ generated by lattices in a pair of opposite horospherical subgroups.

1. Introduction

Let R denote the set of all prime numbers including the infinite prime ∞ and R_f the set of finite prime numbers, i.e., $R_f = R - \{\infty\}$. We set $\mathbb{Q}_\infty = \mathbb{R}$. For each $p \in R$, let G_p be a non-trivial connected semisimple algebraic $\mathbb{Q}_p$-group and for each $p \in R_f$, let D_p be a compact open subgroup of $G_p(\mathbb{Q}_p)$. The adele group of G_p, $p \in R$ with respect to D_p, $p \in R_f$ is defined to be the restricted topological product of the groups $G_p(\mathbb{Q}_p)$ with respect to the distinguished subgroups D_p. We denote this group by $(G_{\mathbb{A}}, \{D_p, p \in R_f\})$ or simply by $(G_{\mathbb{A}}, D_p)$. That is,

$$(G_{\mathbb{A}}, D_p) = \{(g_p) \in \prod_{p \in R} G_p(\mathbb{Q}_p) \mid g_p \in D_p \text{ for almost all } p \in R_f\}.$$

As is well known, the adele group $(G_{\mathbb{A}}, D_p)$ is a locally compact topological group.

If G is a connected semisimple $\mathbb{Q}$-group, then we mean by $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$ the adele group attached to the groups $G_p = G$, $p \in R$ with respect to the subgroups $G(\mathbb{Z}_p)$, $p \in R_f$. It is a well known result of Borel [Bo1] that the diagonal embedding of $G(\mathbb{Q})$ into $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$, which we will identify with $G(\mathbb{Q})$, is a lattice in $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$. Furthermore

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Godement's criterion in an adelic setting, proved by Mostow and Tamagawa [MT] and also independently by Borel [Bo1], implies that G is $\mathbb{Q}$-isotropic if and only if $G(\mathbb{Q})$ is a non-uniform lattice in $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$.

In the spirit of Margulis arithmeticity theorem [Ma1], we show in this paper that any irreducible lattice in an adele group $(G_{\mathbb{A}}, D_p)$ is essentially of the form described as above. We say that a lattice Γ in $(G_{\mathbb{A}}, D_p)$ is *irreducible* if, for any finite subset S of R containing ∞, $\pi_S\left(\Gamma \cap \{(g_p) \in G_{\mathbb{A}} \mid g_p \in D_p \text{ for all } p \notin S\}\right)$ is an irreducible lattice in $\prod_{p\in S} G_p(\mathbb{Q}_p)$ in the usual sense (see [Ma2, Ch III, 5.9] or definition 2.9 below) where π_S denotes the natural projection $(G_{\mathbb{A}}, D_p) \to \prod_{p\in S} G_p(\mathbb{Q}_p)$.

The following is a sample case of our main theorem:

1.1. Theorem. *For each $p \in R$, let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors and D_p a compact open subgroup for almost all $p \in R_f$. Assume that G_∞ is absolutely simple. Then any irreducible non-uniform lattice Γ in $(G_{\mathbb{A}}, D_p)$ is rational in the sense that there exist a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group H and a $\mathbb{Q}_p$-isomorphism $f_p : H \to G_p$ for each $p \in R$ with $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all $p \in R_f$ such that Γ is a subgroup of finite index in $f(H(\mathbb{Q}))$ where f is the restriction of the product map $\prod_{p\in R} f_p$ to $(H_{\mathbb{A}}, H(\mathbb{Z}_p))$. In particular, f provides a topological group isomorphism of $(H_{\mathbb{A}}, H(\mathbb{Z}_p))$ to $(G_{\mathbb{A}}, D_p)$.*

In order to define a rational lattice in an adele group in generality, we first describe arithmetic methods of constructing irreducible lattices in adele groups. Let K be a number field. Let R_K be the set of all (inequivalent) valuations of K. For each $v \in R_K$, K_v denotes the local field which is the completion of K with respect to v and for non-archimedean $v \in R_K$, $\mathcal{O}_v$ denotes the ring of integers of K_v. If $\mathcal{H}$ is a connected absolutely simple K-group, it is a well known fact that the set $\mathcal{T}(\mathcal{H}) = \{v \in R_K \mid \mathcal{H}(K_v) \text{ is compact}\}$ is finite. Let S be a subset of $R_K - \mathcal{T}(\mathcal{H})$ containing all archimedean valuations in $R_K - \mathcal{T}(\mathcal{H})$, and let $(\mathcal{H}_S, \mathcal{H}(\mathcal{O}_v))$ denote the restricted topological product of the groups $\mathcal{H}(K_v)$, $v \in S$ with respect to the subgroups $\mathcal{H}(\mathcal{O}_v)$. Then the subgroup $\mathcal{H}(K(S))$, when identified with its image under the diagonal embedding into $(\mathcal{H}_S, \mathcal{H}(\mathcal{O}_v))$, is a lattice in $(\mathcal{H}_S, \mathcal{H}(\mathcal{O}_v))$ where $K(S)$ denotes the ring of S-integers in K [Bo1]. The group $\mathcal{H}$ being absolutely simple, $\mathcal{H}(K(S))$ is in fact an irreducible lattice in $(\mathcal{H}_S, \mathcal{H}(\mathcal{O}_v))$.

Unless mentioned otherwise, throughout the introduction, we let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group for each $p \in R$ and D_p a compact open subgroup for each $p \in R_f$.

1.2. Definition. We call an irreducible lattice Γ in $(G_{\mathbb{A}}, D_p)$ *rational* if there exist K, $\mathcal{H}$, S as above and a topological group epimorphism $f : (G_{\mathbb{A}}, D_p) \to (\mathcal{H}_S, \mathcal{H}(\mathcal{O}_v))$ with compact kernel such that $f(\Gamma)$ is commensurable with $\mathcal{H}(K(S))$.

Remark.

(1) Since $S \subset R_K - \mathcal{T}(\mathcal{H})$, $\mathcal{H}(K_v)$ is non-compact for each $v \in S$. If we denote by G_p^i the maximal connected normal $\mathbb{Q}_p$-subgroup of G_p without any $\mathbb{Q}_p$-anisotropic factors for each $p \in R$ and let $D_p^i = D_p \cap G_p^i$ for each $p \in R_f$, then in the above definition the quotient $(G_{\mathbb{A}}, D_p)/\ker f$ is isomorphic to $(G_{\mathbb{A}}^i, D_p^i)$. In particular, if $G_p(\mathbb{Q}_p)$ has no compact factors for any $p \in R$, we may assume that f is an isomorphism in Definition 1.2.

(2) If $R_0 = \{p \in R \mid \ G_p(\mathbb{Q}_p) \text{ is non-compact}\}$, then $(G_{\mathbb{A}}^i, D_p^i)$ is naturally identified with the restricted topological product of the groups $G_p^i(\mathbb{Q}_p)$, $p \in R_0$ with respect to the subgroups D_p^i. If R_0 is finite, then $(G_{\mathbb{A}}^i, D_p^i) = \prod_{p \in R_0} G_p^i(\mathbb{Q}_p)$. In this case, the above definition of a rational lattice in $(G_{\mathbb{A}}, D_p)$ coincides with that of an R_0-arithmetic (usually referred to as "S-arithmetic") lattice of $\prod_{p \in R_0} G_p^i(\mathbb{Q}_p)$ given in [Ma2, Ch IX, 1.4].

(3) If Γ is an irreducible lattice in $(G_{\mathbb{A}}, D_p)$, then $pr(\Gamma)$ is an irreducible lattice in $(G_{\mathbb{A}}^i, D_p^i)$ as well where pr denotes the natural projection $(G_{\mathbb{A}}, D_p) \to (G_{\mathbb{A}}^i, D_p^i)$. Then an irreducible lattice Γ in $(G_{\mathbb{A}}, D_p)$ is rational if and only if $pr(\Gamma)$ is a rational lattice in $(G_{\mathbb{A}}^i, D_p^i)$ in the sense of Definition A (or Definition B) in 4.1.

The following is a special case of Corollary 4.10 below.

1.3. Main Theorem. *If $\sum_{p \in R} rank_{\mathbb{Q}_p}(G_p) \geq 2$, any irreducible lattice in $(G_{\mathbb{A}}, D_p)$ is rational.*

That the adele group $(G_{\mathbb{A}}, D_p)$ contains an irreducible lattice imposes a strong restriction not only on the family of the ambient groups G_p but also on the family of distinguished subgroups D_p. The following presents a necessary and sufficient condition on those restriction:

1.4. Theorem. *For each $p \in R$, assume that $G_p(\mathbb{Q}_p)$ has no compact factors. The adele group $(G_{\mathbb{A}}, D_p)$ admits an irreducible lattice if and only if there exist a connected semisimple $\mathbb{Q}$-simple $\mathbb{Q}$-group H such that G_p is $\mathbb{Q}_p$-isomorphic to a connected normal $\mathbb{Q}_p$-subgroup of H for each $p \in R$ and D_p is a subgroup whose volume is maximum among all compact open subgroups of $G_p(\mathbb{Q}_p)$ for almost all $p \in R_f$.*

See Theorem 4.13 below for a more general statement.

Example.

(1) If G_p is $\mathbb{Q}_p$-simple and $\mathbb{Q}_p$-isotropic for each $p \in R$ and $(G_{\mathbb{A}}, D_p)$ admits an irreducible lattice, then all G_p's are typewise homogeneous, that is, their Dynkin types are the same.

(2) Let $n \geq 2$ and $G_p = PGL_n$ for each $p \in R$. Then $(G_{\mathbb{A}}, D_p)$ has an irreducible lattice if and only if D_p is conjugate to $PGL_n(\mathbb{Z}_p)$ for almost all $p \in R_f$.

For $n = 2$, for each $p \in R_f$, there are two conjugacy classes of maximal compact open subgroups of $PGL_2(\mathbb{Q}_p)$, represented by $PGL_2(\mathbb{Z}_p)$ and by

$$L_p = \langle \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in PGL_2(\mathbb{Z}_p) \right\}, \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \rangle$$

respectively. Note that in the Bruhat-Tits tree associated to $PGL_2(\mathbb{Q}_p)$, the conjugacy class of $PGL_2(\mathbb{Z}_p)$ corresponds to the stabilizer of a vertex and the conjugacy class of L_p corresponds to the stabilizer of the middle point of an edge. If we denote by μ_p a Haar measure of $PGL_2(\mathbb{Q}_p)$, then $\mu_p(L_p) = \frac{2}{p+1}\mu_p(PGL_2(\mathbb{Z}_p))$ because the common subgroup $\left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in PGL_2(\mathbb{Z}_p) \right\}$ has index $p+1$ in $PGL_2(\mathbb{Z}_p)$ while it has index 2 in L_p. Hence if D_p is conjugate to L_p for infinitely many primes p, $(G_{\mathbb{A}}, D_p)$ does not admit an irreducible lattice. Furthermore, it follows from Theorem 1.1 and the Hasse principle that, up to automorphism of $(G_{\mathbb{A}}, PGL_2(\mathbb{Z}_p))$, $PGL_2(\mathbb{Q})$ is the unique irreducible non-uniform lattice in $(G_{\mathbb{A}}, PGL_2(\mathbb{Z}_p))$ up to commensurability (see Proposition 6.3 below).

(3) Let $n \geq 1$. If $G_p = PGSp_{2n}$ for each $p \in R$, then $(G_{\mathbb{A}}, D_p)$ admits an irreducible lattice only when D_p is conjugate to $PGSp_{2n}(\mathbb{Z}_p)$ for almost all $p \in R_f$. Up to automorphism of $(G_{\mathbb{A}}, PGSp_{2n}(\mathbb{Z}_p))$, the subgroup $PGSp_{2n}(\mathbb{Q})$ is the unique irreducible non-uniform lattice in $(G_{\mathbb{A}}, PGSp_{2n}(\mathbb{Z}_p))$ up to commensurability (see Proposition 6.3 below).

(4) More generally, if G is a connected absolutely simple $\mathbb{Q}$-group, then for almost all $p \in R_f$, $G(\mathbb{Z}_p)$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$, or equivalently, the volume of $G(\mathbb{Z}_p)$ is the maximum among all compact open subgroups of $G(\mathbb{Q}_p)$ [Ti]. It thus follows from Theorem 1.4 that if we let $G_p = G$, $p \in R$, then $(G_{\mathbb{A}}, D_p)$ admits an irreducible lattice if and only if D_p is conjugate to $G(\mathbb{Z}_p)$ for almost all $p \in R_f$ (see 4.14).

