TEMPERED SUBGROUPS AND REPRESENTATIONS
WITH MINIMAL DECAY OF MATRIX COEFFICIENTS

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ABSTRACT. We present a function $F$ for each simple real linear Lie group $G$ with real rank at least 2 such that $F$ bounds from above all the $K$-matrix coefficients of non-trivial irreducible spherical unitary representations of $G$ where $K$ is a maximal compact subgroup of $G$. This enables us to determine when a closed subgroup $H$ is a $(G,K)$-tempered subgroup of $G$: for example, if the restriction $F|_H$ of $F$ to $H$ lies in $L^{1-\epsilon}(H)$. We also prove that this function $F$ is the best possible for $G$ a real-split group of type $A_n$ or $C_n$ and as a consequence, we obtain that if $H$ is semisimple, then $H$ is a $(G,K)$-tempered subgroup of $G$ if and only if $F|_H$ lies in $L^1(H)$.

Résumé. Nous présentons une fonction $F$ pour chaque $G$, groupe de Lie linéaire réel simple de rang réel au moins 2, telle que $F$ donne une borne supérieure pour tous les coefficients matriciels $K$-finis des représentations unitaires sphériques irréductibles de $G$, avec $K$ un sous-groupe compact maximal de $G$. Ceci nous permet à déterminer quand un sous-groupe fermé $H$ de $G$ est $(G,K)$-tempéré; par exemple, c’est le cas si la restriction de $F$ à $H$ est dans $L^{1-\epsilon}(H)$. Nous prouvons aussi que cette fonction $F$ est la meilleure possible pour $G$ un groupe réel déployé de type $A_n$ ou $C_n$, et comme conséquence, nous obtenons que si $H$ est semisimple, alors $H$ est un sous-groupe $(G,K)$-tempéré de $G$ si et seulement si $F|_H$ est dans $L^1(H)$.

1. Introduction

Let $G$ be a connected semisimple linear Lie group and $K$ a maximal compact subgroup of $G$. We say that a unitary representation of $G$ is spherical if it has a $K$-invariant vector. For a unitary spherical representation $\rho$, we will use the term “$K$-matrix coefficients of $\rho$” to refer to its matrix coefficients with respect to $K$-invariant unit vectors.

In this paper we are interested in the asymptotic behavior of the $K$-matrix coefficients of spherical unitary representations of $G$ when restricted to a closed subgroup $H$ of $G$. One motivation comes from the notion “$(G,K)$-tempered subgroups” of $G$ defined by Margulis [10]. That is, a closed subgroup $H$ of $G$ is called $(G,K)$-tempered if there exists a (positive) function $q \in L^1(H)$ such that for any non-trivial irreducible spherical unitary representation $\rho$ of $G$, $|\langle \rho(h)v,w \rangle| \leq q(h)||v|| ||w||$ for all $h \in H$ and any $K$-fixed vectors $v$ and $w$. Note that any compact subgroup of $G$ is a $(G,K)$-tempered subgroup for a trivial reason. Margulis also showed in [10] that if a closed subgroup $H$ is a $(G,K)$-tempered subgroup, then for any non-compact subgroup $F$ of $H$, the quotient $G/F$ does not allow a compact quotient by a discrete subgroup of $G$ (see [6] for a survey on the general problem).
We denote by $\hat{G}$ (resp. $\hat{G}_K$) the set of equivalence classes of non-trivial irreducible unitary (resp. spherical) representations.

In this paper we first present a “good upper bound function” for $K$-matrix coefficients for all representations in $\hat{G}_K$ for a simple real linear Lie group $G$ with real rank at least 2. Secondly we show that in simple real-split linear Lie group of type $A_n$ or $C_n$ this function is in fact the best possible by exhibiting a spherical representation of $G$ in $\hat{G}_K$ whose $K$-matrix coefficients are bounded below by this function. We now formulate the main results.

The notation $[x]$ denotes the largest integer which is not greater than $x$.

**Theorem A.** Let $G$ be a connected simple real linear Lie group with real rank $n \geq 2$, $K$ a maximal compact subgroup, $B$ a minimal parabolic subgroup, $A \subset B$ a maximal $\mathbb{R}$-split torus, $A^+ \subset A$ the positive Weyl chamber given by the choice of $B$. Denote by $\Phi'$ the set of all non-multipliable roots in the relative root system $\Phi_\mathbb{R}(a, g)$ where $a$ and $g$ are the Lie algebras of $A$ and $G$ respectively. Let $\alpha_1, ..., \alpha_n$ be the basis of $\Phi'$ whose subscripts are determined by the highest weight given in section 2.1.

Then for any $\epsilon > 0$, there exists a constant $C$ (depending on $\epsilon$) such that for any $\rho \in \hat{G}_K$ and $f_0$ a $K$-invariant unit vector of $\rho$,

$$|\langle \rho(g) f_0, f_0 \rangle| \leq C F(g)^{1-\epsilon} \quad \text{for any } g \in G$$

where $F$ is the $K$-bi-invariant function defined on $A^+$ as follows according to the type of $\Phi'$:

$$\Phi' = -\log F$$

$A_n, n \geq 2$

$$\left\{ \begin{array}{ll}
\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \frac{i}{2} \alpha_i + \sum_{i=(n+1)/2}^{n} \frac{i}{2} \alpha_i & \text{for } n \text{ odd} \\
\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{i}{2} \alpha_i + \frac{n}{2} \alpha_{n/2+1} + \sum_{i=n/2+2}^{n} \frac{n-i+1}{2} \alpha_i & \text{for } n \text{ even}
\end{array} \right.$$

$B_n, n \geq 2$

$$\sum_{i=1}^{n-1} i \alpha_i + \sum_{i=\lceil n/2+1 \rceil}^{n} \frac{n}{2} \alpha_i$$

$C_n, n \geq 2$

$$\sum_{i=1}^{\lfloor n/2 \rfloor} i \alpha_i + \frac{n}{2} \alpha_n$$

$D_n, n \geq 4$

$$\sum_{i=1}^{\lfloor n/2 \rfloor} i \alpha_i + \sum_{i=\lceil n/2+1 \rceil}^{n-2} \frac{n}{2} \alpha_i + \frac{n}{4} (\alpha_{n-1} + \alpha_n)$$

$E_6$

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

$E_7$

$$2\alpha_1 + \frac{7}{2} \alpha_2 + 4\alpha_3 + 6\alpha_4 + \frac{9}{2} \alpha_5 + 3\alpha_6 + \frac{3}{2} \alpha_7$$

$E_8$

$$2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 8\alpha_4 + 7\alpha_5 + 5\alpha_6 + 3\alpha_7 + \alpha_8$$

$F_4$

$$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$$

$G_2$

$$2\alpha_1 + \alpha_2$$

**Corollary B.** With the same notation as in Theorem A, let $H$ be a closed subgroup of $G$. If the restriction $F|_{H}$ of $F$ to $H$ is in $L^{1-\epsilon}(H)$ for some $\epsilon > 0$, then $H$ is a $(G, K)$-tempered subgroup of $G$.

