TEMPERED SUBGROUPS AND REPRESENTATIONS WITH MINIMAL DECAY OF MATRIX COEFFICIENTS

HEE OH

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ABSTRACT. We present a function F for each simple real linear Lie group G with real rank at least 2 such that F bounds from above all the K-matrix coefficients of non-trivial irreducible spherical unitary representations of G where K is a maximal compact subgroup of G. This enables us to determine when a closed subgroup H is a (G, K)- tempered subgroup of G: for example, if the restriction $F|_H$ of F to H lies in $L^{1-\epsilon}(H)$. We also prove that this function F is the best possible for G a real-split group of type A_n or C_n and as a consequence, we obtain that if H is semisimple, then H is a (G, K)- tempered subgroup of G if and only if $F|_H$ lies in $L^1(H)$.

RÉSUMÉ . *Nous présentons une fonction F pour chaque G, groupe de Lie linéaire réel simple de rang réel au moins 2, telle que F donne une borne supérieure pour tous les coefficients matriciels K-finis des représentations unitaires sphériques irréductibles de G, avec K un sous-groupe compact maximal de G. Ceci nous permet à determiner quand un sous-groupe fermé H de G est (G, K)-tempéré; par exemple, c'est le cas si la restriction de F à H est dans $L^{1-\epsilon}(H)$. Nous prouvons aussi que cette fonction F est la meilleure possible pour G un groupe réel déployé de type A_n ou C_n, et comme conséquence, nous obtenons que si H est semisimple, alors H est un sous-groupe (G, K)-tempéré de G si et seulement si $F|_H$ est dans $L^1(H)$.*

1. Introduction

Let G be a connected semisimple linear Lie group and K a maximal compact subgroup of G. We say that a unitary representation of G is *spherical* if it has a K-invariant vector. For a unitary spherical representation ρ, we will use the term "K-matrix coefficients of ρ" to refer to its matrix coefficients with respect to K-invariant unit vectors.

In this paper we are interested in the asymptotic behavior of the K-matrix coefficients of spherical unitary representations of G when restricted to a closed subgroup H of G. One motivation comes from the notion "(G, K)-tempered subgroups" of G defined by Margulis [10]. That is, a closed subgroup H of G is called (G, K)-*tempered* if there exists a (positive) function $q \in L^1(H)$ such that for any non-trivial irreducible spherical unitary representation ρ of G, $|\langle \rho(h)v, w\rangle| \le q(h)\|v\|\|w\|$ for all $h \in H$ and any K- fixed vectors v and w. Note that any compact subgroup of G is a (G, K)- tempered subgroup for a trivial reason. Margulis also showed in [10] that if a closed subgroup H is a (G, K)-tempered subgroup, then for any non-compact subgroup F of H, the quotient G/F does not allow a compact quotient by a discrete subgroup of G (see [6] for a survey on the general problem).

We denote by $\hat{G}$ (resp. $\hat{G}_K$) the set of equivalence classes of non-trivial irreducible unitary (resp. spherical) representations.

In this paper we first present a "good upper bound function" for K-matrix coefficients for all representations in $\hat{G}_K$ for a simple real linear Lie group G with real rank at least 2. Secondly we show that in simple real-split linear Lie group of type A_n or C_n this function is in fact the best possible by exhibiting a spherical representation of G in $\hat{G}_K$ whose K-matrix coefficients are bounded below by this function. We now formulate the main results.

The notation $[x]$ denotes the largest integer which is not greater than x.

Theorem A. *Let G be a connected simple real linear Lie group with real rank $n \geq 2$, K a maximal compact subgroup, B a minimal parabolic subgroup, $A \subset B$ a maximal $\mathbb{R}$-split torus, $A^+ \subset A$ the positive Weyl chamber given by the choice of B. Denote by Φ' the set of all non-multipliable roots in the relative root system $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ where $\mathfrak{a}$ and $\mathfrak{g}$ are the Lie algebras of A and G respectively. Let $\alpha_1, ..., \alpha_n$ be the basis of Φ' whose subscripts are determined by the highest weight given in section 2.1.*

Then for any $\epsilon > 0$, there exists a constant C (depending on ϵ) such that for any $\rho \in \hat{G}_K$ and f_0 a K-invariant unit vector of ρ,

$$|\langle \rho(g) f_0, f_0 \rangle| \leqq C\, F(g)^{1-\epsilon} \qquad \text{for any } g \in G$$

where F is the K-bi-invariant function defined on A^+ as follows according to the type of Φ':

Φ'	$-\log F$
$A_n, n \geq 2$	$\begin{cases} \sum_{i=1}^{(n-1)/2} \frac{i}{2}\alpha_i + \sum_{i=(n+1)/2}^{n} \frac{(n-i+1)}{2}\alpha_i & \text{for } n \text{ odd} \\ \sum_{i=1}^{n/2} \frac{i}{2}\alpha_i + \frac{n}{4}\alpha_{n/2+1} + \sum_{i=n/2+2}^{n} \frac{(n-i+1)}{2}\alpha_i & \text{for } n \text{ even} \end{cases}$
$B_n, n \geq 2$	$\sum_{i=1}^{[n/2]} i\alpha_i + \sum_{i=[n/2+1]}^{n} \frac{n}{2}\alpha_i$
$C_n, n \geq 2$	$\sum_{i=1}^{n-1} i\alpha_i + \frac{n}{2}\alpha_n$
$D_n, n \geq 4$	$\sum_{i=1}^{[n/2]} i\alpha_i + \sum_{i=[n/2+1]}^{n-2} \frac{n}{2}\alpha_i + \frac{n}{4}(\alpha_{n-1} + \alpha_n)$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + \frac{7}{2}\alpha_2 + 4\alpha_3 + 6\alpha_4 + \frac{9}{2}\alpha_5 + 3\alpha_6 + \frac{3}{2}\alpha_7$
E_8	$2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 8\alpha_4 + 7\alpha_5 + 5\alpha_6 + 3\alpha_7 + \alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + \alpha_2$

Corollary B. *With the same notation as in Theorem A, let H be a closed subgroup of G. If the restriction $F|_H$ of F to H is in $L^{1-\epsilon}(H)$ for some $\epsilon > 0$, then H is a (G, K)-tempered subgroup of G.*

Remark 1. Suppose further that H is a connected semisimple Lie subgroup of G such that $A \cap H$ is a maximal $\mathbb{R}$-split torus of H and $B \cap H$ is a minimal parabolic subgroup of H.

Let δ_H denote the modular function of $B \cap H$, that is, the product of all positive roots including the multiplicity. Let $\lambda_1, \cdots, \lambda_m$ be the fundamental weights of the Lie algebra of H corresponding to $A \cap H$ and $B \cap H$. For any two weights α and β of the Lie algebra of H, we write $\alpha < \beta$ if $(\alpha, \lambda_j) < (\beta, \lambda_j)$ for all $1 \le j \le m$.

Then the condition $F|_H \in L^{1-\epsilon}(H)$ is equivalent to

$$-\log F|_{A^+ \cap H} > \log \delta_H;$$

which is again equivalent to the condition $F|_H \in L^1(H)$.