If H is a connected semisimple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group, there exists a pair P_1, P_2 of opposite proper $\mathbb{Q}$-parabolic subgroups of H. Then the subgroup $R_u(P_i)(\mathbb{Q})$ is a (uniform) lattice in $(H_{\mathbb{A}}, H(\mathbb{Z}_p)) \cap \prod_{p\in R} R_u(P_i)(\mathbb{Q}_p)$ where $R_u(P_i)$ denotes the unipotent radical of P_i for each $i = 1, 2$ [Bo1]. The notation $H(\mathbb{Q})^+$ denotes the subgroup generated by all unipotent elements contained in $H(\mathbb{Q})$. If H is almost $\mathbb{Q}$-simple, the subgroup generated by these two lattices $R_u(P_1)(\mathbb{Q})$ and $R_u(P_2)(\mathbb{Q})$ coincides with the subgroup $H(\mathbb{Q})^+$ [BT1]. We also show that a *discrete* subgroup of $(G_{\mathbb{A}}, D_p)$ containing *any* lattices in a pair of opposite horospherical subgroups respectively, is essentially of the form $H(\mathbb{Q})^+$ for H as above, under some additional assumptions on G_∞.

A subgroup U of $(G_{\mathbb{A}}, D_p)$ is called a *horospherical* subgroup if $U = (G_{\mathbb{A}}, D_p) \cap \prod_{p\in R} R_u(P_p)$ where P_p is a proper parabolic $\mathbb{Q}_p$-subgroups of G_p for each $p \in R$. Two horospherical subgroups of $(G_{\mathbb{A}}, D_p)$ are called opposite if the corresponding $\mathbb{Q}_p$-parabolic subgroups are opposite for each $p \in R$. For a subgroup $\Gamma \subset (G_{\mathbb{A}}, D_p)$, we denote by Γ^∞ the image of $\Gamma \cap (G_\infty(\mathbb{R}) \times \prod_{p\in R_f} D_p)$ under the natural projection $pr_\infty : (G_{\mathbb{A}}, D_p) \to G_\infty(\mathbb{R})$.

1.5. Theorem. *For each $p \in R$, assume that G_p has no $\mathbb{Q}_p$-anisotropic factors. Assume that rank $(G_\infty) \geq 2$. Let Γ be a subgroup of $(G_{\mathbb{A}}, D_p)$ containing lattices in a pair of opposite horospherical subgroups of $(G_{\mathbb{A}}, D_p)$. Assume moreover $(*)$ that Γ^∞ is a lattice in $G_\infty(\mathbb{R})$. Then Γ is discrete if and only if there exist a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group H and a topological group isomorphism $f : (H_{\mathbb{A}}, H(\mathbb{Z}_p)) \to (G_{\mathbb{A}}, D_p)$ such that $f(H(\mathbb{Q})^+) \subset \Gamma \subset f(H(\mathbb{Q}))$.*

Remark. The above theorem holds without the assumption $(*)$ provided Margulis's conjecture (see [Oh1, Conjecture 0.1]) holds for G_∞. Indeed, for a discrete subgroup Γ as above, Γ^∞ is a discrete subgroup containing lattices in a pair of opposite horospherical subgroups in $G_\infty(\mathbb{R})$. The conjecture says that any such a discrete subgroup is a lattice in $G_\infty(\mathbb{R})$ as long as the real rank of G_∞ is at least 2. See [Oh, Theorem 4.1] and the remark following it for the list of groups for which the conjecture has been settled. For instance, the list includes groups G_∞ which are split over $\mathbb{R}$ and not locally isomorphic to $SL_3(\mathbb{R})$.

We also remark that in an S-arithmetic setting (S finite), i.e., when $G = \prod_{p\in S} G_p(\mathbb{Q}_p)$, the class of discrete subgroup of G containing lattices in a pair of opposite horospherical subgroups coincides with that of non-uniform lattices in G [Oh1]. In an adelic setting this is no more true, since the subgroup $H(\mathbb{Q})^+$ has infinite index in $H(\mathbb{Q})$ in general (cf. Remark 4.12). However $H(\mathbb{Q})^+$ is contained every subgroup of finite index in $H(\mathbb{Q})$ [BT1].

Naturally one may ask how many irreducible lattices an adele group $(G_{\mathbb{A}}, D_p)$ can admit up to commensurability. By the Hasse principle for an adjoint absolutely simple $\mathbb{Q}$-group, Theorem 1.1 implies that for instance, if for some $p \in R$, G_p is not of type $A_n\,(n \geq 2)$, $D_n\,(n \geq 3)$, or E_6, then $(G_{\mathbb{A}}, D_p)$ admits at most one irreducible non-uniform lattice up to commensurability and up to automorphism of $(G_{\mathbb{A}}, D_p)$ (see Proposition 6.3).

In section 2, we set up some notation as well as state some well known facts about algebraic groups. In section 3, we obtain a necessary and sufficient condition on the collection of distinguished subgroups D_p so that $(G_{\mathbb{A}}, D_p)$ admits a lattice contained in $G(\mathbb{Q})$ (Theorem 3.9). Theorem 1.3 immediately follows from Theorem 4.9 (see Corollary 4.10). For the proof of Theorem 4.9, based on the S-arithmeticity theorem and a special case of super-rigidity theorem of Margulis [Ma2], we first obtain a connected $\mathbb{Q}$-simple $\mathbb{Q}$-group H so that, up to $\mathbb{Q}_p$-anisotropic factors, H is isomorphic to G_p, say via f_p, for each $p \in R$ and $H(\mathbb{Q})$ corresponds to Γ under the product map $\prod_{p\in R} f_p$, up to commensurability. Here we embed Γ into $\prod_{p\in R} f_p(H(\mathbb{Q})) \cap (G_{\mathbb{A}}, D_p)$ as well. Up to this part, the proof proceeds exactly the same way as in the Margulis S-arithmeticity theorem for S finite. The difference in the case of S infinite is to handle the compact subgroups D_p's. To ensure the $\mathbb{Q}_p$-isomorphisms f_p's transfer the $\mathbb{Q}$-structure on H to $(G_{\mathbb{A}}, D_p)$ in a compatible way, we has to show that $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all $p \in R_f$. This is based on the strong approximation property of the simply connected covering of H, explained in Section 3.

Similarly the proof of Theorem 1.5 is based on the result analogous in the S-arithmetic setting obtained by the author [Oh1]. These are explained in section 5. In section 6, we relate the Hasse principle with the set of irreducible non-uniform lattices in $(G_{\mathbb{A}}, D_p)$.

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2. Notation and Terminology

We continue the definitions and the notation mentioned in the introduction.

2.1. For any field k, we mean by a linear algebraic k-group M that M is a Zariski closed k-subgroup of GL_N for some N, and we set $M(J) = M \cap GL_N(J)$ for any subring J of k. The term "k-group" will always mean a linear algebraic group with a fixed realization as a k-closed subgroup of GL_N for some N.

2.2. Let K be a number field. Let R_K, K_v and $\mathcal{O}_v$ be as in the introduction. For any $S \subset R_K$, we set $K(S)$ to be the ring of S-integers in K, that is, $K(S) = \{x \in K \mid x \in \mathcal{O}_v \text{ for all } v \in R_f - S\}$. If $K = \mathbb{Q}$, we simply set $R = R_{\mathbb{Q}}$ and we sometimes write $\mathbb{Z}_S$

for $\mathbb{Q}(S)$.

2.3. For each $v \in R_K$, let G_v be a connected semisimple algebraic K_v-group. Fix a compact open subgroup D_v of $G_v(K_v)$ for each non-archimedean $v \in R_K$. For any subset $T \subset R_K$, we denote by (G_T, D_v) the restricted topological product of the groups $G_v(K_v)$, $v \in T$ with respect to the distinguished subgroups D_v. If T is finite, then $(G_T, D_v) = \prod_{v \in T} G_v(K_v)$, which we will then simply denote by G_T. If $T = R_K$, we write $(G_{\mathbb{A}}, D_v) = (G_T, D_v)$ and call an adele group. The topology on (G_T, D_v) is given as follows: a base of open subsets consists of the sets of the form $\prod_{p \in S} U_v \times \prod_{v \notin S} D_v$ where S is a finite subset of T and $U_v \subset G_v(K_v)$ is an open subset for each $v \in S$. The group (G_T, D_v) is a locally compact group with respect to this topology.

2.4. As is well known, for each $v \in R_K$, the local field K_v is a finite extension of a subfield isomorphic to $\mathbb{Q}_p$ for some (unique) $p \in R$. For each $p \in R$, we denote by I_p the set of valuations $v \in R_K$ such that $\mathbb{Q}_p \subset K_v$ (up to isomorphism). Then $H_p = \prod_{v \in I_p} \mathrm{Rest}_{K_v/\mathbb{Q}_p} G_v$ is a connected semisimple $\mathbb{Q}_p$-group and $\prod_{v \in I_p} G_v(K_v)$ is isomorphic to $H_p(\mathbb{Q}_p)$ as topological groups. We denote this isomorphism by Rest^0. The map Rest^0 also extends to an isomorphism of the adele group $(G_{\mathbb{A}}, D_v)$ with the adele group $(H_{\mathbb{A}}, M_p)$ where $M_p = \prod_{v \in I_p} \mathrm{Rest}_{K_v/\mathbb{Q}_p} D_v$ (cf. [Ma2, Ch I, 3.1.4] and [Bo1]).

2.5. If G is a connected semisimple K-group, then we mean by $(G_{\mathbb{A}}, G(\mathcal{O}_v))$ the adele group attached to the groups $G_v = G$ with respect to the subgroups $G(\mathcal{O}_v)$. The diagonal embedding of $G(K)$ into $(G_{\mathbb{A}}, G(\mathcal{O}_v))$, which we will identify with $G(K)$, is a lattice in $(G_{\mathbb{A}}, G(\mathcal{O}_v))$ [Bo1]. Denote by $\mathcal{T}(G)$ the set of all $v \in R_K$ such that $G(K_v)$ is compact. Then $\mathcal{T}(G)$ is finite (cf. [Ma2, Ch I, 3.2.3]). It then follows that if $T \subset R_K$ contains all archimedean valuations in $R_K - \mathcal{T}(G)$, the subgroup $G(K(T))$ is a lattice in $(G_T, G(\mathcal{O}_v))$ when diagonally embedded into $(G_T, G(\mathcal{O}_v))$. Note that

$$G(K(T)) = \{x \in G(K) \mid x \in G(\mathcal{O}_v) \text{ for all non-archimedean } v \in R_K - T\}.$$

The group G is K-isotropic if and only if $G(K(T))$ is a non-uniform lattice in $(G_T, G(\mathcal{O}_v))$.

2.6. For each $p \in R$, we denote by pr_p the natural projection $(G_{\mathbb{A}}, D_p) \to G_p(\mathbb{Q}_p)$. For any $T \subset R$, the notation pr_T denotes the natural projection $(G_{\mathbb{A}}, D_p) \to (G_T, D_p)$. We set

$$G_{\mathbb{A}(T)} = \{(g_p) \in G_{\mathbb{A}} \mid g_p \in D_p \text{ for all } p \in R_f - T\}.$$

For any subgroup H of $(G_{\mathbb{A}}, D_p)$ and for any subset $T \subset R$, we set

$$H^T = pr_T\left(H \cap G_{\mathbb{A}(T)}\right).$$

2.7. For any finite $S \subset R$, let G_p be a connected semisimple algebraic $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors for each $p \in S$. A lattice Λ in $G_S = \prod_{p\in S} G_p(\mathbb{Q}_p)$ is called *irreducible* if for any two connected normal subgroups $H = \prod_{p\in S} H_p(\mathbb{Q}_p)$ and $M = \prod_{p\in S} M_p(\mathbb{Q}_p)$ of G_S such that for each $p \in S$, G_p is an almost direct product of connected normal $\mathbb{Q}_p$-subgroups M_p and H_p, $(\Lambda \cap H) \cdot (\Lambda \cap M)$ has infinite index in Λ (cf. [Ma2, Ch III, 5.9]). This is equivalent to saying that Γ cannot be an almost direct product of two infinite normal subgroups of Γ.