**Remark 1.** Suppose further that $H$ is a connected semisimple Lie subgroup of $G$ such that $A \cap H$ is a maximal $\mathbb{R}$-split torus of $H$ and $B \cap H$ is a minimal parabolic subgroup of $H$. 

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2
Let $\delta_H$ denote the modular function of $B \cap H$, that is, the product of all positive roots including the multiplicity. Let $\lambda_1, \cdots, \lambda_m$ be the fundamental weights of the Lie algebra of $H$ corresponding to $A \cap H$ and $B \cap H$. For any two weights $\alpha$ and $\beta$ of the Lie algebra of $H$, we write $\alpha < \beta$ if $(\alpha, \lambda_j) < (\beta, \lambda_j)$ for all $1 \leq j \leq m$.

Then the condition $F|_H \in L^{1-\epsilon}(H)$ is equivalent to

$$-\log F|_{A^+ \cap H} > \log \delta_H;$$

which is again equivalent to the condition $F|_H \in L^1(H)$.

2. If the restriction $F|_H$ is $L^{k-\epsilon}(H)$-integrable for some $\epsilon > 0$ and some positive integer $k$, then the diagonal embedding $\delta_k(H)$ of $H$ into the group $\prod_{i=1}^k G_i$ is a $(\prod_{i=1}^k G_i, \prod_{i=1}^k K_i)$-tempered subgroup of $\prod_{i=1}^k G_i$ where $G_i = G$ and $K_i = K$ for all $1 \leq i \leq k$. To see this, it is enough to note that for any non-trivial irreducible spherical representation $\rho$ of $\prod_{i=1}^k G_i$, the restrictions of the $K$-matrix coefficients of $\rho$ to $\delta_k(H)$ are bounded by $(F|_H)^{k(1-\epsilon)}$.

For a unitary representation $\rho$ of $G$, $\rho$ is said to be strongly $L^q$ if there is a dense subset $V$ in the Hilbert space attached to $\rho$ such that the matrix coefficients of $\rho$ with respect to the vectors in $V$ lie in $L^q(G)$. Let $p(G)$ be the smallest real number such that for any $\rho \in \hat{G}$, $\rho$ is strongly $L^q$ for any $q > p(G)$ (cf. [7]). Similarly let $p_K(G)$ be the smallest real number such that for any $\rho \in \hat{G}_K$, the $K$-matrix coefficients of $\rho$ are $L^q(G)$-integrable for any $q > p_K(G)$.

The estimate of the Harish-Chandra function $\Xi$ of $G$ shows that $p_K(G)$ is at least 2 (cf. [3]) and hence $G$ cannot be a $(G, K)$-tempered subgroup of itself. The method used in proving Theorem A yields upper bounds for both $p(G)$ and $p_K(G)$.

The following follows from remark (1) after Corollary B.

**Corollary C.** With the same notation as in Theorem $A$, let $\delta_G$ be the modular function of $B$ (cf. Table 3.7). Define

$$r(G) = \max\{\frac{\text{the coefficient of } \alpha_i \text{ in } \log \delta_G}{\text{the coefficient of } \alpha_i \text{ in } -\log F} \mid i = 1, \cdots, n\}. $$

Then $p(G) \leq r(G)$ and $p_K(G) \leq r(G)$.

If $G$ is split over $\mathbb{R}$, $r(G)$ is as follows:

$$\Phi = \Phi': \ A_n \ B_n \ C_n \ D_n \ E_6 \ E_7 \ E_8 \ F_4 \ G_2$$

$$r(G): \ 2n \ 2n \ 2n \ 2(n-1) \ 16 \ 18 \ 58 \ 11 \ 6$$

For $n \geq 3$, Vogan’s classification of unitary duals for $GL_n(D)$ yields that for $G = SL_n(D)$, $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $p(G)$ is $2(n-1), 2(n-1)$ and $2n - 1$ respectively and for $Sp_{2n}(\mathbb{R})$, it follows from Howe’s result in [3] that $p(G) = 2n$. The number $p(G)$ in other classical group cases was calculated by Li [7] and was given an upper bound by Li and Zhu [8] in exceptional split group cases. We remark that the numbers $r(G)$ in Corollary C coincide with $p(G)$ calculated in [7] for all classical real split groups except $B_n$ type. For a split group of type $E_6$, by obtaining $r(G) = 16$, we improve the bound for $p(G)$ in [8].

We state a theorem which yields a necessary and sufficient condition for a closed semisimple subgroup to be a $(G, K)$-tempered subgroup in a simple real split linear Lie group $G$ of type $A_n$ or $C_n$. Let $G$ be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the
Let \( \Phi \) be an irreducible reduced root system with a fixed ordering. Denote by \( \Phi^+ \) the set of positive roots and by \( \Delta = \{\alpha_1, \cdots, \alpha_n\} \) the set of simple roots of \( \Phi \). The subscripts

\[\begin{align*}
\text{bi-linear form } & \left( \begin{array}{cc}
0 & I_n \\
-I_n & 0
\end{array} \right) \text{ where } I_n \text{ denotes the skew diagonal } n \times n \text{-identity matrix. Set } \\
K &= SO_n(\mathbb{R}) \text{ and } Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R}) \text{ respectively. Define the parabolic subgroup } P \text{ of } G \\
\text{as follows: }
\end{align*}\]

for \( G = SL_n(\mathbb{R}) \), \( P = \{ (g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1 \} \),

for \( G = Sp_{2n}(\mathbb{R}) \), \( P = \{ (g_{ij}) \in G \mid g_{i1} = 0, g_{2n+1} = 0 \text{ if } i \neq 1, j \neq 2n \} \).