2. If the restriction $F|_H$ is $L^{k-\epsilon}(H)$-integrable for some $\epsilon > 0$ and some positive integer k, then the diagonal embedding $\delta_k(H)$ of H into the group $\prod_{i=1}^k G_i$ is a $(\prod_{i=1}^k G_i, \prod_{i=1}^k K_i)$-tempered subgroup of $\prod_{i=1}^k G_i$ where $G_i = G$ and $K_i = K$ for all $1 \le i \le k$. To see this, it is enough to note that for any non-trivial irreducible spherical representation ρ of $\prod_{i=1}^k G_i$, the restrictions of the K-matrix coefficients of ρ to $\delta_k(H)$ are bounded by $(F|_H)^{k(1-\epsilon)}$.

For a unitary representation ρ of G, ρ is said to be *strongly* L^q if there is a dense subset V in the Hilbert space attached to ρ such that the matrix coefficients of ρ with respect to the vectors in V lie in $L^q(G)$. Let $p(G)$ be the smallest real number such that for any $\rho \in \hat{G}$, ρ is strongly L^q for any $q > p(G)$ (cf. [7]). Similarly let $p_K(G)$ be the smallest real number such that for any $\rho \in \hat{G}_K$, the K-matrix coefficients of ρ are $L^q(G)$-integrable for any $q > p_K(G)$. The estimate of the Harish-Chandra function Ξ of G shows that $p_K(G)$ is at least 2 (cf. [3]) and hence G cannot be a (G, K)-tempered subgroup of itself. The method used in proving Theorem A yields upper bounds for both $p(G)$ and $p_K(G)$.

The following follows from remark (1) after Corollary B.

Corollary C. *With the same notation as in Theorem A, let δ_G be the modular function of B (cf. Table 3.7). Define*

$$r(G) = \max\{\frac{\text{the coefficient of } \alpha_i \text{ in } \log \delta_G}{\text{the coefficient of } \alpha_i \text{ in } -\log F} \mid i = 1, \cdots, n\}.$$

Then $p(G) \le r(G)$ and $p_K(G) \le r(G)$.

If G is split over $\mathbb{R}$, $r(G)$ is as follows:

$\Phi = \Phi'$:	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$r(G)$:	$2n$	$2n$	$2n$	$2(n-1)$	16	18	58	11	6

For $n \ge 3$, Vogan's classification of unitary duals for $GL_n(D)$ yields that for $G = SL_n(D)$, $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $p(G)$ is $2(n-1)$, $2(n-1)$ and $2n-1$ respectively and for $Sp_{2n}(\mathbb{R})$, it follows from Howe's result in [3] that $p(G) = 2n$. The number $p(G)$ in other classical group cases was calculated by Li [7] and was given an upper bound by Li and Zhu [8] in exceptional split group cases. We remark that the numbers $r(G)$ in Corollary C coincide with $p(G)$ calculated in [7] for all classical real split groups except B_n type. For a split group of type E_6, by obtaining $r(G) = 16$, we improve the bound for $p(G)$ in [8].

We state a theorem which yields a necessary and sufficient condition for a closed semisimple subgroup to be a (G, K)-tempered subgroup in a simple real split linear Lie group G of type A_n or C_n. Let G be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. Set $K = SO_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ respectively. Define the parabolic subgroup P of G as follows:

$$\text{for } G = SL_n(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\},$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{2nj} = 0 \text{ if } i \neq 1, j \neq 2n\}.$$

Note that P is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$. We fix an ordering in the root system of G so that the positive Weyl chamber A^+ is as follows:

$$SL_n(\mathbb{R}), A^+ = \{\text{diag}(a_1, \cdots, a_n) \mid \prod_{i=1}^{n} a_i = 1,\ a_i \geqq a_{i+1} \text{ for all } 1 \leq i \leq n-1\},$$
$$Sp_{2n}(\mathbb{R}), A^+ = \{\text{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq n-1\}.$$

Example. The function F defined in Theorem A is as follows:

$$\text{for } G = SL_n(\mathbb{R}), \qquad F(a) = \begin{cases} \prod_{i=1}^{n/2} {a_i}^{-1} & \text{for n even} \\ \left(\prod_{i=1}^{(n-1)/2} {a_i}^{-1}\right) {a_{(n+1)/2}}^{-1/2} & \text{for n odd} \end{cases}$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \qquad F(a) = \prod_{i=1}^{n} {a_i}^{-1}$$

where $a \in A^+$.

Theorem D. *Let G be $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$ and P, K and A^+ be as above.*

(1) *For any $\epsilon > 0$, there exist constants C_1 and C_2 such that*

$$C_1\, F(a) \leq |\langle \text{Ind}_P^G(I)(a) f_0, f_0 \rangle| \leq C_2\, F(a)^{1-\epsilon}$$

for any $a \in A^+$ and for any K-invariant unit vector f_0 in $\text{Ind}_P^G(I)$.

(2) *If a closed subgroup H of G is (G, K)-tempered, $F|_H$ is in $L^1(H)$.*

(3) *A closed semisimple subgroup H of G is (G, K)-tempered if and only if $F|_H$ is in $L^1(H)$.*

(4) *$p_K(G) = r(G) = p(G)$.*

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2. A maximal system of strongly orthogonal roots in each irreducible root system

2.1. Let Φ be an irreducible reduced root system with a fixed ordering. Denote by Φ^+ the set of positive roots and by $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ the set of simple roots of Φ. The subscripts of α_i's are determined by the following choice of the highest root [2].

Φ	the highest root
A_n	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
B_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$
C_n	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$

We define the number $N(\Phi)$ as follows:

$$N(\Phi) = \begin{cases} [\frac{n+1}{2}], & \text{for } \Phi = A_n \\ 2[\frac{n}{2}], & \text{for } \Phi = D_n \\ 4, & \text{for } \Phi = E_6 \\ \text{rank}(\Phi)\ , & \text{for } \Phi = B, C, F_4, G_2, E_7, E_8 \end{cases}$$

2.2. Construction of some strongly orthogonal roots. Two roots α and β are called *strongly orthogonal* if neither one of $\alpha \pm \beta$ is a root. Consider the family $\mathcal{S}(\Phi)$ of all subsets of Φ^+ whose elements are pairwise strongly orthogonal. We call an element $\mathcal{O}$ in $\mathcal{S}(\Phi)$ a *strongly orthogonal system*. Let f be the function on $\mathcal{S}(\Phi)$ given by $f(\mathcal{O}) = \sum_{\alpha\in\mathcal{O}} \alpha$. The aim of this section is to construct an element $\mathcal{Q}(\Phi) = \{\gamma_1, \cdots, \gamma_{N(\Phi)}\}$ in $\mathcal{S}(\Phi)$ on which f attains its maximum. For simplicity, we set $N(\Phi) = N$.

We define $\mathcal{Q}(\Phi)$ as follows:

Φ	$\mathcal{Q}(\Phi)$
A_n	$\begin{cases} \gamma_i = \alpha_i + \cdots + \alpha_{n-i+1} \text{ for } i \leq N-1 \\ \gamma_N = \begin{cases} \alpha_N & \text{for } n \text{ odd} \\ \alpha_N + \alpha_{N+1} & \text{for } n \text{ even} \end{cases} \end{cases}$
B_n	$\begin{cases} \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_n \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} \text{ for } i \leq [\frac{n}{2}] \\ \gamma_n = \alpha_{(n+1)/2} + \cdots + \alpha_n \text{ for } n \text{ odd} \end{cases}$

$$C_n \quad \begin{cases} \gamma_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n \text{ for } i \leq N-1 \\ \gamma_N = \alpha_n \end{cases}$$

$$D_n \quad \begin{cases} \gamma_1 = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n \\ \gamma_2 = \alpha_1 + \cdots + \alpha_{n-1} \\ \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} \text{ for } 3 \leq i \leq [\frac{n}{2}] \end{cases}$$

$$E_6 \quad \begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \gamma_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_4 = \alpha_2 \end{cases}$$

$$E_7 \quad \begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \gamma_5 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_6 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \\ \gamma_7 = \alpha_2 \end{cases}$$

$$E_8 \quad \begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 \\ \gamma_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_6 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_7 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_8 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \end{cases}$$

$$F_4 \quad \begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \gamma_4 = \alpha_1 \end{cases}$$

$$G_2 \quad \begin{cases} \gamma_1 = 3\alpha_1 + 2\alpha_2 \\ \gamma_2 = \alpha_1 \end{cases}$$

The following lemma can be easily checked.