2.8. For a connected semisimple $\mathbb{Q}_p$-subgroup G_p, we denote by G_p^i (i standing for isotropic) the maximal connected normal subgroup of G_p without any $\mathbb{Q}_p$-anisotropic factors. Note that $G_p(\mathbb{Q}_p)/G_p^i(\mathbb{Q}_p)$ is compact. That G_p has no $\mathbb{Q}_p$-anisotropic factors is equivalent to saying that $G_p(\mathbb{Q}_p)$ has no compact factors.

2.9. Let G_p be a connected semisimple algebraic $\mathbb{Q}_p$-group for each $p \in R$. Fix a compact open subgroup D_p of $G_p(\mathbb{Q}_p)$ for each $p \in R_f$. Let $T \subset R$. A lattice Γ in (G_T, D_p) is called *irreducible* if the projection of Γ^S into $\prod_{p\in S} G_p(\mathbb{Q}_p)^i$ is an irreducible lattice in $\prod_{p\in S} G_p(\mathbb{Q}_p)^i$ for any finite subset $S \subset T$ containing ∞.

2.10. For a connected semisimple algebraic K-group G, the notation $G(K)^+$ denotes the normal subgroup of $G(K)$ generated by the subgroups $R_u(P)(K)$ where P runs through the set of all parabolic K-subgroups of G and $R_u(P)$ denotes the unipotent radical of P. Or equivalently $G(K)^+$ denotes the subgroup generated by all unipotent elements in $G(K)$ [BT1].

If G is almost K-simple and K-isotropic, $G(K)^+$ coincides with the subgroup generated by $R_u(P_1)(K)$ and $R_u(P_2)(K)$ for any pair P_1, P_2 of opposite proper parabolic K-subgroups [BT1]. Recall that two parabolic subgroups are called opposite if their intersection is a common Levi subgroup in both of them.

2.11. We refer to [PR], [Bo1] or [Ma2, Ch I] as a general reference to our terminology regarding algebraic groups.

3. Distinguished subgroups D_p

3.1. Lemma. *Let G_p be a connected semisimple $\mathbb{Q}_p$-group for each $p \in R$. Fix a compact open subgroup D_p of $G_p(\mathbb{Q}_p)$ for each $p \in R_f$. Let S be a finite subset of R. Assume that $\infty \in S$ if $G_\infty(\mathbb{R})$ is non-compact.*

(1) *If Γ is a discrete subgroup in $(G_{\mathbb{A}}, D_p)$, then Γ^S is a discrete subgroup in G_S.*
(2) *If Γ is a (uniform) lattice in $(G_{\mathbb{A}}, D_p)$, then Γ^S is a (uniform) lattice in G_S.*

Proof. For simplicity, set $G_{\mathbb{A}} = (G_{\mathbb{A}}, D_p)$. Denote by pr^S the restriction of pr_S to the subgroup $G_{\mathbb{A}(S)}$. Since the kernel of pr^S is compact, the subgroup Γ^S is discrete if Γ is so. Let Γ be a (uniform) lattice in $G_{\mathbb{A}}$. Since $G_{\mathbb{A}(S)}$ is an open subgroup of $G_{\mathbb{A}}$, the intersection $\Gamma \cap G_{\mathbb{A}(S)}$ is a (uniform) lattice in $G_{\mathbb{A}(S)}$. Consider the natural map $\tilde{pr}^S : G_{\mathbb{A}(S)}/(\Gamma \cap G_{\mathbb{A}(S)}) \to G_S/\Gamma^S$ induced by pr^S. Now if μ is an invariant measure on $G_{\mathbb{A}(S)}/(\Gamma \cap G_{\mathbb{A}(S)})$, then $\tilde{pr}^S_*(\mu)$ is an invariant measure on G_S/Γ^S. Hence if Γ is a uniform lattice in $G_{\mathbb{A}}$, then Γ^S is a uniform lattice in G_S. □

3.2. We recall the following corollary of (a special case of) the strong approximation theorem:

Theorem. *Let G be a connected semisimple simply connected almost $\mathbb{Q}$-simple group. Let S be a finite subset of R such that $\prod_{p\in S} G_p(\mathbb{Q}_p)$ is non-compact. Then $G(\mathbb{Z}_S)$ is dense in the direct product $\prod_{p\in R_f - S} G(\mathbb{Z}_p)$, when diagonally embedded. In particular if Δ is a subgroup of finite index in $G(\mathbb{Z}_S)$, then the closure of Δ has finite index in $\prod_{p\in R_f - S} G(\mathbb{Z}_p)$.*

Proof. Let $G_{\mathbb{A}_S}$ denote the restricted topological product of the $G_p(\mathbb{Q}_p)$ for $p \in R_f - S$ with respect to the subgroups $G(\mathbb{Z}_p)$, $p \in R_f - S$.

By the strong approximation theorem (cf. [PR, Theorem 7.12, P. 427]), $G(\mathbb{Q})$ is dense in $G_{\mathbb{A}_S}$. Since $\prod_{p\in R_f-S} G(\mathbb{Z}_p)$ is an open subgroup of $G_{\mathbb{A}_S}$, $G(\mathbb{Q}) \cap \prod_{p\in R_f-S} G(\mathbb{Z}_p)$ is dense in $\prod_{p\in R_f-S} G(\mathbb{Z}_p)$. Since $G(\mathbb{Z}_S) = G(\mathbb{Q}) \cap \prod_{p\in R_f-S} G(\mathbb{Z}_p)$, this proves the first claim. For the second claim, it suffices to note that $[\overline{G(\mathbb{Z}_S)} : \overline{\Delta}] \leq [G(\mathbb{Z}_S) : \Delta]$ by the first claim. □

3.3. Proposition. *Let $p \in R_f$. Let G be a connected semisimple $\mathbb{Q}_p$-group and let K_1 and K_2 be maximal compact subgroups of $G(\mathbb{Q}_p)$. Assume that there exists a maximal compact subgroup K of $\tilde{G}(\mathbb{Q}_p)$ such that $\pi(K) \subset K_1 \cap K_2$ where $\tilde{G}$ is the simply connected covering of G and $\pi : \tilde{G} \to G$ is the $\mathbb{Q}_p$-isogeny. Then $K_1 = K_2$.*

Proof. Consider the Bruhat-Tits building $\mathcal{B}$ attached to G. In the following proof, we use some results in [Ti] without repeating reference. The group $\tilde{G}(\mathbb{Q}_p)$ acts on $\mathcal{B}$ through the map π. For each $i = 1, 2$, the maximal compact subgroup K_i is the stabilizer $G(\mathbb{Q}_p)^{x_i}$ in $G(\mathbb{Q}_p)$ of some point $x_i \in \mathcal{B}$. Since $\tilde{G}$ is simply connected, there exists a vertex $v \in \mathcal{B}$ such that $\tilde{G}(\mathbb{Q}_p)^v = K$. We claim that $v = x_1 = x_2$, which implies that $K_i \subset G(\mathbb{Q}_p)^v$ for both $i = 1$ and 2. Since K_1 and K_2 are maximal compact subgroups, this implies that $K_1 = K_2 = G(\mathbb{Q}_p)^v$. Suppose that $v \neq x_i$ for some $i \in \{1, 2\}$. Since $\pi(K)$ stabilizes v and x_i, it stabilizes pointwisely the unique geodesic l joining v and x_i. Since $v \neq x_i$, we can find a facet $\mathcal{F}$ whose closure contains v and $\mathcal{F} \cap l$ is non-empty. Fix $z \in \mathcal{F} \cap l$. Note that the dimension of $\mathcal{F}$ is positive. Since $\tilde{G}$ is simply connected, $\tilde{G}(\mathbb{Q}_p)^z = \tilde{G}(\mathbb{Q}_p)^{\mathcal{F}}$ where $\tilde{G}(\mathbb{Q}_p)^{\mathcal{F}}$ is the pointwise stabilizer of $\mathcal{F}$ in $\tilde{G}(\mathbb{Q}_p)$. Therefore $\pi(K) \subset \tilde{G}(\mathbb{Q}_p)^{\mathcal{F}}$. This is a contradiction, since the stabilizer of a facet of positive dimension in $\tilde{G}(\mathbb{Q}_p)$ cannot be a maximal compact subgroup. This finishes the proof. □

3.4. Lemma [Ti, 3.2]**.** *Let G be a connected semisimple $\mathbb{Q}$-group. Then $G(\mathbb{Z}_p)$ is a maximal compact subgroup for almost all $p \in R_f$.*

3.5. Lemma. *Let G be a connected semisimple $\mathbb{Q}$-group, $\tilde{G}$ the simply connected covering of G and $\pi : \tilde{G} \to G$ the central $\mathbb{Q}$-isogeny. Then for almost all $p \in R_f$, $\pi(\tilde{G}(\mathbb{Z}_p)) \subset G(\mathbb{Z}_p)$.*

See [PR, P. 451] for the proof of Lemma 3.5.

3.6. Proposition. *Let G, $\tilde{G}$ and π be as in Lemma 3.5. For each $p \in R_f$, let D_p be a compact open subgroup of $G(\mathbb{Q}_p)$. Assume that $\pi(\tilde{G}(\mathbb{Z}_p)) \subset D_p$ for almost all $p \in R_f$. Then $D_p \subset G(\mathbb{Z}_p)$ for almost all $p \in R_f$.*

Proof. Since every compact open subgroup of $G(\mathbb{Q}_p)$ is contained in a maximal compact open subgroup, we may assume that D_p is a maximal compact open subgroup for all $p \in R_f$. By Lemmas 3.4 and 3.5, there exists a finite subset $S \subset R_f$ such that for each $p \in R_f - S$, both D_p and $G(\mathbb{Z}_p)$ are maximal compact subgroups of $G(\mathbb{Q}_p)$ and $\pi(\tilde{G}(\mathbb{Z}_p)) \subset D_p \cap G(\mathbb{Z}_p)$. Applying Proposition 3.3, we have $D_p = G(\mathbb{Z}_p)$ for all $p \in R_f - S$, proving the claim. □

3.7. The following is a special case of [Ma2, Ch IX, 4.15].

Theorem. *Let G be a connected semisimple almost $\mathbb{Q}$-simple $\mathbb{Q}$-group. Let S be a finite subset of R such that $\sum_{p \in S} \text{rank}_{\mathbb{Q}_p} G \geq 2$. If $G(\mathbb{R})$ is non-compact, we assume that $\infty \in S$. If $\Gamma \subset G(\mathbb{Q})$ and Γ is a lattice in G_S (when diagonally embedded), then Γ and $G(\mathbb{Z}_S)$ are commensurable. In particular Γ contains a subgroup of finite index in $G(\mathbb{Z}_S)$.*

3.8. Set $G(\mathbb{Q}) \cap (G_{\mathbb{A}}, D_p) = \{x \in G(\mathbb{Q}) \mid x \in D_p$ for almost all $p \in R_f\}$. We identify this set with its image under the diagonal embedding into $(G_{\mathbb{A}}, D_p)$.

Theorem. *Let G and S be as in Lemma 3.7. Let D_p be a compact open subgroup of $G(\mathbb{Q}_p)$ for each $p \in R_f$. If Γ is a closed subgroup of $G(\mathbb{Q}) \cap (G_{\mathbb{A}}, D_p)$ such that Γ^S is a lattice in $G_S = \prod_{p \in S} G(\mathbb{Q}_p)$, then $\pi(\tilde{G}(\mathbb{Z}_p)) \subset D_p \subset G(\mathbb{Z}_p)$ for almost all $p \in R_f$, where $\tilde{G}$ and π are as in Lemma 3.5. Furthermore if Γ is a lattice in $(G_{\mathbb{A}}, D_p)$, then $D_p = G(\mathbb{Z}_p)$ for almost all $p \in R_f$.*

Proof. We first consider the case when G is simply connected. By Theorem 3.7, Γ^S contains a subgroup of finite index in $G(\mathbb{Z}_S)$. Let Δ denote the diagonal embedding of the subgroup $\Gamma^S \cap G(\mathbb{Z}_S)$ into $\prod_{p\in R_f - S} G(\mathbb{Z}_p)$. Note that $\Delta \subset \prod_{p\in R_f-S}(G(\mathbb{Z}_p)\cap D_p) \subset \prod_{p\in R_f-S} G(\mathbb{Z}_p)$. By Theorem 3.2, the closure $\overline{\Delta}$ of Δ is a compact open subgroup in $\prod_{p\in R_f-S} G(\mathbb{Z}_p)$, and hence $G(\mathbb{Z}_p) \subset pr_p(\overline{\Delta})$ for almost all $p \in R_f$. On the other hand, since $\prod_{p\in R_f-S}(G(\mathbb{Z}_p)\cap D_p)$ is compact, we have $pr_p(\overline{\Delta}) \subset G(\mathbb{Z}_p)\cap D_p$ for each $p \in R_f - S$. Therefore $G(\mathbb{Z}_p) = D_p$ for almost all $p \in R_f$ by Lemma 3.4.