Note that \( P \) is the maximal parabolic subgroup which stabilizes the line \( \mathbb{R}e_1 \). We fix an ordering in the root system of \( G \) so that the positive Weyl chamber \( A^+ \) is as follows:

\[\begin{align*}
SL_n(\mathbb{R}), A^+ &= \{ \text{diag}(a_1, \cdots , a_n) \mid \prod_{i=1}^n a_i = 1, a_i \geq a_{i+1} \text{ for all } 1 \leq i \leq n-1 \}, \\
Sp_{2n}(\mathbb{R}), A^+ &= \{ \text{diag}(a_1, \cdots , a_n, a_n^{-1}, \cdots , a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq n-1 \}.
\end{align*}\]

**Example.** The function \( F \) defined in Theorem A is as follows:

\[\begin{align*}
\text{for } G = SL_n(\mathbb{R}), & \quad F(a) = \left\{ \begin{array}{ll}
\prod_{i=1}^{n/2} a_i^{-1} & \text{for } n \text{ even} \\
\left( \prod_{i=1}^{(n-1)/2} a_i^{-1} \right) a_{(n+1)/2}^{-1/2} & \text{for } n \text{ odd}
\end{array} \right. \\
\text{for } G = Sp_{2n}(\mathbb{R}), & \quad F(a) = \prod_{i=1}^n a_i^{-1}
\end{align*}\]

where \( a \in A^+ \).

**Theorem D.** Let \( G \) be \( SL_n(\mathbb{R}) \) or \( Sp_{2n}(\mathbb{R}) \) and \( P, K \) and \( A^+ \) be as above.

1. For any \( \epsilon > 0 \), there exist constants \( C_1 \) and \( C_2 \) such that

\[ C_1 F(a) \leq |(\text{Ind}^G_P(I)(a)f_0, f_0)| \leq C_2 F(a)^{1-\epsilon} \]

for any \( a \in A^+ \) and for any \( K \)-invariant unit vector \( f_0 \) in \( \text{Ind}^G_P(I) \).

2. If a closed subgroup \( H \) of \( G \) is \((G, K)\)-tempered, \( F|_H \) is in \( L^1(H) \).

3. A closed semisimple subgroup \( H \) of \( G \) is \((G, K)\)-tempered if and only if \( F|_H \) is in \( L^1(H) \).

4. \( p_K(G) = r(G) = p(G) \).

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**2. A maximal system of strongly orthogonal roots in each irreducible root system**

**2.1.** Let \( \Phi \) be an irreducible reduced root system with a fixed ordering. Denote by \( \Phi^+ \) the set of positive roots and by \( \Delta = \{\alpha_1, \cdots, \alpha_n\} \) the set of simple roots of \( \Phi \). The subscripts
of \(\alpha_i\)'s are determined by the following choice of the highest root \([2]\).

\[
\begin{align*}
\Phi & \quad \text{the highest root} \\
A_n & \quad \alpha_1 + \alpha_2 + \cdots + \alpha_n \\
B_n & \quad \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n \\
C_n & \quad 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n \\
D_n & \quad \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\
E_6 & \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
E_7 & \quad 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \\
E_8 & \quad 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 \\
F_4 & \quad 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \\
G_2 & \quad 3\alpha_1 + 2\alpha_2
\end{align*}
\]

We define the number \(N(\Phi)\) as follows:

\[
N(\Phi) = \begin{cases} 
\frac{[n+1]}{2}, & \text{for } \Phi = A_n \\
2\left\lfloor \frac{n}{2} \right\rfloor, & \text{for } \Phi = D_n \\
4, & \text{for } \Phi = E_6 \\
\text{rank}(\Phi), & \text{for } \Phi = B, C, F_4, G_2, E_7, E_8 
\end{cases}
\]

2.2. Construction of some strongly orthogonal roots. Two roots \(\alpha\) and \(\beta\) are called strongly orthogonal if neither one of \(\alpha \pm \beta\) is a root. Consider the family \(S(\Phi)\) of all subsets of \(\Phi^+\) whose elements are pairwise strongly orthogonal. We call an element \(O\) in \(S(\Phi)\) a strongly orthogonal system. Let \(f\) be the function on \(S(\Phi)\) given by \(f(O) = \sum_{\alpha \in O} \alpha\). The aim of this section is to construct an element \(Q(\Phi) = \{\gamma_1, \ldots, \gamma_{N(\Phi)}\}\) in \(S(\Phi)\) on which \(f\) attains its maximum. For simplicity, we set \(N(\Phi) = N\).

We define \(Q(\Phi)\) as follows:

\[
\begin{align*}
\Phi & \quad Q(\Phi) \\
A_n & \quad \begin{cases} 
\gamma_i = \alpha_i + \cdots + \alpha_{n-i+1} & \text{for } i \leq N-1 \\
\gamma_N = \begin{cases} 
\alpha_N & \text{for } n \text{ odd} \\
\alpha_N + \alpha_{N+1} & \text{for } n \text{ even} 
\end{cases} \\
\gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_{n} \\
\gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} & \text{for } i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\gamma_n = \alpha_{(n+1)/2} + \cdots + \alpha_n & \text{for } n \text{ odd}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
B_n & \quad \begin{cases} 
\gamma_i = \alpha_i + \cdots + \alpha_{n-i} & \text{for } i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\gamma_n = \alpha_{(n+1)/2} + \cdots + \alpha_n & \text{for } n \text{ odd}
\end{cases}
\end{align*}
\]
Lemma. The set $Q(\Phi)$ is a strongly orthogonal system.

2.3. Proposition. $f(Q(\Phi)) = \max_{O \in S(\Phi)} f(O)$, that is, for any $O \in S(\Phi)$, the coefficient of $\alpha_i$ in $f(Q(\Phi))$ is greater than or equal to the coefficient of $\alpha_i$ in $f(O)$ for each $1 \leq i \leq n$, where $n$ is the rank of $\Phi$.

Proof. Let $O$ be any element in $S(\Phi)$. We prove this proposition by induction. We can easily check that the proposition is true for $n = 2$. Suppose that $n \geq 3$.

For $\Phi = A_n$, take any element in $O$, say $\alpha = \alpha_1 + \cdots + \alpha_{j-1}$, $i < j - 1$. Then $\alpha \leq \gamma_1$ since $\gamma_1$ is the highest root. On the other hand $O - \{\alpha\}$ is contained in $\{\alpha_m + \cdots + \alpha_{j-1} \mid m, l \notin \{i, j\}\}$,
which is a root system of type $A_{n-2}$. Note that $\mathcal{Q}(A_n) \cap \{\alpha_m + \cdots + \alpha_{i-1} \mid m, l \notin \{i, j\}\} = \mathcal{Q}(A_{n-2})$. Therefore by the induction assumption, $f(\mathcal{O} - \{\alpha\}) \leq f(\mathcal{Q}(A_{n-2}))$. Hence we have $f(\mathcal{O}) \leq \gamma_1 + f(\mathcal{Q}(A_{n-2})) \leq f(\mathcal{Q}(A_n))$, proving the claim.