Lemma. *The set $\mathcal{Q}(\Phi)$ is a strongly orthogonal system.*

2.3. Proposition. $f(\mathcal{Q}(\Phi)) = \max_{\mathcal{O}\in\mathcal{S}(\Phi)} f(\mathcal{O})$*, that is, for any $\mathcal{O} \in \mathcal{S}(\Phi)$, the coefficient of α_i in $f(\mathcal{Q}(\Phi))$ is greater than or equal to the coefficient of α_i in $f(\mathcal{O})$ for each $1 \leq i \leq n$, where n is the rank of Φ.*

Proof. Let $\mathcal{O}$ be any element in $\mathcal{S}(\Phi)$. We prove this proposition by induction. We can easily check that the proposition is true for $n = 2$. Suppose that $n \geq 3$.

For $\Phi = A_n$, take any element in $\mathcal{O}$, say $\alpha = \alpha_i + \cdots + \alpha_{j-1}$, $i < j-1$. Then $\alpha \leq \gamma_1$ since γ_1 is the highest root. On the other hand $\mathcal{O} - \{\alpha\}$ is contained in $\{\alpha_m + \cdots + \alpha_{l-1} \mid m, l \notin \{i, j\}\}$, which is a root system of type A_{n-2}. Note that $\mathcal{Q}(A_n) \cap \{\alpha_m + \cdots + \alpha_{l-1} \mid m, l \notin \{i, j\}\} = \mathcal{Q}(A_{n-2})$. Therefore by the induction assumption, $f(\mathcal{O} - \{\alpha\}) \leq f(\mathcal{Q}(A_{n-2}))$. Hence we have $f(\mathcal{O}) \leq \gamma_1 + f(\mathcal{Q}(A_{n-2})) \leq f(\mathcal{Q}(A_n))$, proving the claim.

For $\Phi = B_n$, note that for any $\alpha \in \Phi^+$, we have that the coefficient of α_1 in α is at most 1. Write $\mathcal{O}$ as $\mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of α_1 in β is 1 and $\mathcal{O}_2 = {\mathcal{O}_1}^c$. It is not difficult to check that if three positive roots in B_n are mutually strongly orthogonal, then the coefficient of α_1 in at least one of them is 0. Therefore $|\mathcal{O}_1| \leq 2$. We can easily see that for any two strongly orthogonal roots $\beta_1, \beta_2 \in \Phi^+$ such that the coefficient of α_1 in β_i is 1 for both $i = 1, 2$, we have $\beta_1 + \beta_2 \leq \sum_{i=1}^n 2\alpha_i$; hence $\sum_{\beta \in \mathcal{O}_1} \beta \leq \gamma_1 + \gamma_2$, because $|\mathcal{O}_1| \leq 2$ and $\gamma_1 + \gamma_2 = \sum_{i=1}^n 2\alpha_i$. For $\theta \subset \Delta$, the notation $[\theta]$ denotes the set of the roots in Φ which can be expressed as integral combinations of the roots in θ. Since $\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, $\gamma_3 + \gamma_4 = \sum_{i=2}^n 2\alpha_i$ and $[\alpha_2, \cdots, \alpha_n]$ is a root system of type B_{n-1}, we can proceed by induction as before.

The argument for D_n is exactly the same as the one for B_n; so we omit it.

If Φ is of type C_n, write $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of α_1 in β is positive and $\mathcal{O}_2 = {\mathcal{O}_1}^c$. It is easy to see that $|\mathcal{O}_1| \leq 1$. Therefore $\sum_{\alpha \in \mathcal{O}_1} \alpha \leq \gamma_1$, for γ_1 is the highest root in Φ. Since $\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, it remains to use induction process.

For exceptional root system cases, we can prove the proposition by checking each root system case by case. $\square$

As a corollary of the above proposition, we obtain that $\mathcal{Q}(\Phi)$ is a maximal element in $\mathcal{S}(\Phi)$ with respect to the inclusion ordering.

Remark. I learned from E. Vinberg that this construction of a strongly orthogonal system coincides with the so called Kostant's cascade construction (cf. [9]), if Φ is one of the types A_n, C_n or G_2. But in all cases the cardinalities of the sets in Kostant's cascade construction coincide with the numbers $N(\Phi)$, which are the cardinalities of $\mathcal{Q}(\Phi)$ in our construction. We note that not all maximal strongly orthogonal systems in Φ have the same cardinality. For example, $\{\alpha_2, \alpha_4, 2\alpha_2 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$ is a maximal strongly orthogonal system in the root system of F_4.

We remark that if Φ is none of A_n, C_n or G_2, the function f attains its maximum in our construction but not in Kostant's cascade construction.

3. An upper bound function for matrix coefficients in simple non-compact linear Lie groups

3.1. Let G be a connected semisimple non-compact linear Lie group, B a minimal parabolic subgroup, A a maximal $\mathbb{R}$-split torus contained in B, A^+ the positive Weyl chamber and K a maximal compact subgroup. Consider a Cartan decomposition of G: $G = KA^+K$. Since the K-matrix coefficients of a spherical unitary representation are K-bi-invariant, they are determined by their restrictions to the A^+-part. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{a}$ the Lie algebra of A. We denote by $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ the set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$, which is endowed with the ordering given by B. If G is split over $\mathbb{R}$, then $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ will be simply denoted by $\Phi(\mathfrak{a}, \mathfrak{g})$. If we fix a Haar measure dg on G, then the modular function δ_G of B

is given as

$$\delta_G = \prod_{\alpha \in \Phi^+_{\mathbb{R}}(\mathfrak{a},\mathfrak{g})} \exp \alpha.$$

It is well known (cf. [3]) that the induced representation $\mathrm{Ind}_B^G(I)$ of the trivial representation of B is irreducible and has a unique (up to a sign) K-invariant unit vector, say f_0. The matrix coefficient of $\mathrm{Ind}_B^G(I)$ defined by $g \mapsto \langle \mathrm{Ind}_B^G(I)(g) f_0, f_0 \rangle$ is called the *Harish-Chandra function* of G, which we will denote by Ξ_G. When there is no confusion, Ξ_G will simply be denoted by Ξ.

Harish-Chandra has shown the following:

Proposition. *(*cf. [3]*) For any $\epsilon > 0$, there exist constants c_1 and c_2 such that*

$$c_1\, \delta_G^{-1/2}(a) \leq \Xi_G(a) \leq c_2\, \delta_G^{-1/2+\epsilon}(a)$$

for all $a \in A^+$.

Moreover the value of Harish-Chandra function Ξ of $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$ at $\begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$ for $a_0 > 1$ is asymptotically $(\log a_0)/a_0$ up to some constant multiple.