If G is not simply connected, consider the simply connected covering $\tilde{G}$ and the $\mathbb{Q}$-isogeny $\pi : \tilde{G} \to G$. Denote by π_p the restriction of π to $\tilde{G}(\mathbb{Q}_p)$. By Lemma 3.5, we have that $\tilde{G}(\mathbb{Z}_p) = \pi_p^{-1}G(\mathbb{Z}_p)$ for almost all $p \in R_f$. Set $\tilde{\Gamma} = \tilde{G}(\mathbb{Q})\cap(\tilde{G}_{\mathbb{A}}, \pi_p^{-1}(D_p))$. Since the kernel of $\prod_{p\in S}\pi_p$ is finite, it is clear that $\tilde{\Gamma}^S$ is a lattice in $\prod_{p\in S}\tilde{G}(\mathbb{Q}_p)$. Therefore, by the previous simply connected case, we have $\pi_p^{-1}(D_p) = \tilde{G}(\mathbb{Z}_p)$; hence $\pi_p(\tilde{G}(\mathbb{Z}_p)) \subset D_p$ for almost all $p \in R_f$. Therefore for almost $p \in R_f$, $\pi_p(\tilde{G}(\mathbb{Z}_p)) \subset D_p \subset G(\mathbb{Z}_p)$ by Proposition 3.6.

Now assume that Γ is a lattice in $(G_{\mathbb{A}}, D_p)$. Recall that $G(\mathbb{Q})$ is a lattice in $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$. Denote by μ_1 and μ_2 the Haar measures on $(G_{\mathbb{A}}, D_p)$ and $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$ normalized so that $\mu_1(D_p) = 1$ and $\mu(G(\mathbb{Z}_p)) = 1$ for all $p \in R_f$, respectively. For each finite $S \subset R$ containing ∞, Γ^S is a subgroup of finite index in $G(\mathbb{Z}_S)$ and both are lattices in G_S. Hence $\mu_2(G_S/\Gamma^S) \geq \mu_2(G_S/G(\mathbb{Z}_S))$. Note that

$$\mu_2(G_S/\Gamma^S) = \frac{\mu_1(G_S/\Gamma^S)}{\prod_{p\in R_f\cap S}[G(\mathbb{Z}_p) : D_p]}.$$

For any increasing sequence S_i, $i = 1, 2, \cdots$ such that $R = \cup_i S_i$, the measures $\mu_2(G_{S_i}/G(\mathbb{Z}_{S_i}))$ and $\mu_1(G_{S_i}/\Gamma^{S_i})$ converge to the measures $\mu_2((G_{\mathbb{A}}, G(\mathbb{Z}_p))/G(\mathbb{Q}))$ and $\mu_1((G_{\mathbb{A}}, D_p)/\Gamma)$ respectively. Hence $\lim_{i\to\infty}\prod_{p\in R_f\cap S_i}[G(\mathbb{Z}_p) : D_p]$ should be bounded. Hence $D_p = G(\mathbb{Z}_p)$ for almost all $p \in R_f$. □

3.9. Theorem. *Let G be a connected semisimple almost $\mathbb{Q}$-simple group and D_p a compact open subgroup of $G(\mathbb{Q}_p)$ for each $p \in R_f$. If G is not simply connected, assume that D_p is a maximal compact open subgroup for almost all $p \in R_f$. Then the following are equivalent.*

1. *The adele group $(G_{\mathbb{A}}, D_p)$ admits a lattice contained in $G(\mathbb{Q}) \cap (G_{\mathbb{A}}, D_p)$.*
2. *There exists a finite subset $S \subset R$ such that $\sum_{p\in S}$ rank $_{\mathbb{Q}_p} G \geq 2$, and Γ^S is a lattice in $\prod_{p\in S} G(\mathbb{Q}_p)$ where $\Gamma = G(\mathbb{Q}) \cap (G_{\mathbb{A}}, D_p)$.*
3. *$D_p = G(\mathbb{Z}_p)$ for almost all $p \in R_f$.*

Proof. To show (1) $\Rightarrow$ (2), since $G(\mathbb{Q}_p)$ is non-compact for almost all $p \in R$ (see 2.5), there exists a finite subset $S \subset R$ such that $\sum_{p\in S} \text{rank}_{\mathbb{Q}_p} G \geq 2$. If $G_\infty(\mathbb{R})$ is non-compact, we assume $\infty \in S$. Let Δ be a lattice in $(G_{\mathbb{A}}, D_p)$ contained in $G(\mathbb{Q})$. It is not difficult to check that Γ is a discrete subgroup of $(G_{\mathbb{A}}, D_p)$. Since $\Delta \subset \Gamma$, the subgroup Γ is a lattice in $(G_{\mathbb{A}}, D_p)$ as well. Hence Γ^S is a lattice in $\prod_{p\in S} G(\mathbb{Q}_p)$ by Lemma 3.1. The direction (2) $\Rightarrow$ (3) follows from Theorem 3.8. If (3) holds, then $(G_{\mathbb{A}}, D_p) = (G_{\mathbb{A}}, G(\mathbb{Z}_p))$ in the sense that the identity map provides a topological group isomorphism between them, and hence $G(\mathbb{Q})$ is a lattice in $(G_{\mathbb{A}}, D_p)$. □

3.10. The second claim in Theorem 3.8 combined with Theorem 3.9 yields the following:

Theorem. *Let G be a connected semisimple almost $\mathbb{Q}$-simple group and D_p a compact open subgroup of $G(\mathbb{Q}_p)$ for each $p \in R_f$. Then the following are equivalent.*

1. *The adele group $(G_{\mathbb{A}}, D_p)$ admits a lattice contained in $G(\mathbb{Q}) \cap (G_{\mathbb{A}}, D_p)$.*
2. *$D_p = G(\mathbb{Z}_p)$ for almost all $p \in R_f$.*

4. Rationality of an irreducible lattice in $G_{\mathbb{A}}$

4.1. In the following we give two definitions of a rational lattice. It is convenient for our purpose to understand the equivalence of the two definitions.

In both definitions, let $T \subset R$ and let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors for each $p \in T$ and D_p a compact open subgroup of $G(\mathbb{Q}_p)$ for almost all finite $p \in T$.

Definition A. An irreducible lattice Γ in (G_T, D_p) is called a *rational* lattice if there exist:

1. a connected semisimple adjoint $\mathbb{Q}$-simple $\mathbb{Q}$-group H;
2. if $\infty \notin T$, $H(\mathbb{R})$ is compact;
3. for each $p \in T$, a decomposition $H = H_p^1 \times H_p^2$ where H_p^1 and H_p^2 are connected semisimple adjoint $\mathbb{Q}_p$-groups;
4. for each $p \in T$, a maximal compact open subgroup $M_p \subset H_p^2(\mathbb{Q}_p)$ with $M_p = H(\mathbb{Z}_p) \cap H_p^2(\mathbb{Q}_p)$ for almost all finite $p \in T$; and
5. a family of $\mathbb{Q}_p$-epimorphisms $f_p : H \to G_p$, $p \in T$ with $\ker f_p = H_p^2$ and $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all finite $p \in T$

such that Γ is commensurable with the subgroup

$$f\left(H(\mathbb{Q}(T)) \cap \prod_{p\in T} (H_p^1(\mathbb{Q}_p) \times M_p)\right)$$

where $f = \prod_{p \in T} f_p$.

Definition B. An irreducible lattice in (G_T, D_p) is called a *rational* lattice if there exist a number field K, a connected absolutely simple K-group $\mathcal{H}$ and a subset $B \subset R_K - \mathcal{T}(\mathcal{H})$ containing all archimedean valuations in $R_K - \mathcal{T}(\mathcal{H})$ such that there exists a topological group isomorphism $f : (\mathcal{H}_B, \mathcal{H}(\mathcal{O}_v)) \to (G_T, D_p)$ and $f(\mathcal{H}(K(B))$ is commensurable with Γ.

Remark. When T is finite, Definition A (essentially) coincides with that of an S-arithmetic lattice given in [Zi, Theorem 10.1.12] and Definition B coincides with that of an S-arithmetic lattice given in [Ma2, Ch IX]

4.2. Proposition. *Definitions A and B are equivalent.*

Proof. Assume that Γ is a rational lattice in (G_T, D_p) as in Definition A. For any H as in (1), there exist a number field K and a connected absolutely simple K-group $\mathcal{H}$ such that $H = \mathrm{Rest}_{K/\mathbb{Q}} \mathcal{H}$. For each $p \in T$, let $I_p \subset R_K$ be as in 2.4. Then considering H as a $\mathbb{Q}_p$-group and $\mathcal{H}$ as a K_v-group for each $v \in I_p$, we have the decomposition $H = \prod_{v \in I_p} \mathrm{Rest}_{K_v/\mathbb{Q}_p} \mathcal{H}$ over $\mathbb{Q}_p$. Since $\mathcal{H}$ is a connected absolutely simple K_v-group, each $\mathrm{Rest}_{K_v/\mathbb{Q}_p} \mathcal{H}$ is a connected semisimple adjoint $\mathbb{Q}_p$-simple $\mathbb{Q}_p$-group. For each $p \in T$, we can find a partition of I_p into $I_p^1 \cup I_p^2$ such that $H_p^1 = \prod_{v \in I_p^1} \mathrm{Rest}_{K_v/\mathbb{Q}_p} \mathcal{H}$ and $H_p^2 = \prod_{v \in I_p^2} \mathrm{Rest}_{K_v/\mathbb{Q}_p} \mathcal{H} = \ker f_p$. Set $B = \cup_{p \in T} I_p^1$. Note that if $\infty \in T$, $H_\infty^2(\mathbb{R})$ is compact, since otherwise $H_\infty^2(\mathbb{R})$ does not admit a compact open subgroup. Hence if $v \in R_K$ is an archimedean valuation with $\mathcal{H}(K_v)$ non-compact, then $v \notin I_\infty^2$. That is, I_∞^1 and hence B contains all archimedean valuations in $R_K - \mathcal{T}(\mathcal{H})$. If $\infty \notin T$, $H(\mathbb{R})$ is compact, and hence for all archimedean $v \in I_\infty$, $\mathcal{H}(K_v)$ is compact, that is, $v \in \mathcal{T}(\mathcal{H})$. Since $G_p(\mathbb{Q}_p)$ has no compact factors, $B \subset R_K - \mathcal{T}(\mathcal{H})$.

Since $M_p = H(\mathbb{Z}_p) \cap H_p^2(\mathbb{Q}_p)$ for almost all finite $p \in T$ and M_∞ has finite index in $H_\infty^2(\mathbb{R})$ in the case when $\infty \in T$, the subgroup $\mathrm{Rest}_{K/\mathbb{Q}} \mathcal{H}(K(B))$ is commensurable with

$$\{x \in H(\mathbb{Q}(T)) \mid x \in H_p^1(\mathbb{Q}_p) \times M_p \text{ for each } p \in T\}.$$

Via the map Rest^0 (see 2.4), the group $(\mathcal{H}_B, \mathcal{H}(\mathcal{O}_v))$ is isomorphic to $(H_T^1, H(\mathbb{Z}_p) \cap H^1(\mathbb{Q}_p))$ and the lattice $\mathcal{H}(K(B))$ is mapped to a subgroup commensurable with the subgroup

$$pr_1 \circ \delta_T \left(\{x \in H(\mathbb{Q}(T)) \mid x \in H_p^1(\mathbb{Q}_p) \times M_p \text{ for each } p \in T\}\right)$$

where δ_T denotes the diagonal embedding of $H(\mathbb{Q})$ into $(H_T, H(\mathbb{Z}_p))$ and pr_1 denotes the canonical projection $(H_T, H(\mathbb{Z}_p)) \to (H_T^1, H(\mathbb{Z}_p) \cap H_p^1(\mathbb{Q}_p))$.