For $\Phi = B_n$, note that for any $\alpha \in \Phi^+$, we have that the coefficient of $\alpha_1$ in $\alpha$ is at most 1. Write $\mathcal{O}$ as $\mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of $\alpha_1$ in $\beta$ is 1 and $\mathcal{O}_2 = \mathcal{O}_1^c$. It is not difficult to check that if three positive roots in $B_n$ are mutually strongly orthogonal, then the coefficient of $\alpha_1$ in at least one of them is 0. Therefore $|\mathcal{O}_1| \leq 2$. We can easily see that for any two strongly orthogonal roots $\beta_1, \beta_2 \in \Phi^+$ such that the coefficient of $\alpha_1$ in $\beta_i$ is 1 for both $i = 1, 2$, we have $\beta_1 + \beta_2 \leq \sum_{i=1}^{n} 2\alpha_i$; hence $\sum_{\beta \in \mathcal{O}_1} \beta \leq \gamma_1 + \gamma_2$, because $|\mathcal{O}_1| \leq 2$ and $\gamma_1 + \gamma_2 = \sum_{i=1}^{n} 2\alpha_i$. For $\theta \subset \Delta$, the notation $[\theta]$ denotes the set of the roots in $\Phi$ which can be expressed as integral combinations of the roots in $\theta$. Since $\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, $\gamma_3 + \gamma_4 = \sum_{i=2}^{n} 2\alpha_i$ and $[\alpha_2, \cdots, \alpha_n]$ is a root system of type $B_{n-1}$, we can proceed by induction as before.

The argument for $D_n$ is exactly the same as the one for $B_n$; so we omit it.

If $\Phi$ is of type $C_n$, write $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of $\alpha_1$ in $\beta$ is positive and $\mathcal{O}_2 = \mathcal{O}_1^c$. It is easy to see that $|\mathcal{O}_1| \leq 1$. Therefore $\sum_{\alpha \in \mathcal{O}_1} \alpha \leq \gamma_1$, for $\gamma_1$ is the highest root in $\Phi$. Since $\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, it remains to use induction process.

For exceptional root system cases, we can prove the proposition by checking each root system case by case. □

As a corollary of the above proposition, we obtain that $\mathcal{Q}(\Phi)$ is a maximal element in $\mathcal{S}(\Phi)$ with respect to the inclusion ordering.

**Remark.** I learned from E. Vinberg that this construction of a strongly orthogonal system coincides with the so called Kostant’s cascade construction (cf. [9]), if $\Phi$ is one of the types $A_n, C_n$ or $G_2$. But in all cases the cardinalities of the sets in Kostant’s cascade construction coincide with the numbers $N(\Phi)$, which are the cardinalities of $\mathcal{Q}(\Phi)$ in our construction. We note that not all maximal strongly orthogonal systems in $\Phi$ have the same cardinality. For example, $\{\alpha_2, \alpha_4, 2\alpha_2 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$ is a maximal strongly orthogonal system in the root system of $F_4$.

We remark that if $\Phi$ is none of $A_n, C_n$ or $G_2$, the function $f$ attains its maximum in our construction but not in Kostant’s cascade construction.

### 3. An upper bound function for matrix coefficients

**in simple non-compact linear Lie groups**

**3.1.** Let $G$ be a connected semisimple non-compact linear Lie group, $B$ a minimal parabolic subgroup, $A$ a maximal $\mathbb{R}$-split torus contained in $B$, $A^+$ the positive Weyl chamber and $K$ a maximal compact subgroup. Consider a Cartan decomposition of $G$: $G = KA^+K$. Since the $K$-matrix coefficients of a spherical unitary representation are $K$-bi-invariant, they are determined by their restrictions to the $A^+$-part. Denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathfrak{a}$ the Lie algebra of $A$. We denote by $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ the set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$, which is endowed with the ordering given by $B$. If $G$ is split over $\mathbb{R}$, then $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ will be simply denoted by $\Phi(\mathfrak{a}, \mathfrak{g})$. If we fix a Haar measure $dg$ on $G$, then the modular function $\delta_G$ of $B$
is given as
\[ \delta_G = \prod_{\alpha \in \Phi^+_R(a, g)} \exp \alpha. \]

It is well known (cf. [3]) that the induced representation \( \text{Ind}_{B}^{G}(I) \) of the trivial representation of \( B \) is irreducible and has a unique (up to a sign) \( K \)-invariant unit vector, say \( f_0 \). The matrix coefficient of \( \text{Ind}_{B}^{G}(I) \) defined by \( g \mapsto (\text{Ind}_{B}^{G}(I)(g)f_0, f_0) \) is called the Harish-Chandra function of \( G \), which we will denote by \( \Xi_G \). When there is no confusion, \( \Xi_G \) will simply be denoted by \( \Xi \).

Harish-Chandra has shown the following:

**Proposition.** (cf. [3]) For any \( \epsilon > 0 \), there exist constants \( c_1 \) and \( c_2 \) such that
\[ c_1 \delta_G^{-1/2}(a) \leq \Xi_G(a) \leq c_2 \delta_G^{-1/2+\epsilon}(a) \]
for all \( a \in A^+ \).

Moreover the value of Harish-Chandra function \( \Xi \) of \( SL_2(\mathbb{R}) \) or \( PSL_2(\mathbb{R}) \) at \( \left( \begin{array}{cc} a_0 & 0 \\ 0 & a_0^{-1} \end{array} \right) \)
for \( a_0 > 1 \) is asymptotically \( (\log a_0)/a_0 \) up to some constant multiple.

3.2. We can write the Haar measure \( dg \) of \( G \) in terms of the Cartan decomposition \( KA^+K \) as follows:
\[ dg = \Delta(a)dk_1 da dk_2 \]
where \( \Delta(a) \) is a positive function on \( A^+ \) satisfying \( d_1(t)\delta(a) \leq \Delta(a) \leq d_2\delta(a) \) for all \( a \in \{ a \in A^+ | |\alpha(a)| \geq t \) for all \( \alpha \in \Phi^+_R(a, g) \} \) and for some constants \( d_1(t) \) and \( d_2 \) if \( t > 1 \) (cf. [3]).

For a \( K \)-matrix coefficient \( \phi(g) = \langle \rho(g)v, w \rangle \) of \( \rho \in \hat{G}_K \), it is well known that \( \phi \in L^p(G) \) if and only if \( \int_{A^+} |\phi(a)|^p \delta(a) da < \infty \).