3.2. We can write the Haar measure dg of G in terms of the Cartan decomposition KA^+K as follows:

$$dg = \Delta(a) dk_1\, da\, dk_2$$

where $\Delta(a)$ is a positive function on A^+ satisfying $d_1(t)\delta(a) \leqq \Delta(a) \leqq d_2\delta(a)$ for all $a \in \{a \in A^+ \mid |\alpha(a)| \geq t \text{ for all } \alpha \in \Phi^+_{\mathbb{R}}(\mathfrak{a},\mathfrak{g})\}$ and for some constants $d_1(t)$ and d_2 if $t > 1$ (cf. [3]).

For a K-matrix coefficient $\phi(g) = \langle \rho(g)v, w \rangle$ of $\rho \in \hat{G}_K$, it is well known that $\phi \in L^p(G)$ if and only if $\int_{A^+} |\phi(a)|^p \delta(a) da < \infty$.

3.3. Proposition. *Let H be $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$. Suppose that for some $k \geq 2$, H acts on $\mathbb{R}^k$ by a rational representation so that the only H-invariant vector is the origin. Let $H \ltimes \mathbb{R}^k$ be the associated semidirect product. Let ρ be a unitary representation of $H \ltimes \mathbb{R}^k$ without any $\mathbb{R}^k$-invariant vectors. Then we have*

$$|\langle \rho|_H(h)v, w \rangle| \leq \Xi_H(h) (\dim\langle K \cdot v \rangle \dim\langle K \cdot w \rangle)^{1/2}$$

where $h \in H$, $K = SO_2(\mathbb{R})$ and v and w are K-finite unit vectors of ρ.

Moreover if ρ is spherical, then the K-matrix coefficients of $\rho|_H$ are bounded by Ξ_H.

Proof. By [12, Theorem 7.3.9], the restriction $\rho|_H$ of ρ to H is weakly contained in the infinite sum of the regular representation of H. It is well known (cf. [4, Ch V, Theorem 3.2 1]) that the K-finite (or K-fixed) matrix coefficients of the regular representation of H satisfy the above inequality. □

In the spirit of Howe's strategy (see [7]) we state the following proposition.

The notation $\mathfrak{u}_\alpha$ for $\alpha \in \Phi_{\mathbb{R}}(\mathfrak{a},\mathfrak{g})$ denotes the root space in $\mathfrak{g}$ corresponding to α.

3.4. Proposition. *Let G be a connected simple real split linear Lie group. Let $\{\beta_1, \cdots, \beta_m\} \subset \Phi^+(\mathfrak{a}, \mathfrak{g})$ be a strongly orthogonal system. Then for any $\epsilon > 0$, there exists a constant C such that for any $\rho \in \hat{G}_K$ and K-fixed unit vectors v and w of ρ, we have*

$$|\langle \rho(a)v, w\rangle| \leq C \prod_{i=1}^{m} \exp\left((-\frac{1}{2} + \epsilon)\beta_i\right)(a)$$

for any $a \in A^+$.

Proof. For each $1 \leq i \leq m$, let H_i be the connected closed subgroup of G whose Lie algebra is generated by $\mathfrak{u}_{\pm\beta_i}$. Note that for each $1 \leq i \leq m$, (1) H_i is isomorphic to $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$; (2) the subgroups $H_i \cap B$, $H_i \cap A$ and $H_i \cap K$ are a minimal parabolic subgroup, a maximal $\mathbb{R}$-split torus and a maximal compact subgroup of H_i respectively; (3) the positive Weyl chamber $A^+(H_i)$ of H_i is contained in A^+.

It is not difficult to see that $A^+ \subset \prod_{i=1}^m A^+(H_i) C_G(\prod_{i=1}^m H_i)$ where $C_G(\prod_{i=1}^m H_i)$ denotes the centralizer of $\prod_{i=1}^m H_i$. Since $\{\beta_1, \cdots, \beta_m\}$ is a strongly orthogonal system, it follows that $x_i x_j = x_j x_i$ for all $i \neq j$, $x_i \in H_i$ and $x_j \in H_j$. By looking at the root system, it is not difficult to see that for each H_i, there exists an abelian unipotent subgroup U_i of G of dimension at least 2 such that H_i normalizes U_i and $C_G(H_i) \cap U_i$ is trivial. It follows that for each $1 \leq i \leq m$, $H_i \ltimes \mathbb{R}^{k_i}$ can be considered to be a subgroup of G where the semidirect product is as described in Proposition 3.3 and $k_i = \dim U_i$.

Let $\rho \in \hat{G}_K$, and v and w be K-fixed unit vectors. The restriction $\rho|_{\prod_{i=1}^m H_i}$ can be written as a direct integral $\int_X \rho_\alpha d\mu(\alpha)$ where X is some Borel measure space with measure μ, $\alpha \in X$ and ρ_α is an irreducible representation of $\prod_{i=1}^m H_i$ for all $\alpha \in X$ (cf. [3]). Without loss of generality, we may assume that all ρ_α's are non-trivial spherical representations. Any element $a \in A^+$ can be uniquely written as $a = a_1 \cdots a_m c$ where $a_i \in A^+(H_i)$ and $c \in C_G(\prod_{i=1}^m H_i)$. Write $\rho(c)v$ and w as $\int v_\alpha d\mu(\alpha)$ and $\int w_\alpha d\mu(\alpha)$ respectively where v_α and w_α are vectors in ρ_α. Since c centralizes each H_i, $\rho(c)v$ is $K \cap H_i$-fixed for all $1 \leq i \leq m$. Therefore there is no loss of generality in assuming that for all $\alpha \in X$, v_α and w_α are $K \cap H_i$-fixed for all $1 \leq i \leq m$. Fix $\alpha \in X$. Then $\rho_\alpha|_{\prod_{i=1}^m H_i} = \otimes_{i=1}^m \rho_{\alpha i}$, $v_\alpha = \otimes_{i=1}^m v_{\alpha i}$ and $w_\alpha = \otimes_{i=1}^m w_{\alpha i}$, where $\rho_{\alpha i}$ is a spherical irreducible representation of H_i and $v_{\alpha i}$ and $w_{\alpha i}$ are $K \cap H_i$-fixed vectors for each $1 \leq i \leq m$. By Moore's theorem (cf. [12, Theorem 2.1.9]), for each $1 \leq i \leq m$, the representation $\rho_{\alpha i}$ is non-trivial and $\rho_{\alpha i}$ has no U_i-invariant vector.