Note that $\text{Rest}^0(\prod_{v\in I_p^1}\mathcal{H}(\mathcal{O}_v)) = H(\mathbb{Z}_p)\cap H_p^1(\mathbb{Q}_p)$ for each finite $p\in T$. Hence if f_1 denotes the restriction of the map f to $(H_T^1, H(\mathbb{Z}_p)\cap H^1(\mathbb{Q}_p))$, then $f_1\circ \text{Rest}^0$ is an isomorphism of $(\mathcal{H}_B, \mathcal{H}(\mathcal{O}_v))$ to (G_T, D_p) and the image of $\mathcal{H}(K(B))$ under this isomorphism is commensurable with Γ. Hence Γ is a rational lattice in Definition B as well.

To see the converse, if we let $H = \text{Rest}_{K/\mathbb{Q}}\mathcal{H}$, then H is a connected semisimple adjoint $\mathbb{Q}$-simple $\mathbb{Q}$-group. Let $T=\{p\in R \mid \text{ for some } v\in B,\ K_v \text{ is a finite extension of } \mathbb{Q}_p\}$. Note that $H(\mathbb{R})$ is non-compact if and only if $\mathcal{H}(K_v)$ is non-compact for an archimedean valuation $v\in R_K$. Hence if $H(\mathbb{R})$ is non-compact, then $\infty\in T$, since B contains all archimedean valuations in $R_K-\mathcal{T}(\mathcal{H})$. Let $I_p^1 = I_p\cap B$ and $I_p^2 = I_p - I_p^1$. Set $H_p^1 = \prod_{v\in I_p^1}\text{Rest}_{K_v/\mathbb{Q}_p}\mathcal{H}$ and $H_p^2=\prod_{v\in I_p^2}\text{Rest}_{K_v/\mathbb{Q}_p}\mathcal{H}$. Note that if $\infty\in T$, $H_\infty^2(\mathbb{R})$ is compact. It follows from the lemma below that there exist $\mathbb{Q}_p$-isomorphisms $h_p: H_p^1\to G_p$, $p\in T$ such that $h_p(H_p^1(\mathbb{Q}_p)\cap H_p(\mathbb{Z}_p)) = D_p$ for almost all finite $p\in T$ and $f=\prod_{p\in T}h_p\circ\text{Rest}^0$ where $f:(\mathcal{H}_B,\mathcal{H}(\mathcal{O}_v))\to(G_T,D_p)$ is the given topological group isomorphism and $\text{Rest}^0:\prod_{v\in I_p^1}\mathcal{H}(K_v)\to H_p^1(\mathbb{Q}_p)$ as in 2.4. If pr_p denotes the natural projection $H\to H_p^1$, then the map $f_p = h_p\circ pr_p$ is a $\mathbb{Q}_p$-epimorphism from $H\to G_p$ with $\ker f_p = H_p^2$ and $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all finite $p\in T$. Set $M_p = H(\mathbb{Z}_p)\cap H_p^2(\mathbb{Q}_p)$ for each finite $p\in T$. If $\infty\in T$, set $M_\infty = H_\infty^2(\mathbb{R})$. Then $\mathcal{H}(K(B))$, is commensurable to the subgroup

$$\{x\in\mathcal{H}(K(B_0)) \mid \prod_{v\in I_p}\text{Rest}_{K_v/\mathbb{Q}_p}x\in H_p^1\times M_p \text{ for each } p\in T\}$$

where $B_0=\cup_{p\in T}I_p$. Therefore via the map $\prod_{p\in T}f_p$, Γ is commensurable to

$$\{x\in H(\mathbb{Q}(T)) \mid x\in H_p^1(\mathbb{Q}_p)\times M_p \text{ for each } p\in T\}.$$

Hence Γ is rational as in Definition A. □

We formulate the lemma used in the above proof.

4.3. Lemma. *Let $S, T\subset R$. Let G_p, $p\in S$ (resp. H_p, $p\in T$) be connected semisimple adjoint $\mathbb{Q}_p$-groups without any $\mathbb{Q}_p$-anisotropic factors and $M_p\subset G_p(\mathbb{Q}_p)$ (resp. $L_p\subset H_p(\mathbb{Q}_p)$) compact open subgroups for each finite prime $p\in S$ (resp. $p\in T$). Assume that M_p and L_p are maximal compact subgroups for almost all finite $p\in S$. If $f:(G_S, M_p)\to(H_T, L_p)$ is a topological group isomorphism, then $S=T$, there exist $\mathbb{Q}_p$-isomorphisms $f_p: G_p\to H_p$, $p\in S$ such that $f_p(M_p)=L_p$ for almost all finite $p\in S$ and $f(x)=\prod_{p\in S}f_p(x)$ for any $x\in(G_S, M_p)$.*

Proof. For each $p \in S$ and $r \in T$, consider the map $f_{pr} : G_p(\mathbb{Q}_p) \to H_r(\mathbb{Q}_r)$ defined by $f_{pr}(g) = pr_r(f(g))$ for each $g \in G_p(\mathbb{Q}_p)$. Here $pr_r : (H_T, L_p) \to H_r(\mathbb{Q}_r)$ denotes the natural projection map. Then f_{pr} is a continuous homomorphism for each p and r. Since f is an isomorphism, for each $p \in S$, $f_{pr}(G_r(\mathbb{Q}_r)) \neq \{e\}$ for some $r \in T$. By [Ma2, Ch I, Proposition 2.6.1], we have $f_{pr}(G_r(\mathbb{Q}_r)) \neq \{e\}$ if and only if $p = r$, and when $p = r$, the topological group isomorphism $f_{pp} : G_p(\mathbb{Q}_p) \to H_p(\mathbb{Q}_p)$ extends to a rational $\mathbb{Q}_p$-isomorphism $f_p : G_p \to H_p$. It follows that $S = T$. Since the restriction of f to $(G_{S\cap R_f}, M_p)$ induces an isomorphism $f' : (G_{S\cap R_f}, M_p) \to (H_{T\cap R_f}, L_p)$, the image $f'(\prod_{p\in S\cap R_f} M_p)$ is an open compact subgroup of $(H_{T\cap R_f}, L_p)$. Since $f'(\prod_{p\in S\cap R_f} M_p) = \prod_{p\in S\cap R_f} f_p(M_p)$ and $f_p(M_p) \subset H_p(\mathbb{Q}_p)$ for each $p \in S \cap R_f$, $L_p \subset f_p(M_p)$ for almost all finite $p \in S$. Since M_p and L_p are maximal compact for almost all finite $p \in S$, we have that $L_p = f_p(M_p)$ for almost all finite $p \in S$. □

4.4. Margulis's S-arithmeticity theorem states:

Theorem. *Let $S \subset R$ be a finite subset and let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors for each $p \in S$. If $\sum_{p\in S} rank_{\mathbb{Q}_p}(G_p) \geq 2$, any irreducible lattice in G_S is an S-arithmetic lattice in G_S.*

See [Ma2, Ch IX, Theorem 1.11 and the remark 1.3. (iii)] or [Zi, Theorem 10.1.12].

4.5. Before we give a proof of rationality theorem which works for uniform and non-uniform lattices simultaneously, we give an instructive simpler proof for an irreducible non-uniform lattice assuming that G_∞ is absolutely simple. Theorem 1.1 immediately follows from the following:

Theorem. *For each $p \in R$, let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors. For each $p \in R_f$, let $D_p \subset G_p(\mathbb{Q}_p)$ be a compact open subgroup. Assume that G_∞ is absolutely simple. Fix a finite subset $S_0 \subset R$ containing ∞ such that $\sum_{p\in S_0} rank_{\mathbb{Q}_p}(G_p) \geq 2$. Let Γ be a subgroup of $(G_{\mathbb{A}}, D_p)$ such that Γ^S is an irreducible non-uniform lattice in G_S for any finite $S \subset T$ including S_0. Then there exist a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group H and a $\mathbb{Q}_p$-isomorphism $f_p : H \to G_p$ for each $p \in R$ with $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all $p \in R_f$ such that $\Gamma \subset f(H(\mathbb{Q}))$ where f is the restriction of $\prod_{p\in R} f_p$ to $(H_{\mathbb{A}}, H(\mathbb{Z}_p))$.*

Proof. Set $\Omega = \{S \subset R \mid S_0 \subset S, |S| < \infty\}$.

Step 1. Obtain $\mathbb{Q}$-forms H_S for each $S \in \Omega$. For any $S \in \Omega$, by Theorem 4.4 and Definition A, there exist a connected absolutely simple $\mathbb{Q}$-group H_S, $\mathbb{Q}_p$-isomorphisms

$r_{Sp} : H_S \to G_p$, $p \in S$ such that Γ^S is commensurable with the subgroup

$$\{(r_{Sp}(x)) \mid x \in H_S(\mathbb{Z}_S)\}.$$

Since the groups G_p, $p \in S$ and H_S are adjoint, we may assume that Γ_S is a finite index subgroup of $\{(r_{Sp}(x)) \mid x \in H_S(\mathbb{Z}_S)\}$.

Step 2. The $\mathbb{Q}$-forms H_S are all $\mathbb{Q}$-isomorphic. Set $H = H_{S_0\infty}$ and $r = r_{S_0\infty}$. By the assumption and Lemma 3.1, Γ^∞ is a lattice in $G_\infty(\mathbb{R})$ and hence is Zariski dense in G_∞ by Borel density theorem. For any $S \in \Omega$, since $\Gamma^\infty \subset r_{S\infty}(H_S(\mathbb{Q})) \cap r(H(\mathbb{Q}))$ and $r \circ r_{S,\infty}^{-1}(\Gamma^\infty) \subset H_S(\mathbb{Q})$, the map $r \circ r_{S\infty}^{-1} : H \to H_S$ is defined over $\mathbb{Q}$ [Ma2, Ch I, 0.11]. Since both H and H_S are absolutely simple, the map $r \circ r_{S\infty}^{-1}$ is indeed a $\mathbb{Q}$-isomorphism.

Step 3. Define a $\mathbb{Q}_p$-isomorphism $f_p : H \to G_p$ for each $p \in R$. For each $p \in R$, we define a map $f_p : H \to G_p$ by $f_p = r_{Sp} \circ (r_{S\infty})^{-1} \circ r$ for any $S \in \Omega$ containing p. To show that this is independent of the choice of S, we claim that for any $p \in R$ and for any S_1, $S_2 \in \Omega$ such that $p \in S_1 \cap S_2$, $r_{S_1p} \circ r_{S_1\infty}^{-1} = r_{S_2p} \circ r_{S_2\infty}^{-1}$. Since

$$\Gamma^{\infty,p} \subset \{(r_{S_1\infty}(x), r_{S_1p}(x)) \mid x \in H_{S_1}(\mathbb{Q})\} \cap \{(r_{S_2\infty}(x), r_{S_2p}(x)) \mid x \in H_{S_2}(\mathbb{Q})\},$$

we have that $r_{S_1p} \circ r_{S_1\infty}^{-1}(z) = r_{S_2p} \circ r_{S_2\infty}^{-1}(z)$ for any $z \in pr_\infty(\Gamma^{\{\infty,p\}})$. Since $pr_\infty(\Gamma^{\{\infty,p\}})$ is a Zariski dense subset in G_∞, $r_{S_1p} \circ r_{S_1\infty}^{-1} = r_{S_2p} \circ r_{S_2\infty}^{-1}$ and hence the map f_p is well defined for each $p \in R$. Since r_{Sp} is a $\mathbb{Q}_p$-isomorphism and $(r_{S\infty})^{-1} \circ r$ is a $\mathbb{Q}$-isomorphism, f_p is a $\mathbb{Q}_p$-isomorphism.

Step 4. Show $\Gamma \subset f(H(\mathbb{Q}))$ where $f = \prod_{p\in R} f_p$. We now claim that $\Gamma \subset f(H(\mathbb{Q}))$ where $f = \prod_{p\in R} f_p$ and $f(H(\mathbb{Q})) = \{(f_p(x)) \mid x \in H(\mathbb{Q})\}$. It suffices to show that $\Gamma^S \subset f_S(H(\mathbb{Q})) = \{(f_p(x))_{p\in S} \mid x \in H(\mathbb{Q})\}$ for each $S \in \Omega$. But $\Gamma^S \subset \{(r_{Sp}(x))_{p\in S} \mid x \in H_S(\mathbb{Q})\}$. If $x \in H_S(\mathbb{Q})$, then there exists a unique $y \in H(\mathbb{Q})$ such that $x = r_{S\infty}^{-1} \circ r(y)$. Hence $r_{Sp}(x) = f_p(y)$ for each $p \in S$. Therefore $\Gamma^S \subset f_S(H(\mathbb{Q}))$ for any $S \in \Omega$.