3.3. **Proposition.** Let \( H \) be \( SL_2(\mathbb{R}) \) or \( PSL_2(\mathbb{R}) \). Suppose that for some \( k \geq 2 \), \( H \) acts on \( \mathbb{R}^k \) by a rational representation so that the only \( H \)-invariant vector is the origin. Let \( H \ltimes \mathbb{R}^k \) be the associated semidirect product. Let \( \rho \) be a unitary representation of \( H \ltimes \mathbb{R}^k \) without any \( \mathbb{R}^k \)-invariant vectors. Then we have
\[ |\langle \rho|_H(h)v, w \rangle| \leq \Xi_H(h)(\dim(K \cdot v) \dim(K \cdot w))^{1/2} \]
where \( h \in H, K = SO_2(\mathbb{R}) \) and \( v \) and \( w \) are \( K \)-finite unit vectors of \( \rho \).

Moreover if \( \rho \) is spherical, then the \( K \)-matrix coefficients of \( \rho|_H \) are bounded by \( \Xi_H \).

**Proof.** By [12, Theorem 7.3.9], the restriction \( \rho|_H \) of \( \rho \) to \( H \) is weakly contained in the infinite sum of the regular representation of \( H \). It is well known (cf. [4, Ch V, Theorem 3.2 1]) that the \( K \)-finite (or \( K \)-fixed) matrix coefficients of the regular representation of \( H \) satisfy the above inequality. □

In the spirit of Howe’s strategy (see [7]) we state the following proposition.

The notation \( u_\alpha \) for \( \alpha \in \Phi_R(a, g) \) denotes the root space in \( g \) corresponding to \( \alpha \).
3.4. Proposition. Let $G$ be a connected simple real split linear Lie group. Let $\{\beta_1, \cdots, \beta_m\} \subset \Phi^+(a, g)$ be a strongly orthogonal system. Then for any $\epsilon > 0$, there exists a constant $C$ such that for any $\rho \in \hat{G}_K$ and $K$-fixed unit vectors $v$ and $w$ of $\rho$, we have

$$|\langle \rho(a)v, w \rangle| \leq C \prod_{i=1}^{m} \exp\left((-\frac{1}{2} + \epsilon)\beta_i(a)\right)$$

for any $a \in A^+$.

Proof. For each $1 \leq i \leq m$, let $H_i$ be the connected closed subgroup of $G$ whose Lie algebra is generated by $u_{\pm \alpha}$. Note that for each $1 \leq i \leq m$, (1) $H_i$ is isomorphic to $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$; (2) the subgroups $H_i \cap B, H_i \cap A$ and $H_i \cap K$ are a minimal parabolic subgroup, a maximal $\mathbb{R}$-split torus and a maximal compact subgroup of $H_i$ respectively; (3) the positive Weyl chamber $A^+(H_i)$ of $H_i$ is contained in $A^+$. It is not difficult to see that $A^+ \subset \prod_{i=1}^{m} A^+(H_i)C_G(\prod_{i=1}^{m} H_i)$ where $C_G(\prod_{i=1}^{m} H_i)$ denotes the centralizer of $\prod_{i=1}^{m} H_i$. Since $\{\beta_1, \cdots, \beta_m\}$ is a strongly orthogonal system, it follows that $x_i x_j = x_j x_i$ for all $i \neq j$, $x_i \in H_i$ and $x_j \in H_j$. By looking at the root system, it is not difficult to see that for each $H_i$, there exists an abelian unipotent subgroup $U_i$ of $G$ of dimension at least 2 such that $H_i$ normalizes $U_i$ and $C_G(H_i) \cap U_i$ is trivial. It follows that for each $1 \leq i \leq m$, $H_i \ltimes \mathbb{R}^{k_i}$ can be considered to be a subgroup of $G$ where the semidirect product is as described in Proposition 3.3 and $k_i = \text{dim} U_i$.

Let $\rho \in \hat{G}_K$, and $v$ and $w$ be $K$-fixed unit vectors. The restriction $\rho|_{\prod_{i=1}^{m} U_i}$ can be written as a direct integral $\int_X \rho_\alpha d\mu(\alpha)$ where $X$ is some Borel measure space with measure $\mu$, $\alpha \in X$ and $\rho_\alpha$ is an irreducible representation of $\prod_{i=1}^{m} H_i$ for all $\alpha \in X$. Without loss of generality, we may assume that all $\rho_\alpha$’s are non-trivial spherical representations. Any element $a \in A^+$ can be uniquely written as $a = a_1 \cdots a_m c$ where $a_i \in A^+(H_i)$ and $c \in C_G(\prod_{i=1}^{m} H_i)$. Write $\rho(c)v$ and $w$ as $\int v_\alpha d\mu(\alpha)$ and $\int w_\alpha d\mu(\alpha)$ respectively where $v_\alpha$ and $w_\alpha$ are vectors in $\rho_\alpha$. Since $c$ centralizes each $H_i$, $\rho(c)v$ is $K \cap H_i$-fixed for all $1 \leq i \leq m$. Therefore there is no loss of generality in assuming that for all $\alpha \in X$, $v_\alpha$ and $w_\alpha$ are $K \cap H_i$-fixed for all $1 \leq i \leq m$. Fix $\alpha \in X$. Then $\rho_\alpha|_{\prod_{i=1}^{m} U_i} = \bigotimes_{i=1}^{m} \rho_{\alpha i}$, $v_\alpha = \bigotimes_{i=1}^{m} v_{\alpha i}$ and $w_\alpha = \bigotimes_{i=1}^{m} w_{\alpha i}$, where $\rho_{\alpha i}$ is a spherical irreducible representation of $H_i$ and $v_{\alpha i}$ and $w_{\alpha i}$ are $K \cap H_i$-fixed vectors for each $1 \leq i \leq m$. By Moore’s theorem (cf. [12, Theorem 2.1.9]), for each $1 \leq i \leq m$, the representation $\rho_{\alpha i}$ is non-trivial and $\rho_{\alpha i}$ has no $U_i$-invariant vector.