By Proposition 3.3, we obtain that for each $1 \leq i \leq m$,

$$|\langle \rho_{\alpha i}(a_i) v_{\alpha i}, w_{\alpha i}\rangle| \leq \Xi_{H_i} \|v_{\alpha i}\| \cdot \|w_{\alpha i}\|.$$

Hence

$$
\begin{aligned}
|\langle \rho(a)v, w\rangle| &\le \int_\alpha |\langle \rho_\alpha(\prod_{i=1}^m a_i)v_\alpha, w_\alpha\rangle|\, d\mu(\alpha) \\
&\le \int_\alpha \prod_{i=1}^m |\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i}\rangle|\, d\mu(\alpha) \\
&\le \int_\alpha \prod_{i=1}^m (\Xi_{H_i}(a_i)\|v_{\alpha i}\|\cdot\|w_{\alpha i}\|)\ d\mu(\alpha) \\
&= \prod_{i=1}^m \Xi_{H_i}(a_i)\|\rho(c)v\|\cdot\|w\| \\
&\le \prod_{i=1}^m \Xi_{H_i}(a_i)\|v\|\cdot\|w\| = \prod_{i=1}^m \Xi_{H_i}(a_i).
\end{aligned}
$$

Note that the modular function δ_{H_i} of $H_i \cap B$ is equal to $\exp(\beta_i)$. Hence by Proposition 3.1, for each i, there exists a constant C_i (not depending on a) such that

$$
\Xi_{H_i}(a_i) \le C_i\ \exp(-\frac{1}{2}+\epsilon)\beta_i(a).
$$

This proves the proposition. □

3.5. Proof of Theorem A. It is well known ([1], Theorem 7.2) that G contains a connected simple closed subgroup G_0 such that G_0 is split over $\mathbb{R}$, rank $G_0 = \mathbb{R}$-rank G and Φ' is isomorphic to $\Phi(\mathfrak{g}_0, \mathfrak{g}_0 \cap \mathfrak{a})$ where Φ' is the set of all non-multipliable roots in $\Phi_{\mathbb{R}}(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_0$ is the Lie algebra of G_0. Recall the strongly orthogonal system $\mathcal{Q}(\Phi') = \{\gamma_1, \cdots, \gamma_{N(\Phi')}\}$ of Φ' we constructed in section 2.2.

The notation $\mathfrak{u}_\alpha$ for $\alpha \in \Phi'$ denotes the one-dimensional root subalgebra of $\mathfrak{g}_0$. Set $N = N(\Phi')$. For each $1 \le i \le N$, we define H_i to be the connected closed subgroup of G_0 whose Lie algebra is generated by $\mathfrak{u}_{\pm\gamma_i}$.

Case ($\Phi' \ne D_{n=2k+1}$): We note that the restriction $F|_{A^+}$ of the function F to A^+ in Theorem A is equal to $\prod_{i=1}^N {\delta_{H_i}}^{-1/2}$ or equivalently,

$$
F|_{A^+} = \prod_{i=1}^N \exp(-\frac{1}{2}\gamma_i).
$$

Then Theorem A follows from Proposition 3.4.

Case ($\Phi' = D_{n=2k+1}$): In this case, we define H'_N to be the connected closed subgroup of G_0 whose Lie algebra is generated by $\mathfrak{u}_{\pm(\alpha_{k+1}+2(\alpha_{k+2}+\cdots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n)}$, $\mathfrak{u}_{\pm\alpha_k}$ and $\mathfrak{u}_{\pm\alpha_{k+1}}$. Note that the Lie algebra of H'_N is isomorphic to that of $SO(3,3)$. We have that

$$
\delta_{H'_N} = \exp(4\alpha_k)\left(\prod_{i=k+1}^{n-2}\exp(6\alpha_i)\right)\exp(3\alpha_{n-1})\exp(3\alpha_n).
$$

By [7, Lemma 4.1], the restriction $\rho|_{H'_N}$ is strongly $L^{4+\epsilon}$ for any $\rho \in \hat{G}$. This implies (see [3, Corollary 7.2]) that the restrictions to H'_N of the K-finite matrix coefficients (with respect to unit vectors) of ρ are bounded by $\Xi_{H_N'}^{1/2}$. It is not difficult to see from the proof of Proposition 3.4 that, when we replace H_N by $H_N{}'$, a statement similar to Proposition 3.4 holds, that is, the K-matrix coefficients of ρ for any $\rho \in \hat{G}_K$ are bounded above by $(\prod_{i=1}^{N-1} \delta_{H_i}^{-1/2})\delta_{H_N'}^{-1/4}$. Therefore it remains to observe that the function F in Theorem A is given by

$$F|_{A^+} = \left(\prod_{i=1}^{n-2} \exp(-\frac{1}{2}\gamma_i)\right) \exp(\alpha_k) \left(\prod_{i=k+1}^{n-2} \exp(\frac{3}{2}\alpha_i)\right) \exp(\frac{3}{4}\alpha_{n-1}) \exp(\frac{3}{4}\alpha_n),$$

which is equal to $(\prod_{i=1}^{n-2} \delta_{H_i}^{-1/2})\delta_{H_N'}^{-1/4}$, to complete the proof.

Remark. The results in section 2.2 show that the function F is the best possible upper bound for K-matrix coefficients, which can be obtained using Proposition 3.4 when $\Phi' \neq D_{n=2k+1}$. Note that when $\Phi' = D_{n=2k+1}$, we improved F by replacing one $SL_2(\mathbb{R})$ by $SO(3,3)$.

3.6. Corollary. *With G, Φ' and $\alpha_1, \cdots, \alpha_n$ as in section 3.5, suppose that $\mathcal{O} = \{\beta_1, \cdots, \beta_t\}$ is a strongly orthogonal system of Φ' and that for some number r, the coefficient of α_j in $\sum_{i=1}^t r\beta_i$ is strictly bigger than the coefficient of α_j in $2\log(\delta_G)$ for each $1 \leq j \leq n$. Then we have*

$$p(G) \leq r \ \text{and} \ p_K(G) \leq r.$$

3.7. In each simple real-split Lie group G, the modular function δ_G of B is given as below (cf. [2]), from which the remark after Corollary C follows.

Φ	$\log \delta$
A_n	$\sum_{i=1}^{n} i(n-i+1)\alpha_i$
B_n	$\left(\sum_{i=1}^{n-1} (2ni - i^2)\alpha_i\right) + n^2\alpha_n$
C_n	$\left(\sum_{i=1}^{n} (2ni - i^2 + i)\alpha_i\right) + \frac{n(n+1)}{2}\alpha_n$
D_n	$\left(\sum_{i=1}^{n-2} (2ni - i^2 - i)\alpha_i\right) + \frac{n(n-1)}{2}(\alpha_{n-1} + \alpha_n)$
E_6	$16\alpha_1 + 22\alpha_2 + 30\alpha_3 + 42\alpha_4 + 30\alpha_5 + 16\alpha_6$
E_7	$34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7$
E_8	$92\alpha_1 + 136\alpha_2 + 182\alpha_3 + 270\alpha_4 + 220\alpha_5 + 168\alpha_6 + 114\alpha_7 + 58\alpha_8$
F_4	$16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4$
G_2	$10\alpha_1 + 6\alpha_2$

4. Spherical unitary representations with minimal decay in $SL_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$

4.1. In this section we will show that the upper bound function F we obtained in Theorem A is the best possible when $G = SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. This will be proved by showing that there exists a spherical unitary representation of G whose K-matrix coefficients are bounded from below by a constant multiple of F. Those representations are the induced representations $\mathrm{Ind}_P^G(I)$ of the trivial representation where P is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$.

4.2. For the rest of section 4, let G be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. Set $K = SO_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ respectively. Define the maximal parabolic subgroup P of G as follows:

$$\text{for } G = SL_n(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\};$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{(2n)j} = 0 \text{ if } i \neq 1, j \neq 2n\}.$$

We fix an ordering in the root system of G so that the positive Weyl chamber A^+ is as follows:

$$SL_n(\mathbb{R}), A^+ = \{\mathrm{diag}(a_1, \cdots, a_n) \mid \prod_{i=1}^{n} a_i = 1,\ a_i \geqq a_{i+1} \text{ for all } 1 \leq i \leq n-1\};$$

$$Sp_{2n}(\mathbb{R}), A^+ = \{\mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1{}^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq n-1\}.$$

4.3. We recall the formula for the matrix coefficients of the induced representation $\mathrm{Ind}_P^G(I)$ (cf. [5]). Consider the Langlands decomposition of P: $P = MA_PN$. Denote by $\bar{N}$ the unipotent radical of the opposite parabolic subgroup to P with the common Levi subgroup MA_P.