Step 5. Show $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all $p \in R_f$. The product map f induces a topological group isomorphism from $(H_\mathbb{A}, f_p^{-1}(D_p))$ to $(G_\mathbb{A}, D_p)$. Note that $f^{-1}(\Gamma) \subset H(\mathbb{Q}) \cap (H_\mathbb{A}, f_p^{-1}(D_p))$. Since $f^{-1}(\Gamma)^S$ is a lattice in $\prod_{p\in S} H(\mathbb{Q}_p)$ for any $S \in \Omega$, by Theorem 3.10, we have $f_p^{-1}(D_p) = H(\mathbb{Z}_p)$, or equivalently $f_p(H(\mathbb{Z}_p)) = D_p$, for almost all $p \in R_f$. This finishes the proof. □

4.6. The proof of Theorem 1.3 is more involved in general cases. We will need the following preparation before giving its proof. In view of the equivalence of the two definitions given in 4.1, the following is a direct corollary of [Ma2, Ch VIII, Theorem 3.6]:

Proposition. *Let H be a connected semisimple adjoint $\mathbb{Q}$-simple $\mathbb{Q}$-group. Let S be a finite subset of R. Assume that $\infty \in S$ if $H(\mathbb{R})$ is non-compact. For each $p \in S$, let $H = H_p^1 \times H_p^2$ where the subgroups H_p^1 and H_p^2 are connected normal $\mathbb{Q}_p$-subgroups of H, H_p^1 has no $\mathbb{Q}_p$-anisotropic factors and $M_p \subset H_p^2(\mathbb{Q}_p)$ a compact open subgroup. Let F be a connected adjoint semisimple $\mathbb{Q}$-group, Λ a subgroup of H commensurable with $\{x \in H(\mathbb{Q}(S)) \mid x \in (H_p^1(\mathbb{Q}_p) \times M_p)$ for each $p \in S\}$ and $\delta : \Lambda \to F(\mathbb{Q})$ a homomorphism with a Zariski dense image in F. Assume that $\sum_{p \in S} rank_{\mathbb{Q}_p}(H_p^1) \geq 2$. Then there exists a (unique) $\mathbb{Q}$-isomorphism $j : H \to F$ which extends δ.*

4.7. Lemma. *Let S and G_p be as on Theorem 4.4. If Γ is an irreducible lattice in G_S. Then for any $p \in S$, the restriction of $pr_p : G_S \to G_p$ to Γ is injective.*

Proof. If $S = \{p\}$, the statement is trivial. Suppose not. Set $N = \{\gamma \in \Gamma \mid pr_p(\gamma) = e\}$. Then N is a normal subgroup of Γ. Since the lattice Γ is irreducible, the image of Γ under pr_p is infinite. Hence N is not commensurable with Γ. By Margulis's normal subgroup theorem [Ma2, Ch VIII, Theorem 2.6], N is contained in the center of G_S. Since the groups G_p are adjoint, the center of G_S is trivial, proving the claim. □

4.8. Lemma. *For any $p \in R$, let G be a connected reductive $\mathbb{Q}_p$-group. Then any compact open subgroup of $G_p(\mathbb{Q}_p)$ is contained in only a finitely number of compact subgroups of $G_p(\mathbb{Q}_p)$.*

Proof. If $G(\mathbb{R})$ contains a compact open subgroup, say U, it follows that $G(\mathbb{R})$ itself is compact and U has a finite index in $G(\mathbb{R})$. Hence the claim follows. For a finite prime p, see [PR, Proposition 3.6, P 136]. □

4.9. We are now ready to prove the main theorem:

Theorem. *Let $T \subset R$. For each $p \in T$, let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors. For almost all finite $p \in T$, let $D_p \subset G_p(\mathbb{Q}_p)$ be a maximal compact open subgroup. Fix a finite subset $S_0 \subset T$ (containing ∞ if $\infty \in T$) such that $\sum_{p \in S_0} rank_{\mathbb{Q}_p}(G_p) \geq 2$. Let Γ be a subgroup of (G_T, D_p) such that Γ^S is an irreducible lattice in G_S for any finite $S \subset T$ including S_0. Then Γ is contained in some rational lattice in (G_T, D_p).*

Proof. Set $p_0 = \infty$ if $\infty \in T$, and otherwise let p_0 be any fixed prime in S_0. Set $\Omega = \{S \subset T \mid S_0 \subset S, |S| < \infty\}$.

Step 1. Obtain the $\mathbb{Q}$-forms H_S, $S \in \Omega$. For each $S \in \Omega$, we denote by H_S, H_{Sp}^1, H_{Sp}^2, M_{Sp}, f_{Sp}, pr_{Sp} as in Theorem 4.4 and Definition A in 4.1. Also set r_{Sp} to be the composition map $f_{Sp}\circ pr_{Sp} : H_S \to G_p$. Since the groups G_p, $p \in S$ and H_S are adjoint, we may assume that Γ^S is a subgroup of finite index in

$$\{(r_{Sp}(x))_{p\in S} \mid x \in H_S(\mathbb{Z}_S) \cap (H^1_{Sp} \times M_{Sp}) \text{ for each } p \in S\}.$$

Step 2. The $\mathbb{Q}$-forms H_S are all $\mathbb{Q}$-isomorphic. Set $H = H_{S_0}$ and $r = r_{S_0 p_0}$. We first claim that the group H_S is $\mathbb{Q}$-isomorphic to H for any $S \in \Omega$. Consider the maps $r : H \to G_{p_0}$ and $r_{Sp_0} : H_S \to G_\infty$. For simplicity, we set $pr_{p_0} = pr_0$. Since $pr_0(\Gamma^S) \subset r_{Sp_0}(H_S(\mathbb{Z}_S))$ for any $S \in \Omega$, the set $r^{-1}_{Sp_0}(pr_0(\Gamma^S)) = \{x \in H_S(\mathbb{Z}_S) \mid r(x) \in pr_{Sp_0}(\Gamma^S)\}$ is contained in $H_S(\mathbb{Z}_S)$.

Since the map r_{Sp_0} is injective over $r^{-1}_{Sp_0}(pr_0(\Gamma^{S_0}))$ by Lemma 4.7, the composition map $r^{-1}_{Sp_0}\circ r$ is well defined on $r^{-1}(pr_0(\Gamma^{S_0}))$, which we denote by $j_{Sp_0} : r^{-1}(pr_0(\Gamma^{S_0})) \to H_S(\mathbb{Q})$. By Proposition 4.6, the map j_{Sp_0} extends to a $\mathbb{Q}$-rational isomorphism $j_S : H \to H_S$, proving our claim.

Step 3. Define $\mathbb{Q}_p$-epimorphisms $f_p : H \to G_p$. For each $p \in T$, we define a map $f_p : H \to G_p$ by $f_p = r_{Sp} \circ j_S$ for any $S \in \Omega$ containing p. To show that this is independent of the choice of S, we claim that for any $p \in T$ and for any S_1, $S_2 \in \Omega$ such that $p \in S_1 \cap S_2$, $r_{S_1 p} \circ j_{S_1} = r_{S_2 p} \circ j_{S_2}$. Note that r is injective over $H(\mathbb{Z}_{S_0})$ by Lemma 4.7. We let $r^{-1}(\Gamma^{S_0}) = \{x \in H(\mathbb{Z}_{S_0}) \mid r(x) \in \Gamma^{S_0}\}$.

Since H_{S_0} is $\mathbb{Q}$-simple, it follows that $r^{-1}(pr_0(\Gamma^{S_0}))$ is Zariski dense in H_{S_0}. Hence it suffices to verify this equality for any $x \in r^{-1}(pr_0(\Gamma^{S_0}))$. There exists a unique (see Lemma 4.7) element $y \in \Gamma^{S_0}$ such that $r(x) = pr_0(y)$, and there exist elements $z_i \in H_{S_i}(\mathbb{Q})$, $i = 1, 2$ such that $y = pr_{S_0}(r_{S_1 q}(z_1)) = pr_{S_0}(r_{S_2 q}(z_2))$. Again by Lemma 4.7, we have $r_{S_1 p}(z_1) = r_{S_2 p}(z_2)$. Then $r_{S_1 p} \circ j_{S_1}(x) = r_{S_1 p} \circ r^{-1}_{S_1 p_0} \circ r(x) = r_{S_1 p}(z_1)$ which is equal to $r_{S_2 p}(z_2) = r_{S_2 p} \circ r^{-1}_{S_2 p_0} \circ r(x) = r_{S_2 p} \circ j_{S_2}(x)$. This proves our claim, yielding that f_p is well defined for each $p \in T$. Since r_{Sp} is a $\mathbb{Q}_p$-epimorphism and j_S is a $\mathbb{Q}$-isomorphism, f_p is a $\mathbb{Q}_p$-epimorphism.

Step 4. The groups H^1_p, H^2_p and M_p. Note that $\ker f_p$ is a connected semisimple adjoint $\mathbb{Q}_p$-group, as is any connected normal $\mathbb{Q}_p$-subgroup of H. Letting $H^2_p = \ker f_p$ for each $p \in T$, there exists a connected semisimple adjoint $\mathbb{Q}_p$-group H^1_p such that $H = H^1_p \times H^2_p$. Note that the restriction $f_p : H^1_p \to G_p$ is a $\mathbb{Q}_p$-isomorphism. Since j_S is a $\mathbb{Q}$-isomorphism, $H^1_p = j_S^{-1}(H^1_{Sp})$ and $H^2_p = j_S^{-1}(H^2_{Sp})$ for each $S \in \Omega$ containing p.

We claim that there exists a compact open subgroup, say, M_p, of $H^2_p(\mathbb{Q}_p)$ such that $f_p^{-1}(pr_p(\Gamma^S)) \subset M_p$ for each $S \in \Omega$ containing p. Note that $f_p^{-1}(pr_p(\Gamma^{S_0})) \subset f_p^{-1}(pr_p(\Gamma^S))$ for each $S \in \Omega$ containing p. On the other hand, the latter subgroup is contained in the compact open subgroup $j_S^{-1}(M_{Sp}) \subset H^2_{Sp}(\mathbb{Q}_p)$. Since $f_p^{-1}(pr_p(\Gamma^{S_0})) \subset \cap_{S\in\Omega} j_S^{-1}(M_{Sp})$, $\cap_{S\in\Omega} j_S^{-1}(M_{Sp})$ is a compact open subgroup of $H^2_p(\mathbb{Q}_p)$. Hence by Lemma 4.8, $\cup_{p\in S\in\Omega} j_S^{-1}(M_{Sp}) = j_{S_m}^{-1}(M_{S_m p})$ for some $S_m \in \Omega$. It suffices to set M_p to be a maximal compact open subgroup of $H^2_p(\mathbb{Q}_p)$ containing $j_{S_m}^{-1}(M_{S_m p})$.

Step 5. Show $\Gamma \subset f(H(\mathbb{Q}))$ where $f = \prod_{p\in T} f_p$. We now claim that $\Gamma \subset f(H(\mathbb{Q}) \cap \prod_{p\in T}(H^1_p \times M_p))$ where $f = \prod_{p\in T} f_p$. It suffices to show that $\Gamma^S \subset f_S(H(\mathbb{Q}) \cap \prod_{p\in R}(H^1_p \times M_p))$ for each $S \in \Omega$, where $f_S = \prod_{p\in S} f_p$. For any $\gamma \in \Gamma^S$, we have $\gamma = (r_{Sp}(x))_{p\in S}$ for some $x \in H_S(\mathbb{Q})$ and in the decomposition $x = x^1_p x^2_p$ for $x^1_p \in H^1_{Sp}$ and H^2_{Sp}, we have $x^2_p \in M_p$. Since $x \in H_S(\mathbb{Q})$, then there exists a unique $y \in H(\mathbb{Q})$ such that $x = j_S(y)$. Hence $r_{Sp}(x) = f_p(y)$ and $y = j_S^{-1}(x^1_p) j_S^{-1}(x^2_p)$ where $j_S^{-1}(x^1_p) \in H^1_p$ and $j_S^{-1}(x^2_p) \in j_S^{-1}(\ker pr_{Sp}) \subset M_p$ for each $p \in S$. Therefore $\Gamma^S \subset f_S(H(\mathbb{Q}) \cap \prod_{p\in R}(H^1_p \times M_p))$ for any $S \in \Omega$.