By Proposition 3.3, we obtain that for each $1 \leq i \leq m$,

$$|\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i} \rangle| \leq \Xi_{H_i} \|v_{\alpha i}\| \cdot \|w_{\alpha i}\|.$$
Hence

\[ |\langle \rho(a)v, w \rangle| \leq \int_\alpha |\langle \rho_\alpha (\prod_{i=1}^{m} a_i)v_\alpha, w_\alpha \rangle| \, d\mu(\alpha) \]

\[ \leq \int_\alpha \prod_{i=1}^{m} |\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i} \rangle| \, d\mu(\alpha) \]

\[ \leq \prod_{i=1}^{m} \langle \Xi_{H_i}(a_i)\cdot \|v_{\alpha i}\| \cdot \|w_{\alpha i}\| \rangle \, d\mu(\alpha) \]

\[ = \prod_{i=1}^{m} \Xi_{H_i}(a_i)\|v\| \cdot \|w\| = \prod_{i=1}^{m} \Xi_{H_i}(a_i). \]

Note that the modular function \( \delta_{H_i} \) of \( H_i \cap B \) is equal to \( \exp(\beta_i) \). Hence by Proposition 3.1, for each \( i \), there exists a constant \( C_i \) (not depending on \( a \)) such that

\[ \Xi_{H_i}(a_i) \leq C_i \exp\left(-\frac{1}{2} + \epsilon\right) \beta_i(a). \]

This proves the proposition. □

3.5. Proof of Theorem A. It is well known ([1], Theorem 7.2) that \( G \) contains a connected simple closed subgroup \( G_0 \) such that \( G_0 \) is split over \( \mathbb{R} \), rank \( G_0 = \mathbb{R} \)-rank \( G \) and \( \Phi' \) is isomorphic to \( \Phi(g_0, g_0 \cap a) \) where \( \Phi' \) is the set of all non-multipliable roots in \( \Phi_\mathbb{R}(g, a) \) and \( g_0 \) is the Lie algebra of \( G_0 \). Recall the strongly orthogonal system \( Q(\Phi') = \{ \gamma_1, \cdots, \gamma_N(\Phi') \} \) of \( \Phi' \) we constructed in section 2.2.

The notation \( u_\alpha \) for \( \alpha \in \Phi' \) denotes the one-dimensional root subalgebra of \( g_0 \). Set \( N = N(\Phi') \). For each \( 1 \leq i \leq N \), we define \( H_i \) to be the connected closed subgroup of \( G_0 \) whose Lie algebra is generated by \( u_{\pm \gamma_i} \).

Case (\( \Phi' \neq D_{n=2k+1} \)): We note that the restriction \( F|_{A^+} \) of the function \( F \) to \( A^+ \) in Theorem A is equal to \( \prod_{i=1}^{N} \delta_{H_i}^{-1/2} \) or equivalently,

\[ F|_{A^+} = \prod_{i=1}^{N} \exp\left(-\frac{1}{2} \gamma_i \right). \]

Then Theorem A follows from Proposition 3.4.

Case (\( \Phi' = D_{n=2k+1} \)): In this case, we define \( H_N' \) to be the connected closed subgroup of \( G_0 \) whose Lie algebra is generated by \( u_{\pm(\alpha_{k+1}+2(\alpha_{k+2}+\cdots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n)} \), \( u_{\pm \alpha_k} \) and \( u_{\pm \alpha_k+1} \).

Note that the Lie algebra of \( H_N' \) is isomorphic to that of \( SO(3,3) \). We have that

\[ \delta_{H_N'} = \exp(4\alpha_k) \left( \prod_{i=k+1}^{n-2} \exp(6\alpha_i) \right) \exp(3\alpha_{n-1}) \exp(3\alpha_n). \]

By [7, Lemma 4.1], the restriction \( \rho|_{H_N'} \) is strongly \( L^{4+\epsilon} \) for any \( \rho \in \tilde{G} \). This implies (see [3, Corollary 7.2]) that the restrictions to \( H_N' \) of the \( K \)-finite matrix coefficients (with
respect to unit vectors) of $\rho$ are bounded by $\Xi^{1/2}/H_N'$. It is not difficult to see from the proof of Proposition 3.4 that, when we replace $H_N$ by $H_N'$, a statement similar to Proposition 3.4 holds, that is, the $K$-matrix coefficients of $\rho$ for any $\rho \in \hat{G}_K$ are bounded above by $(\prod_{i=1}^{N-1} \delta_{H_i}^{-1/2}) \delta_{H_N'}^{-1/4}$. Therefore it remains to observe that the function $F$ in Theorem A is given by

$$F|_{A^+} = \left(\prod_{i=1}^{n-2} \exp\left(-\frac{1}{2} \gamma_i\right)\right) \exp(\alpha_k) \left(\prod_{i=k+1}^{n-2} \exp\left(\frac{3}{2} \alpha_i\right)\right) \exp\left(\frac{3}{4} \alpha_{n-1}\right) \exp\left(\frac{3}{4} \alpha_n\right),$$

which is equal to $(\prod_{i=1}^{n-2} \delta_{H_i}^{-1/2}) \delta_{H_N'}^{-1/4}$, to complete the proof.

Remark. The results in section 2.2 show that the function $F$ is the best possible upper bound for $K$-matrix coefficients, which can be obtained using Proposition 3.4 when $\Phi' \neq D_{n=2k+1}$. Note that when $\Phi' = D_{n=2k+1}$, we improved $F$ by replacing one $SL_2(\mathbb{R})$ by $SO(3,3)$.

3.6. Corollary. With $G$, $\Phi'$ and $\alpha_1, \cdots, \alpha_n$ as in section 3.5, suppose that $O = \{\beta_1, \cdots, \beta_t\}$ is a strongly orthogonal system of $\Phi'$ and that for some number $r$, the coefficient of $\alpha_j$ in $\sum_{i=1}^{t} r\beta_i$ is strictly bigger than the coefficient of $\alpha_j$ in $2 \log(\delta_G)$ for each $1 \leq j \leq n$. Then we have

$$p(G) \leq r \text{ and } p_K(G) \leq r.$$

3.7. In each simple real-split Lie group $G$, the modular function $\delta_G$ of $B$ is given as below (cf. [2]), from which the remark after Corollary C follows.

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>$\log \delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\sum_{i=1}^{n} i(n-i+1)\alpha_i$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\sum_{i=1}^{n-1} (2ni-i^2)\alpha_i + n^2\alpha_n$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\sum_{i=1}^{n} (2ni-i^2+i)\alpha_i + \frac{n(n+1)}{2}\alpha_n$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\sum_{i=1}^{n-2} (2ni-i^2-i)\alpha_i + \frac{n(n-1)}{2}(\alpha_{n-1} + \alpha_n)$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$16\alpha_1 + 22\alpha_2 + 30\alpha_3 + 42\alpha_4 + 30\alpha_5 + 16\alpha_6$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$92\alpha_1 + 136\alpha_2 + 182\alpha_3 + 270\alpha_4 + 220\alpha_5 + 168\alpha_6 + 114\alpha_7 + 58\alpha_8$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$10\alpha_1 + 6\alpha_2$</td>
</tr>
</tbody>
</table>
4. Spherical unitary representations with minimal decay in $SL_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$

4.1. In this section we will show that the upper bound function $F$ we obtained in Theorem A is the best possible when $G = SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. This will be proved by showing that there exists a spherical unitary representation of $G$ whose $K$-matrix coefficients are bounded from below by a constant multiple of $F$. Those representations are the induced representations $\text{Ind}_G^P(I)$ of the trivial representation where $P$ is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$.