If g decomposes under the decomposition $G = KMA_PN$, we denote by $\exp H(g)$ the A_P-component of g. It is well known that the representation space of $\mathrm{Ind}_P^G(I)$ of the trivial representation I of P can be realized as $L^2(\bar{N}, dx)$. If g decomposes under $\bar{N}MA_PN$ as

$$g = \bar{n}(g)m(g)a(g)n(g),$$

then the action is given by

$$\mathrm{Ind}_P^G(I)(g)f(x) = e^{-\delta_0(\log a(g^{-1}x))} f(\bar{n}(g^{-1}x)) \text{ for any } f \in L^2(\bar{N}, dx) \text{ and } x \in \bar{N}$$

where δ_0 is the half sum of positive N-roots.

Define the vector f_0 of $\mathrm{Ind}_P^G(I)$ as follows:

$$f_0(x) = e^{-\delta_0(H(x))}.$$

It is not difficult to see that f_0 is K-fixed and the matrix coefficient of $\mathrm{Ind}_P^G(I)$ with respect to f_0 is as follows:

$$\langle \mathrm{Ind}_P^G(I)(g) f_0, f_0 \rangle = \int_{\bar{N}} e^{-\delta_0(\log a(g^{-1}x))} e^{-\delta_0(H(\bar{n}(g^{-1}x)))} e^{-\delta_0(H(x))} dx.$$

4.4. Theorem D follows from the following proposition and Theorem A.

Proposition. *There exists a constant C such that*

$$C\,F(a) \le |\langle \mathrm{Ind}_P^G(I)(a) f_0, f_0\rangle|$$

where $a \in A^+$ and F is as in Theorem A

Proof of Proposition 4.4.

Case: $G = SL_n(\mathbb{R}), n \ge 3$

Denote by $\tilde{a}$ the matrix $\mathrm{diag}(a_1, \cdots, a_n) \in SL_n(\mathbb{R})$ and by x the matrix in $\bar{N}$ whose first column is $(1, x_2, \cdots, x_n)$, that is, $x.e_1 = (1, x_2, \cdots, x_n)$. To simplify notation, set $x_1 = 1$.

The decomposition of $\tilde{a}^{-1}x$ under $\bar{N}MA_PN$ is as follows:

$$a(\tilde{a}^{-1}x) = \mathrm{diag}(a_1, a_1^{-1/(n-1)}, \cdots, a_1^{-1/(n-1)}) \text{ and } \bar{n}(\tilde{a}^{-1}x).e_1 = \left(1, \frac{a_1}{a_2}x_2, \cdots, \frac{a_1}{a_n}x_n\right).$$

Then $\delta_0(a(\tilde{a}^{-1}x)) = a_1^{-n/2}$ and $H(x) = \|x.e_1\| = \sqrt{\sum_{i=1}^n {x_i}^2}$.

Therefore

$$\begin{aligned}\langle \mathrm{Ind}_P^G(I)(\tilde{a}) f_0, f_0\rangle &= \int_{\bar{N}} {a_1}^{n/2} \|\bar{n}(\tilde{a}^{-1}x).e_1\|^{-n/2} \|x.e_1\|^{-n/2}\, dx \\ &= \int_{\mathbb{R}^{n-1}} \left(\sum_{i=1}^n \left(\frac{1}{a_i}\right)^2 x_i^2\right)^{-n/4} \left(\sum_{i=1}^n x_i^2\right)^{-n/4} dm\end{aligned}$$

where dm is the standard measure in $\mathbb{R}^{n-1}$.

Set $k = [\frac{n+1}{2}]$ and let T be the following set:

$$\{(x_2, \cdots, x_n) \mid 0 \le x_i \le 1 \text{ for } 2 \le i \le k-1, 1 \le x_k \le 2, x_i \le \frac{a_i}{a_k}x_k \text{ for } k+1 \le i \le n\}.$$

Note that if $(x_2, \cdots, x_n) \in T$, then for each $1 \le i \le n$, we have

$$x_i \le 2 \quad \text{and} \quad \frac{x_i}{a_i} \le \frac{x_k}{a_k}.$$

Thus for $(x_2, \cdots, x_n) \in T$, we have

$$\left(\sum_{i=1}^n \left(\frac{1}{a_i}\right)^2 x_i^2\right)^{-n/4} \left(\sum_{i=1}^n x_i^2\right)^{-n/4} \ge C\, {a_k}^{n/2}$$

for some constant $C > 0$.

Therefore

$$|\langle \mathrm{Ind}_P^G(I)(\tilde{a}) f_0, f_0\rangle| \ge C \int_T {a_k}^{n/2} dm \ge C\, {a_k}^{n/2} \prod_{i=k+1}^n \left(\frac{a_i}{a_k}\right) \ge C\, F(\tilde{a}).$$

Case: $G = Sp_{2n}(\mathbb{R}), n \ge 2$

For $\tilde{a} = \text{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \in Sp_{2n}(\mathbb{R})$, we have

$$\langle \text{Ind}_P^G(I)(\tilde{a})f_0, f_0\rangle = \int_{\mathbb{R}^{2n-1}} \left(\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^2 + \sum_{i=1}^{n}(a_i y_i)^2\right)^{-n/2}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2\right)^{-n/2} dm$$

where $x_1 = 1$. Let T be the following set:

$$\{(x_1, \cdots, x_n, y_n, \cdots, y_1) \mid 1 \le y_n \le 2, y_i \le \frac{a_n}{a_i} y_n, 0 \le x_i \le 1, 0 \le y_i \le 2 \text{ for } 1 \le i \le n\}.$$

Note that if $(x_1, \cdots, x_n, y_n, \cdots, y_1) \in T$, then

$$x_i \le a_i a_n y_n$$

since $a_i \ge 1$ for all $1 \le i \le n$. Therefore

$$|\langle \text{Ind}_P^G(I)(\tilde{a})f_0, f_0\rangle| \ge C \int_T (a_n y_n)^{-n} dm \ge C\, F(\tilde{a})$$

where C is some positive constant, finishing the proof.

5. (G,K)-TEMPERED SUBGROUPS AND FINITE DIMENSIONAL REPRESENTATIONS

5.1. Let H be a linear connected non-compact semisimple Lie group. Let B_H be a minimal parabolic subgroup, A_H a maximal $\mathbb{R}$-split torus contained in B_H and K_H a maximal compact subgroup of H. Consider a Cartan decomposition of H: $H = K_H A_H^+ K_H$. Let A be the torus of $SL_n(\mathbb{R})$ consisting of all the diagonal elements and A^+ the positive Weyl chamber of $SL_n(\mathbb{R})$ given by

$$A^+ = \{\text{diag}(a_1, \cdots, a_n) \mid a_i \geqq a_{i+1} \text{ for all } 1 \le i \le n-1\}.$$

Let π be a representation of H to $SL_n(\mathbb{R})$ such that $\pi(A_H) \subset A$. For each $1 \le i \le n$, we define a weight β_i of $d\pi$ by

$$\beta_i(X) = (i,i)\text{-entry of the matrix } d\pi(X) \text{ for } X \in \log A_H,$$

where $d\pi$ denotes the differential of π. Denote by W the Weyl group of $SL_n(\mathbb{R})$. Using the well known isomorphism of W with the symmetric group on n letters, we can consider the action of W on $\{\beta_1, \cdots, \beta_n\}$ by $w(\beta_i) = \beta_{w(i)}$ for each $1 \le i \le n$.