Step 6. $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all finite $p \in T$. For each $p \in T$, set $D'_p = \{x \in H^1_p(\mathbb{Q}_p) \mid f_p(x) \in D_p\}$. For $p \in T$, set $L_p = D'_p \times M_p$ and for $p \in R_f - T$, $L_p = H(\mathbb{Z}_p)$. Consider the adele group $(H_{\mathbb{A}}, L_p)$. Now the subgroup $\{x \in H(\mathbb{Q}) \mid f_T(x) \in \Gamma\}$ satisfies the property (2) in Theorem 3.8 where $f_T = \prod_{p\in T} f_p$. Hence $L_p = H(\mathbb{Z}_p)$ for almost all $p \in R_f$, and hence we have $D'_p \times M_p = H(\mathbb{Z}_p)$ and $M_p = H(\mathbb{Z}_p) \cap H^2_p(\mathbb{Q}_p)$ for almost all finite $p \in T$. Therefore $f_p(H(\mathbb{Z}_p)) = D_p$ for almost all finite $p \in T$. Therefore we have constructed (H, f_p, M_p) as required in Definition A. □

4.10. Corollary. *Let $T \subset R$. For each $p \in T$, let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors. For each finite $p \in T$, let $D_p \subset G_p(\mathbb{Q}_p)$ be a compact open subgroup. If $\sum_{p\in T} \text{rank}_{\mathbb{Q}_p}(G_p) \geq 2$. then any irreducible lattice in (G_T, D_p) is rational.*

Proof. The condition on maximality of D_p's was used only in Step 6 in the above proof. Here instead of referring to Theorem 3.9, it suffices to refer to Theorem 3.10 to deduce $L_p = H(\mathbb{Z}_p)$ for almost all $p \in R_f$. Then the rest proceeds exactly the same way. □

4.11. Without the assumption of G_p being adjoint, we can deduce the following from Theorem 4.5 and Theorem 3.9:

Proposition. *For each $p \in R$, let G_p be a connected semisimple $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors and let G_∞ be absolutely almost simple. Let D_p be a compact open subgroup of $G_p(\mathbb{Q}_p)$ for each $p \in R_f$. If Γ is an irreducible non-uniform lattice in $(G_{\mathbb{A}}, D_p)$, then there exists a connected absolutely simple $\mathbb{Q}$-group H and a $\mathbb{Q}_p$-isogeny $f_p : G_p \to H$ for each $p \in R$ such that $\pi(\tilde{H}(\mathbb{Z}_p)) \subset f_p(D_p) \subset H(\mathbb{Z}_p)$ for almost $p \in R$*

and $\prod_{p\in R} f_p(\Gamma) \subset H(\mathbb{Q})$ *where* $\tilde{H}$ *is the simply connected covering of* H *and* $\pi : \tilde{H} \to H$ *is the* $\mathbb{Q}$*-isogeny.*

Example. Let $n \geq 2$ and $G_p = SL_n$ for each $p \in R$. Let D_p be a (not necessarily maximal) compact open subgroup of $SL_n(\mathbb{Q}_p)$ for each $p \in R$. If $(G_{\mathbb{A}}, D_p)$ has an non-uniform irreducible lattice, then for almost all $p \in R_f$, D_p is conjugate to $SL_n(\mathbb{Z}_p)$ by an element of $GL_n(\mathbb{Q}_p)$.

4.12. Remark. We remark that the subgroup Γ need not be a lattice in $G_{\mathbb{A}}$ to satisfy the assumptions in Theorem 4.5 or 4.9. Let G be a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group. If $G(\mathbb{Q})^+ \subset \Lambda \subset G(\mathbb{Q})$, Λ^S is an irreducible lattice in G_S for any finite set S containing ∞ such that $\sum_{p\in S} \text{rank}_{\mathbb{Q}_p}(G_p) \geq 2$. Indeed, Λ^S is a discrete subgroup of G_S such that ${G(\mathbb{Q})^+}^S \subset \Lambda^S \subset {G(\mathbb{Q})}^S$. Note that $G(\mathbb{Z}_S) = {G(\mathbb{Q})}^S$ and ${G(\mathbb{Q})^+}^S$ is an infinite normal subgroup of $G(\mathbb{Z}_S)$. Hence by Margulis's normal subgroup theorem, ${G(\mathbb{Q})^+}^S$ has finite index in $G(\mathbb{Z}_S)$ ([Ma2, Ch VIII, Theorem 2.6]). Therefore Λ^S is a lattice in G_S. From the assumption that G is absolutely simple, the subgroup $G(\mathbb{Z}_S)$ and hence Λ^S is an *irreducible* lattices in G_S.

However $G(\mathbb{Q})^+$ does not have finite index in $G(\mathbb{Q})$ in general. If we denote by $\tilde{G}$ the simply connected covering of G and $\pi : \tilde{G} \to G$ is the $\mathbb{Q}$-isogeny, then $G(\mathbb{Q})^+ = \pi(\tilde{G}(\mathbb{Q}))$. Suppose that $H^1(\mathbb{Q}, \tilde{G})$ is trivial, this happens for example if $\tilde{G} = SL_n$. Let C denote the kernel of π. From the exact sequence $1 \to C \to \tilde{G} \to G \to 1$ it follows that

$$G(\mathbb{Q})/\ G(\mathbb{Q})^+ = G(\mathbb{Q})/\pi(\tilde{G}(\mathbb{Q})) \approx H^1(\mathbb{Q}, C).$$

If $G = PGL_n$ and $\tilde{G} = SL_n$, then

$$H^1(\mathbb{Q}, C) = H^1(\mathbb{Q}, \mu_n) = \mathbb{Q}^*/(\mathbb{Q}^*)^n$$

where μ_n is the n-th root of unity.

4.13. For a connected semisimple $\mathbb{Q}$-group H, for almost all $p \in R_f$, H is unramified over $\mathbb{Q}_p$, that is, quasi-split over $\mathbb{Q}_p$ and split over an unramified extension of $\mathbb{Q}_p$. For such primes $p \in R_f$, $H(\mathbb{Z}_p)$ is a hyperspecial subgroup of $H(\mathbb{Q}_p)$ or equivalently, a compact subgroup whose volume is maximum among all compact subgroups of $H(\mathbb{Q}_p)$. Hyperspecial subgroups of $H(\mathbb{Q}_p)$ are conjugate to each other by an element of $H^{ad}(\mathbb{Q}_p)$ where H^{ad} is the adjoint group of H [Ti, 3.8].

Theorem. *Let* $T \subset R$ *and let* G_p *be a connected semisimple adjoint* $\mathbb{Q}_p$*-group without any* $\mathbb{Q}_p$*-anisotropic factors for each* $p \in T$*. Assume that* $\sum_{p\in T} \text{rank}_{\mathbb{Q}_p}(G_p) \geq 2$*. Then* *the group* (G_T, D_p) *admits an irreducible lattice if and only if there exist a connected semisimple* $\mathbb{Q}$*-simple* $\mathbb{Q}$*-group* H *such that* G_p *is* $\mathbb{Q}_p$*-isomorphic to a connected normal* $\mathbb{Q}_p$*-subgroup of* H *for each* $p \in T$ *and* D_p *is a subgroup whose volume is maximum among all compact subgroups of* $G_p(\mathbb{Q}_p)$ *for almost all finite* $p \in T$*.*

Proof. The "only if" direction follows from Corollary 4.10, Definition A in 4.1 and the above remark. To see the other direction, denote by $f_p : H \to G_p$ a $\mathbb{Q}_p$-epimorphism for each $p \in T$. Let S be a finite subset of R_f such that for any $p \in R_f - S$, H unramified over $\mathbb{Q}_p$ and $H(\mathbb{Z}_p)$ is a hyperspecial subgroup of $H(\mathbb{Q}_p)$. By the hypothesis on D_p, we can find a hyperspecial subgroup $D'_p \subset H(\mathbb{Q}_p)$ such that $D_p \subset f(D'_p)$ for each $p \in (R_f - S) \cap T$. Hence for each $p \in (R_f - S) \cap T$, there exists $g_p \in H^{ad}(\mathbb{Q}_p)$ such that $g_p D'_p g_p^{-1} = H(\mathbb{Z}_p)$. Let $\phi_p = \mathrm{int} g_p$ if $p \in (R_f - S) \cap T$ and $\phi_p = id$ if $p \in R \cap \{S, \infty\} \cap T$. Then $\phi = \prod_{p \in T} \phi_p$ yields a topological group isomorphism between (H_T, D'_p) and $(H_T, H(\mathbb{Z}_p))$. Since $H(\mathbb{Q}(T))$ is an irreducible (H being almost $\mathbb{Q}$-simple) lattice in $(H_T, H(\mathbb{Z}_p))$, $\phi^{-1}(H(\mathbb{Q}(T)))$ is an irreducible lattice in (H_T, D'_p). For each $p \in T$, write $H = H_p^1 \times \ker f_p$ and $D'_p = M_p^1 \times M_p^2$ so that $M_p^1 \subset H_p^1(\mathbb{Q}_p)$ and $M_p^2 \subset \ker f_p(\mathbb{Q}_p)$. Since $\prod_{p \in T}(H_p^1(\mathbb{Q}_p) \times M_p^2) \cap (H_T, D'_p)$ is an open subgroup of (H_T, D'_p), the intersection $\phi^{-1}(H(\mathbb{Q})) \cap \prod_{p \in T}(H_p^1(\mathbb{Q}_p) \times M_p^2)$ is a lattice in $\prod_{p \in T}(H_p^1(\mathbb{Q}_p) \times M_p^2) \cap (H_T, D'_p)$. Now the canonical projection $pr : \prod_{p \in T}(H_p^1(\mathbb{Q}_p) \times M_p^2) \cap (H_T, D'_p) \to (H_T^1, M_p^1)$ has compact kernel, the subgroup $pr(\phi^{-1}(H(\mathbb{Q})) \cap \prod_{p \in T}(H_p^1(\mathbb{Q}_p) \times M_p^2))$ is a lattice in (H_T^1, M_p^1). Since the restriction of $\prod_{p \in T} f_p$ provides a topological group isomorphism from (H_T^1, M_p^1) onto (G_T, D_p), we obtain a lattice in (G_T, D_p). Since H is $\mathbb{Q}$-simple, the lattice obtained this way is irreducible, otherwise, it would yield a proper $\mathbb{Q}$-subgroup of H. □

4.14. Corollary. *Let* H *be a connected absolutely simple* $\mathbb{Q}$*-group. Let* $D_p \subset H(\mathbb{Q}_p)$ *be a compact open subgroup for each* $p \in R_f$*. If* $(H_{\mathbb{A}}, D_p)$ *admits an irreducible lattice, then* D_p *is conjugate to* $H(\mathbb{Z}_p)$ *for almost all* $p \in R_f$*.*

5. Discrete subgroups containing lattices in horospherical subgroups

In the whole section 5, for each $p \in R$, let G_p be a connected semisimple adjoint $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors and D_p a maximal compact open subgroup for almost all $p \in R_f$. We will say that $(G_{\mathbb{A}}, D_p)$ has a $\mathbb{Q}$-form (resp. $\mathbb{Q}$-isotropic form) if there exists a connected semisimple adjoint (resp. $\mathbb{Q}$-isotropic) $\mathbb{Q}$-group H and a $\mathbb{Q}_p$-isomorphism $f_p : H \to G_p$ for each $p \in R$ such that $f_p(D_p) = H(\mathbb{Z}_p)$ for almost all $p \in R_f$. If $(G_{\mathbb{A}}, D_p)$ has a $\mathbb{Q}$-form, we denote by $G_{\mathbb{A}}(\mathbb{Q})$ (resp. $G_{\mathbb{A}}(\mathbb{Q})^+$) the image of $H(\mathbb{Q})$ (resp. $H(\mathbb{Q})^+$) under the restriction of $\prod_{p \in R} f_p$ to $(G_{\mathbb{A}}, D_p)$.