4.2. For the rest of section 4, let $G$ be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the bi-linear form $egin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ where $I_n$ denotes the skew diagonal $n \times n$-identity matrix. Set $K = SO_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ respectively. Define the maximal parabolic subgroup $P$ of $G$ as follows:

$$\text{for } G = SL_n(\mathbb{R}), \quad P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\};$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \quad P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{(2n)j} = 0 \text{ if } i \neq 1, j \neq 2n\}.$$

We fix an ordering in the root system of $G$ so that the positive Weyl chamber $A^+$ is as follows:

$$SL_n(\mathbb{R}), A^+ = \{\text{diag}(a_1, \cdots, a_n) \mid \prod_{i=1}^n a_i = 1, a_i \geq a_{i+1} \text{ for all } 1 \leq i \leq n-1\};$$

$$Sp_{2n}(\mathbb{R}), A^+ = \{\text{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq n-1\}.$$

4.3. We recall the formula for the matrix coefficients of the induced representation $\text{Ind}_G^P(I)$ (cf. [5]). Consider the Langlands decomposition of $P$: $P = MAPN$. Denote by $\tilde{N}$ the unipotent radical of the opposite parabolic subgroup to $P$ with the common Levi subgroup $MAP$.

If $g$ decomposes under the decomposition $G = KMAPN$, we denote by $\exp H(g)$ the $A_P$-component of $g$. It is well known that the representation space of $\text{Ind}_G^P(I)$ of the trivial representation $I$ of $P$ can be realized as $L^2(\tilde{N}, dx)$. If $g$ decomposes under $\tilde{N}MAPN$ as

$$g = \tilde{n}(g)m(g)a(g)n(g),$$

then the action is given by

$$\text{Ind}_G^P(I)(g)f(x) = e^{-\delta_0(\log a(g^{-1}x))}f(\tilde{n}(g^{-1}x)) \text{ for any } f \in L^2(\tilde{N}, dx) \text{ and } x \in \tilde{N}$$

where $\delta_0$ is the half sum of positive $N$-roots.

Define the vector $f_0$ of $\text{Ind}_G^P(I)$ as follows:

$$f_0(x) = e^{-\delta_0(H(x))}.$$

It is not difficult to see that $f_0$ is $K$-fixed and the matrix coefficient of $\text{Ind}_G^P(I)$ with respect to $f_0$ is as follows:

$$\langle \text{Ind}_G^P(I)(g)f_0, f_0 \rangle = \int_{\tilde{N}} e^{-\delta_0(\log a(g^{-1}x))}e^{-\delta_0(H(\tilde{n}(g^{-1}x)))}e^{-\delta_0(H(x))}dx.$$
4.4. Theorem D follows from the following proposition and Theorem A.

**Proposition.** There exists a constant $C$ such that

$$C F(a) \leq |\langle \text{Ind}_G^P(I)(a)f_0, f_0 \rangle|$$

where $a \in A^+$ and $F$ is as in Theorem A.

**Proof of Proposition 4.4.**

**Case:** $G = SL_n(\mathbb{R}), n \geq 3$

Denote by $\tilde{a}$ the matrix $\text{diag}(a_1, \cdots , a_n) \in SL_n(\mathbb{R})$ and by $x$ the matrix in $\bar{N}$ whose first column is $(1, x_2, \cdots , x_n)$, that is, $x.e_1 = (1, x_2, \cdots , x_n)$. To simplify notation, set $x_1 = 1$.

The decomposition of $\tilde{a}^{-1}x$ under $\bar{N}MAPN$ is as follows:

$$a(\tilde{a}^{-1}x) = \text{diag}(a_1a_1^{-1/(n-1)}, \cdots , a_1a_1^{-1/(n-1)})$$

and $\bar{n}(\tilde{a}^{-1}x).e_1 = \left(1, \frac{a_1}{a_2}x_2, \cdots , \frac{a_1}{a_n}x_n\right)$.

Then $\delta_0(a(\tilde{a}^{-1}x)) = a_1^{-n/2}$ and $H(x) = \|x.e_1\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Therefore

$$\langle \text{Ind}_G^P(I)(\tilde{a})f_0, f_0 \rangle = \int_{\bar{N}} a_1^{n/2} \|\bar{n}(\tilde{a}^{-1}x).e_1\|^{-n/2} \|x.e_1\|^{-n/2} dx$$

$$= \int_{\mathbb{R}^{n-1}} \left(\sum_{i=1}^{n} \left(\frac{1}{a_i}\right)^2 x_i^2\right)^{-n/4} \left(\sum_{i=1}^{n} x_i^2\right)^{-n/4} dm$$

where $dm$ is the standard measure in $\mathbb{R}^{n-1}$.

Set $k = \lceil \frac{n+1}{2} \rceil$ and let $T$ be the following set:

$$\{(x_2, \cdots , x_n) \mid 0 \leq x_i \leq 1 \text{ for } 2 \leq i \leq k-1, 1 \leq x_k \leq 2, x_i \leq \frac{a_i}{a_k} x_k \text{ for } k+1 \leq i \leq n\}.$$

Note that if $(x_2, \cdots , x_n) \in T$, then for each $1 \leq i \leq n$, we have

$$x_i \leq 2 \quad \text{and} \quad \frac{x_i}{a_i} \leq \frac{x_k}{a_k}.$$

Thus for $(x_2, \cdots , x_n) \in T$, we have

$$\left(\sum_{i=1}^{n} \left(\frac{1}{a_i}\right)^2 x_i^2\right)^{-n/4} \left(\sum_{i=1}^{n} x_i^2\right)^{-n/4} \geq C a_k^{-n/2}$$

for some constant $C > 0$.