For each $w \in W$, set

$$\mathfrak{a}_w = \{X \in \log(A_H^+) \mid w(\beta_i)(X) \ge w(\beta_{i+1})(X) \text{ for all } 1 \le i \le n-1\}.$$

Note that since $\mathfrak{a}_w = \{X \in \log(A_H^+) \mid d\pi(X) \in w^{-1}(\log A^+)w\}$, we have that $\log(A_H^+) = \cup_{w\in W}\mathfrak{a}_w$. It is not difficult to see that we can choose a subset $W_0 \subset W$ (not unique) so that $\log(A_H^+) = \cup_{w\in W_0}\mathfrak{a}_w$, the interior of $\mathfrak{a}_w$ is non-empty for each $w \in W_0$, and the interiors of $\mathfrak{a}_w$'s, $w \in W_0$ are disjoint. For example, if $\pi(A_H^+) \subset A^+$, then we can choose W_0 to consist of only the identity element of W.

We keep the above notation, such as H, A^+, π, $\beta_1, \cdots, \beta_n$, W_0, $\mathfrak{a}_w$, etc., for the rest of chapter 5. Recall also that δ_H denotes the modular function of B_H.

The following is an application of Theorem D when $G = SL_n(\mathbb{R})$.

Corollary. *The subgroup $\pi(H)$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup if and only if the following holds: for each $w \in W_0$ and all $X \in \mathfrak{a}_w$*

$$\begin{cases} w(\beta_1)(X) + \cdots + w(\beta_{n/2})(X) > \log(\delta_H)(X) & \text{if } n \text{ is even} \\ w(\beta_1)(X) + \cdots + w(\beta_{(n-1)/2})(X) + \frac{1}{2} w(\beta_{(n+1)/2})(X) > \log(\delta_H)(X) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Note that

$$\int_{A_H^+} (F \circ \pi)\, \delta_H d\, a = \sum_{w \in W_0} \int_{\exp \mathfrak{a}_w} (F \circ \pi)\, \delta_H d\, a.$$

On the other hand, on each $\mathfrak{a}_w$, the restriction of $-\log F \circ \pi$ to $\log A_H^+$ is equal to the function in the left in the above inequality (see Example before Theorem D). This proves the claim by Theorem D. □

Example. If H is simple and Ad is the adjoint representation of H, we can consider $Ad(H)$ to be a subgroup of $SL_n(\mathbb{R})$ where $n = \dim(\mathrm{Lie}(H))$. Since the restriction of $-\log F \circ Ad$ to $\log A_H^+$ is equal to $\log \delta_H$, we have that $Ad(H)$ is not an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup by the above corollary.

5.2. Let $\lambda_1, \ldots, \lambda_k$ the fundamental weights of the Lie algebra of H corresponding to A_H^+. For any weights γ_1 and γ_2 of the Lie algebra of H, we define a partial order $>$ so that $\gamma_1 > \gamma_2$ if and only if $(\gamma_1, \lambda_j) > (\gamma_2, \lambda_j)$ for all $1 \le j \le k$. This is equivalent to saying that the coefficient of each simple root in $\gamma_1 - \gamma_2$ is positive, or $\gamma_1(X) > \gamma_2(X)$ for all $X \in \log A_H^+$.

If λ is the highest weight of an irreducible representation, then the lowest weight, which we will denote by $\Lambda(\lambda)$, is given by

$$(\Lambda(\lambda), \lambda_j) = -(\lambda, i(\lambda_j)) \text{ for each } 1 \le j \le k,$$

where i is the opposition involution of the root system of $\mathrm{Lie}(H)$ (cf. [11]).

Corollary. *Let H be a linear connected semisimple Lie group and π an irreducible representation with the highest weight λ. Suppose that*

$$\lambda - \Lambda(\lambda) > 2 \log \delta_H.$$

Then $\pi(H)$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup.

Proof. Let $w \in W_0$. Since λ and $\Lambda(\lambda)$ are the highest weight and the lowest weight of π respectively, it follows from the definition of $\mathfrak{a}_w$ that

$$w(\beta_1) = \lambda \text{ and } w(\beta_n) = \Lambda(\lambda).$$

Let X be any element in $\mathfrak{a}_w$. Since $w(\beta_i)(X) \ge w(\beta_{i+1})(X)$ for each $1 \le i \le n-1$, we have that if n is even,

$$2 \sum_{i=1}^{n/2} w(\beta_i)(X) \ge 2w(\beta_1)(X) + \sum_{i=2}^{n-1} w(\beta_i)(X),$$

and if n is odd

$$\sum_{i=1}^{(n-1)/2} w(\beta_i)(X) + w(\beta_{(n+1)/2})(X) \ge 2w(\beta_1)(X) + \sum_{i=2}^{n-1} w(\beta_i)(X).$$

On the other hand, since $\sum_{i=1}^{n} \beta_i = 0$,

$$2w(\beta_1) + \sum_{i=2}^{n-1} w(\beta_i) = w(\beta_1) - w(\beta_n),$$

which is equal to $\lambda - \Lambda(\lambda)$. Therefore the assumption that $\lambda - \Lambda(\lambda) > 2\log \delta_H$ implies the inequalities in Corollary 5.1, finishing the proof. □

Remark. By the remark prior to Corollary 5.2 and the fact that

$$(\log \delta_H, \lambda_j) = (\log \delta_H, i(\lambda_j)) \text{ for each } 1 \leq j \leq k,$$

we have that if $\lambda > \log \delta_H$, then $\lambda - \Lambda(\lambda) > 2\log \delta_H$; so the hypothesis of the above corollary is satisfied.

Example. If $H = SL_{k+1}(\mathbb{R})$ in Corollary 5.2, then

$$\lambda - \Lambda(\lambda) > 2\log \delta_H$$

is equivalent to the following:

$$c_j + c_{k+1-j} > 2j(k+1-j) \text{ for } 1 \leq j \leq k$$

where $c_j = (\lambda, \lambda_j)$.

5.3. Examples. The following examples are applications of Corollary 5.1.

1. If π is an irreducible representation of $SL_2(\mathbb{R})$ into $SL_n(\mathbb{R})$, then it is well known that $(\lambda, \lambda_1) = \frac{n-1}{2}$; whereas $(\log \delta_H, \lambda_1) = 1$. Therefore $\pi(SL_2(\mathbb{R}))$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup if and only if $n \geqq 4$.
2. The embedding of $SL_k(\mathbb{R})$ as the first k by k diagonal block matrix in $SL_n(\mathbb{R})$ is not an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup for any positive integers k and n.
3. For matrices A of order m and B of order k, the Kronecker product $A \otimes B$ of A and B is the matrix of order mk such that the (ij)-matrix block of $A \otimes B$ is $a_{ij}B$ where a_{ij} is the (ij)-entry of A.

The group $SL_m(\mathbb{R}) \otimes I_k$ is an $(SL_{mk}(\mathbb{R}), SO_{mk}(\mathbb{R}))$-tempered subgroup if and only if $k > 2(m-1)$.

5.4. In this section we consider the case when π is symplectic or orthogonal. It is worthwhile to state the following fact, which enables us to tell when an irreducible representation π with the highest weight λ has such a property.