5.1. Theorem. *Assume that* $\mathrm{rank}_{\mathbb{R}} G_\infty \geq 2$. *Let* P_{1p}, P_{2p} *be a pair of proper opposite parabolic* $\mathbb{Q}_p$*-subgroups of* G_p *for each* $p \in R$. *Let* Γ *be a subgroup of* $(G_{\mathbb{A}}, D_p)$ *containing lattices in* $R_u(P_1)_{\mathbb{A}}$ *and* $R_u(P_2)_{\mathbb{A}}$ *respectively, where* $R_u(P_i)_{\mathbb{A}} = (G_{\mathbb{A}}, D_p) \cap \prod_{p \in R} R_u(P_{ip})$ *for each* $i = 1, 2$. *Assume that* $(*)$ Γ^∞ *is a lattice in* $G_\infty(\mathbb{R})$. *If* Γ *is discrete, then* $(G_{\mathbb{A}}, D_p)$ *has a* $\mathbb{Q}$*-isotropic form such that* $G_{\mathbb{A}}(\mathbb{Q})^+ \subset \Gamma \subset G_{\mathbb{A}}(\mathbb{Q})$.

Proof. Set $F_i = \Gamma \cap R_u(P_i)_{\mathbb{A}}$. By the assumption, F_i is a lattice in $R_u(P_i)_{\mathbb{A}}$. For any finite subset S of R containing ∞, Γ^S is a discrete subgroup of G_S, and F_i^S is a lattice in $R_u(P_i)_{\mathbb{A}} \cap G_S$ for each $i = 1, 2$ by Lemma 3.1. Under the hypothesis, it follows from [Oh1] that the subgroup Γ^S is a non-uniform S-arithmetic lattice in G_S. Applying Theorem 4.5, we obtain a $\mathbb{Q}$-form on $G_{\mathbb{A}}$ such that $\Gamma \subset G_{\mathbb{A}}(\mathbb{Q})$. Without loss of generality, we may assume that there exists a connected absolutely simple $\mathbb{Q}$-group H such that $\Gamma \subset H(\mathbb{Q})$ and both P_{1p} and P_{2p} are parabolic subgroups of H defined over $\mathbb{Q}_p$ for each $p \in R$. Since $pr_p(\gamma) = pr_q(\gamma)$ for any $\gamma \in \Gamma$ and for any $p, q \in R$, we have

$$pr_p(\Gamma \cap (R_u(P_i))_{\mathbb{A}}) \subset R_u(P_{ip}) \cap R_u(P_{iq}).$$

On the other hand, since $\Gamma \cap (R_u(P_i))_{\mathbb{A}}$ is a lattice in $(R_u(P_i))_{\mathbb{A}}$, it follows that $pr_p(\Gamma \cap (R_u(P_i))_{\mathbb{A}})$ is Zariski dense in $R_u(P_{ip})$ for each $p \in R$ (c.f. [Lemma 2.3, Oh1]). Hence $R_u(P_{ip}) = R_u(P_{iq})$ for any p, $q \in R$. For some fixed prime $p \in R$, set $U_1 = R_u(P_{1p})$ and $U_2 = R_u(P_{2p})$. Then $pr_p(\Gamma \cap U_i)$ is Zariski dense in U_i, and hence U_i is defined over $\mathbb{Q}$. It follows that the $\mathbb{Q}$-form on H is isotropic. Since $\Gamma \cap (U_i)_{\mathbb{A}} \subset U_i(\mathbb{Q})$ and both are lattices in $(U_i)_{\mathbb{A}}$, $\Gamma \cap (U_i)_{\mathbb{A}}$ has a finite index in $U_i(\mathbb{Q})$. But $U_i(\mathbb{Q})$ is a unipotent group, and hence it has no finite index subgroup except itself. Therefore $\Gamma \cap (U_i)_{\mathbb{A}} = U_i(\mathbb{Q})$. It is then well known that the subgroup of $H(\mathbb{Q})$ generated by $U_1(\mathbb{Q})$ and $U_2(\mathbb{Q})$ coincides with $H(\mathbb{Q})^+$ (see [BT1]). Therefore Γ contains $H(\mathbb{Q})^+$. This finishes the proof. □

5.2. Corollary. *Let* P_1, P_2 *and* G_∞ *be same as in the above theorem. Let* F_i *be a lattice in* $R_u(P_i)_{\mathbb{A}}$ *for each* $i = 1, 2$. *Denote by* Γ_{F_1, F_2} *the subgroup of* $(G_{\mathbb{A}}, D_p)$ *generated by* F_1 *and* F_2. *Assume that* $(*)$ $\Gamma^\infty_{F_1, F_2}$ *is a lattice in* $G_\infty(\mathbb{R})$. *Then* Γ_{F_1, F_2} *is discrete if and only if there exists a* $\mathbb{Q}$*-form on* $G_{\mathbb{A}}$ *such that* $F_i = G_{\mathbb{A}}(\mathbb{Q}) \cap R_u(P_i)_{\mathbb{A}}$ *for each* $i = 1, 2$ *and* $\Gamma_{F_1, F_2} = G_{\mathbb{A}}(\mathbb{Q})^+$.

Remark 5.3. Theorem 5.1 and Corollary 5.2 hold without the assumption $(*)$ for any group G_∞ for which Margulis's conjecture (see [Oh1, Conjecture 0.1]) has been verified. Indeed, for the subgroup Γ in Theorem 5.1 (or for Γ_{F_1, F_2} in Corollary 5.2), Γ^∞ is a discrete subgroup containing lattices in a pair of opposite horospherical subgroups in $G_\infty(\mathbb{R})$. See the remark following Theorem 1.5.

Hence we obtain the following:

5.4. Corollary. *Let G_∞ be $\mathbb{R}$-split, rank $G_\infty \geq 2$ and $G_\infty(\mathbb{R})$ not locally isomorphic to $SL_3(\mathbb{R})$. Then the following sets are all equal:*

(1) *discrete subgroups in $G_{\mathbb{A}}$ containing lattices in opposite horospherical subgroups of $G_{\mathbb{A}}$;*
(2) *subgroups generated by all unipotent elements of a non-uniform irreducible lattice in $G_{\mathbb{A}}$;*
(3) *subgroups of $f(H(\mathbb{Q}))$ containing $f(H(\mathbb{Q}))^+$ for some H and f as in Theorem 1.1.*

The above sets are non-empty only when the adele group $(G_{\mathbb{A}}, D_p)$ is isomorphic to $(H_{\mathbb{A}}, H(\mathbb{Z}_p))$ as a topological group for some connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group H.

6. Lattices in $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$.

6.1. Let G be a connected absolutely almost simple $\mathbb{Q}$-group. Recall that for any field k, the non-isomorphic k-forms of a k-algebraic variety M are parametrized by the first Galois cohomology set $H^1(k, \mathrm{Aut}(M))$ (cf. [PR, 2.2.3]). Therefore the number of non-isomorphic $\mathbb{Q}$-forms of G is determined by the following question on the Hasse principle for $\mathrm{Aut}(G)$: what is the size of the kernel (the Shafarevich-Tate group of G) of the natural map

$$(*) \qquad H^1(\mathbb{Q}, \mathrm{Aut}(G)) \to \prod_{p \in R} H^1(\mathbb{Q}_p, \mathrm{Aut}(G))?$$

As remarked in [Se, 4.6], a theorem of Borel [Bo, Theorem 6.8] implies that the above kernel is always finite.

If G does not have any outer automorphism, for instance, if G is not of type $A_n\,(n \geq 2), D_n\,(n \geq 4)$ or E_6, then $\mathrm{Aut}(G) = \mathrm{Int}(G)$, which is canonically isomorphic to G^{ad}. Then the Hasse principle for an adjoint $\mathbb{Q}$-group (see [PR, Theorem 6.22]) says that the kernel of the above map is trivial. We say that a connected absolutely almost simple $\mathbb{Q}$-group H is a $\mathbb{Q}$-form of $G_{\mathbb{A}}$ if for each $p \in R$, H and G are isomorphic over $\mathbb{Q}_p$. Hence we summarize:

Proposition. *Let G be a connected absolutely almost simple $\mathbb{Q}$-group. Then $G_{\mathbb{A}}$ admits only finitely many non-isomorphic $\mathbb{Q}$-forms. Moreover if G is not of type $A_n\,(n \geq 2), D_n\,(n \geq 4)$ or E_6, then there is a unique $\mathbb{Q}$-form on $G_{\mathbb{A}}$ up to $\mathbb{Q}$-isomorphism.*

We remark that there exists two central simple division algebras over $\mathbb{Q}$ of degree at least 2, say, D_1 and D_2 such that $PSL_1(D_1)$ and $PSL_1(D_2)$ are not isomorphic over $\mathbb{Q}$, but isomorphic over all $\mathbb{Q}_p$, $p \in R$. Hence the adele group associated with $PSL_1(D_1)$ has (at least) two non-isomorphic $\mathbb{Q}$-forms.

6.2. If G is a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group, G has no $\mathbb{Q}_p$-anisotropic factors for each $p \in R$. Hence by Lemma 4.3, we have:

Proposition. *Let G be a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group. Let f be a topological group automorphism of $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$. Then there exist $\mathbb{Q}_p$-automorphisms f_p of G_p's, $p \in R$ with $f_p(G(\mathbb{Z}_p)) = G(\mathbb{Z}_p)$ for almost all $p \in R_f$ such that f is the restriction of $\prod_{p\in R} f_p$.*

By Theorem 1.1, the above proposition yields the following:

6.3. By Theorem 1.1 and Proposition 6.1, we have:

Proposition. *Let G be a connected absolutely simple $\mathbb{Q}$-isotropic $\mathbb{Q}$-group. Then up to automorphism of $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$, the adele group $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$ admits only finitely many non-uniform irreducible lattices up to commensurability. Moreover if G is not of type $A_n\,(n \geq 2), D_n\,(n \geq 4)$ or E_6. then $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$ admits a unique non-uniform irreducible lattice up to commensurability and up to automorphism of $(G_{\mathbb{A}}, G(\mathbb{Z}_p))$.*

6.4. Recall that for a linear algebraic $\mathbb{R}$-group G, $(H, f : H \to G)$ is called an $\mathbb{R}/\mathbb{Q}$-form of G or simply $\mathbb{Q}$-form of G if H is a linear algebraic $\mathbb{Q}$-group and f is an isomorphism defined over $\mathbb{R}$. For any connected semisimple $\mathbb{R}$-group G with $G(\mathbb{R})$ non-compact, $G(\mathbb{R})$ admits a $\mathbb{Q}$-form with $\text{rank}_{\mathbb{Q}}(G) = 0$; hence a uniform (arithmetic) lattice $G(\mathbb{Z})$, as constructed by Borel [Bo1]. It also admits a $\mathbb{Q}$-form with the same $\mathbb{Q}$-rank as $\text{rank}_{\mathbb{R}}(G)$; hence a non-uniform arithmetic lattice $G(\mathbb{Z})$. This readily follows from [Oh1, Proposition 1.4.2] whose proof is due to Prasad. His proof was not delivered clearly therein. We take this opportunity to give a short proof provided by him.

Proposition (cf. [Oh1, Proposition 1.4.2]**).** *Let G be connected adjoint semisimple linear algebraic group defined over $\mathbb{R}$. Then for a given minimal parabolic $\mathbb{R}$-subgroup P, there exists a $\mathbb{Q}$-form on G with respect to which every parabolic $\mathbb{R}$-subgroup containing P is defined over $\mathbb{Q}$.*

Proof. It is not difficult to reduce to the case when G is absolutely simple (see [Oh1]). Let G^q be the adjoint $\mathbb{Q}$-split group if the $\mathbb{R}$-form of G is inner or the quasi-split $\mathbb{Q}$-form of G, splitting over $k = \mathbb{Q}(i)$ otherwise. Let P^q be the corresponding parabolic subgroup to P of G^q. By [Se, Proposition 37], G is then obtained from G^q by twisting by a P^q-valued cocycle, say, c on $\text{Gal}(\mathbb{C}/\mathbb{R})$. By [PR, Proposition 6.17], the natural map

$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), P^q) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), P^q)$ is surjective. Hence there exists a P^q-valued cocycle d on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ whose restriction to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is is cohomologous to c. Naturally we may regard this cocycle d as an element of $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G^q)$. Then the twist of G^q by the cocycle d coincides with the $\mathbb{R}$-form of G over $\mathbb{R}$ and its distinguished orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contain all the distinguished orbits of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of the $\mathbb{R}$-form of G (cf. [Oh1, 1.4.1]). This proves our claim. □

If G is a connected non-compact semisimple linear algebraic group over $\mathbb{Q}_p$, Tamagawa showed that $G(\mathbb{Q}_p)$ does not admit any non-uniform lattice [Ta]. However it always admits a uniform lattice as shown by Borel and Harder [BH].

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