Theorem. *(*[11]*, Ch 3, Theorem 2.15) The representation π is self-dual if and only if $\lambda = -\Lambda(\lambda)$. In such cases, π is orthogonal (resp. symplectic) if* $\sum_{j=1}^{k} (\log \delta_H, \lambda_j)(\lambda, \lambda_j)$ *is even (resp. odd).*

We remark that all finite dimensional irreducible representations of H are self-dual unless H is of type A_n, D_{2k+1} or E_6.

5.5. We have the following corollary of Theorem D when $G = Sp_n(\mathbb{R})$, which is analogous to Corollaries 5.1 and 5.2.

We use the same realization of $Sp_n(\mathbb{R})$ as in section 4.2 so that a positive Weyl chamber of $Sp_n(\mathbb{R})$ is the following:

$$Sp_n(\mathbb{R}) \cap A^+ = \{\mathrm{diag}(a_1, \cdots, a_{n/2}, a_{n/2}^{-1}, \cdots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq \frac{n}{2} - 1\}.$$

Corollary. *Let H be a linear connected semisimple Lie group and π a representation such that $\pi(H) \subset Sp_n(\mathbb{R})$.*

(1) *The subgroup $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered subgroup if and only if for each $w \in W_0$,*

$$w(\beta_1)(X) + \cdots + w(\beta_{n/2})(X) > \log \delta_H(X) \text{ for all } X \in \mathfrak{a}_w.$$

(2) *Furthermore assume that π is irreducible with the highest weight λ. Suppose that*

$$\lambda > \log \delta_H.$$

Then $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered subgroup.

Proof. The proof of the first claim is similar to that of Corollary 5.1; so we will omit it. Since λ is the highest weight, $w(\beta_1) = \lambda$ for each $w \in W_0$. Since $w(\beta_i)(X) \geq 0$ for any $X \in \mathfrak{a}_w$ and each $1 \leq i \leq \frac{n}{2}$, we have $\sum_{i=1}^{n/2} w(\beta_i)(X) \geq \lambda(X)$. Now the second claim follows from the first one. □

5.6. We consider a realization of $SO(m, n-m)$, $m = [\frac{n}{2}]$ so that a positive Weyl chamber of $SO(m, n-m)$ is given by $SO(m, n-m) \cap A^+$, that is, if n is even,

$$\{\mathrm{diag}(a_1, \cdots, a_m, a_m^{-1}, \cdots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq m-1\}$$

and if n is odd,

$$\{\mathrm{diag}(a_1, \cdots, a_m, 1, a_m^{-1}, \cdots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq m-1\}.$$

Corollary. *Let H be a linear connected semisimple Lie group and π an n-dimensional irreducible representation with the highest weight λ such that $\pi(H) \subset SO(m, n-m)$ where $m = [\frac{n}{2}]$. Suppose that*

$$\lambda > \log \delta_H.$$

Then $\pi(H)$ is an $(SO(m, n-m), SO(m, n-m) \cap SO_n(\mathbb{R}))$-tempered subgroup.

Proof. Consider the case when n is even. Let $p = [\frac{n}{4}]$. Then for any $w \in W_0$ and any $X \in \mathfrak{a}_w$, the function F in Theorem A is such that

$$-\log F \circ \pi(X) = w(\beta_1)(X) + \cdots + w(\beta_p)(X).$$

Therefore by the same argument as in the previous corollary, it is enough to show that

$$w(\beta_1)(X) + \cdots + w(\beta_p)(X) > \log \delta_H(X).$$

This is true since $w(\beta_i)(X) \geq 0$ for all $1 \leq i \leq p$ and $w(\beta_1) = \lambda$. The proof in the case when n is odd is similar. □

Example. If $H = SL_{k+1}(\mathbb{R})$ and $c_j = (\lambda, \lambda_j)$ for $1 \le j \le k$, then π is self-dual if and only if $c_j = c_{k+1-j}$ for $1 \le j \le k$, and the condition $\lambda > \log \delta_H$ is equivalent to the condition $c_j > 2j(k+1-j)$ for each $j = 1, \cdots, k$. Therefore with these two conditions satisfied, if $\sum_{i=1}^k i(k+1-i)c_i$ is even, then $\pi(SL_{k+1}(\mathbb{R}))$ is an $(SO(m,n-m), SO(m,n-m)\cap SO_n(\mathbb{R}))$-tempered subgroup where $m = [\frac{n}{2}]$, and if $\sum_{i=1}^k i(k+1-i)c_i$ is odd, then $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered subgroup.

Moreover in the case when $H = SL_2(\mathbb{R})$ and π is an n-dimensional irreducible representation with $n \ge 4$ (cf. Example 5.3), the subgroup $\pi(SL_2(\mathbb{R}))$ is $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered if n is even; otherwise it is $(SO(m,n-m), SO(m,n-m) \cap SO_n(\mathbb{R}))$-tempered.

5.7. Unipotent tempered subgroups. Lastly we give examples of some unipotent tempered subgroups of $G = SL_n(\mathbb{R})$. In order to apply Theorem D when H is not semisimple, we need to know how each element of H decomposes under the Cartan decomposition of G.

Consider the decomposition of the element $v_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ as $k_1 a k_2$ under the Cartan decomposition of $SL_2(\mathbb{R})$ with $K = SO_2(\mathbb{R})$ and A the torus consisting of all the diagonal elements. Since $v_s(v_s)^t = k_1 a^2 k_1^{-1}$, the eigenvalues of a^2 coincide with those of $v_s(v_s)^t$. If $a = \mathrm{diag}(b, b^{-1})$, then $b = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2+4})}$.

Consider the one parameter unipotent subgroup U_{ij} of $SL_n(\mathbb{R})$ consisting of the elements $u_{ij}(s) = I + sE_{ij}$, $s \in \mathbb{R}$, where $i \ne j$ and E_{ij} is the elementary matrix whose non-zero entry is 1 only at (i,j). We keep the same notation as in section 5.1. Then the A^+-component of $u_{ij}(s)$ under the Cartan decomposition of $SL_n(\mathbb{R})$ is $\mathrm{diag}(b, 1, \cdots, 1, b^{-1})$ where $b = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2+4})}$ by the previous argument.

Therefore $F(u_{ij}(s))$ is equal to $(\sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2+4})})^{-1}$.

Proposition. *Let $n \ge 2$ and $i \ne j$.*

(1) *For any $\epsilon > 0$, the restriction $F|_{U_{ij}}$ is $L^{1+\epsilon}(U_{ij})$-integrable; hence U_{ij} is not an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup.*
(2) *The diagonal embedding $\delta(U_{ij}) = \{(g,g) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \mid g \in U_{ij}\}$ is an $(SL_n(\mathbb{R}) \times SL_n(\mathbb{R}), SO_n(\mathbb{R}) \times SO_n(\mathbb{R}))$-tempered subgroup.*

Proof. The part (1) is clear. For the second claim, see the remark following Corollary B. □

Now consider the unipotent one-parameter subgroup U of $SL_4(\mathbb{R})$ consisting of the elements $U(s) = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$. It is easy to see that the following proposition holds.

5.8. Proposition. *The subgroup U is an $(SL_4(\mathbb{R}), SO_4(\mathbb{R}))$-tempered subgroup.*

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Mathematics department, Yale University, New Haven, CT 06520, U. S. A
Current address: Institute of Mathematics, The Hebrew University, Jerusalem 91904, ISRAEL
E-mail address: heeoh@math.huji.ac.il