Therefore

$$|\langle \text{Ind}_G^P(I)(\tilde{a})f_0, f_0 \rangle| \geq C \int_T a_k^{-n/2} dm \geq C a_k^{-n/2} \prod_{i=k+1}^{n} \left(\frac{a_i}{a_k}\right) \geq C F(\tilde{a}).$$

**Case:** $G = Sp_{2n}(\mathbb{R}), n \geq 2$
For $\hat{a} = \text{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \in Sp_{2n}(\mathbb{R})$, we have
\[
\langle \text{Ind}_P^G(I)(\hat{a})f_0, f_0 \rangle = \int_{\mathbb{R}^{2n-1}} \left( \sum_{i=1}^{n} \left( \frac{x_i}{a_i} \right)^2 + \sum_{i=1}^{n} (a_i y_i)^2 \right)^{-n/2} \left( \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \right)^{-n/2} \ d\mu
\]
where $x_1 = 1$. Let $T$ be the following set:
\[
\{(x_1, \cdots, x_n, y_n, \cdots, y_1) \mid 1 \leq y_n \leq 2, y_1 \leq \frac{a_n}{a_i} y_n, 0 \leq x_i \leq 1, 0 \leq y_i \leq 2 \text{ for } 1 \leq i \leq n\}.
\]
Note that if $(x_1, \cdots, x_n, y_n, \cdots, y_1) \in T$, then
\[
x_i \leq a_i a_n y_n
\]
since $a_i \geq 1$ for all $1 \leq i \leq n$. Therefore
\[
|\langle \text{Ind}_P^G(I)(\hat{a})f_0, f_0 \rangle| \geq C \int_T (a_n y_n)^{-n} \ d\mu \geq C F(\hat{a})
\]
where $C$ is some positive constant, finishing the proof.

5. \((G, K)\)-tempered subgroups and finite dimensional representations

5.1. Let $H$ be a linear connected non-compact semisimple Lie group. Let $B_H$ be a minimal parabolic subgroup, $A_H$ a maximal $\mathbb{R}$-split torus contained in $B_H$ and $K_H$ a maximal compact subgroup of $H$. Consider a Cartan decomposition of $H$: $H = K_H A_H^+ K_H$. Let $A$ be the torus of $SL_n(\mathbb{R})$ consisting of all the diagonal elements and $A^+$ the positive Weyl chamber of $SL_n(\mathbb{R})$ given by
\[
A^+ = \{ \text{diag}(a_1, \cdots, a_n) \mid a_i \geq a_{i+1} \text{ for all } 1 \leq i \leq n-1 \}.
\]
Let $\pi$ be a representation of $H$ to $SL_n(\mathbb{R})$ such that $\pi(A_H) \subset A$. For each $1 \leq i \leq n$, we define a weight $\beta_i$ of $d\pi$ by
\[
\beta_i(X) = (i, i)\text{-entry of the matrix } d\pi(X) \text{ for } X \in \log A_H,
\]
where $d\pi$ denotes the differential of $\pi$. Denote by $W$ the Weyl group of $SL_n(\mathbb{R})$. Using the well known isomorphism of $W$ with the symmetric group on $n$ letters, we can consider the action of $W$ on $\{\beta_1, \cdots, \beta_n\}$ by $w(\beta_i) = \beta_{w(i)}$ for each $1 \leq i \leq n$.

For each $w \in W$, set
\[
a_w = \{ X \in \log(A_H^+) \mid w(\beta_i)(X) \geq w(\beta_{i+1})(X) \text{ for all } 1 \leq i \leq n-1 \}.
\]
Note that since $a_w = \{ X \in \log(A_H^+) \mid d\pi(X) \in w^{-1}(\log A^+)w \}$, we have that $\log(A_H^+) = \bigcup_{w \in W} a_w$. It is not difficult to see that we can choose a subset $W_0 \subset W$ (not unique) so that $\log(A_H^+) = \bigcup_{w \in W_0} a_w$, the interior of $a_w$ is non-empty for each $w \in W_0$, and the interiors of $a_w$'s, $w \in W_0$ are disjoint. For example, if $\pi(A_H^+) \subset A^+$, then we can choose $W_0$ to consist of only the identity element of $W$.

We keep the above notation, such as $H$, $A^+$, $\pi$, $\beta_1, \cdots, \beta_n$, $W_0$, $a_w$, etc., for the rest of chapter 5. Recall also that $\delta_H$ denotes the modular function of $B_H$.

The following is an application of Theorem D when $G = SL_n(\mathbb{R})$. 

14
Corollary. The subgroup \( \pi(H) \) is an \((SL_n(\mathbb{R}), SO_n(\mathbb{R}))\)-tempered subgroup if and only if the following holds: for each \( w \in W_0 \) and all \( X \in a_w \)
\[
\begin{align*}
\begin{cases}
w(\beta_1)(X) + \cdots + w(\beta_{n/2})(X) > \log(\delta_H)(X) & \text{if } n \text{ is even} \\
w(\beta_1)(X) + \cdots + w(\beta_{(n-1)/2})(X) + \frac{1}{2}w(\beta_{(n+1)/2})(X) > \log(\delta_H)(X) & \text{if } n \text{ is odd}.
\end{cases}
\end{align*}
\]

Proof. Note that
\[
\int_{A^+_H} (F \circ \pi) \delta_H \, da = \sum_{w \in W_0} \int_{\exp a_w} (F \circ \pi) \delta_H \, da.
\]
On the other hand, on each \( a_w \), the restriction of \(-\log F \circ \pi \) to \( \log A^+_H \) is equal to the function in the left in the above inequality (see Example before Theorem D). This proves the claim by Theorem D. \( \square \)

Example. If \( H \) is simple and \( Ad \) is the adjoint representation of \( H \), we can consider \( Ad(H) \) to be a subgroup of \( SL_n(\mathbb{R}) \) where \( n = \dim(\text{Lie}(H)) \). Since the restriction of \(-\log F \circ Ad \) to \( \log A^+_H \) is equal to \( \log \delta_H \), we have that \( Ad(H) \) is not an \((SL_n(\mathbb{R}), SO_n(\mathbb{R}))\)-tempered subgroup by the above corollary.

5.2. Let \( \lambda_1, \ldots, \lambda_k \) the fundamental weights of the Lie algebra of \( H \) corresponding to \( A^+_H \). For any weights \( \gamma_1 \) and \( \gamma_2 \) of the Lie algebra of \( H \), we define a partial order \( > \) so that \( \gamma_1 > \gamma_2 \) if and only if \( (\gamma_1, \lambda_j) > (\gamma_2, \lambda_j) \) for all \( 1 \leq j \leq k \). This is equivalent to saying that the coefficient of each simple root in \( (\gamma_1 - \gamma_2, X) \) is positive, or \( \gamma_1(X) > \gamma_2(X) \) for all \( X \in \log A^+_H \).

If \( \lambda \) is the highest weight of an irreducible representation, then the lowest weight, which we will denote by \( \Lambda(\lambda) \), is given by
\[
(\Lambda(\lambda), \lambda_j) = -(\lambda, i(\lambda_j)) \text{ for each } 1 \leq j \leq k,
\]
where \( i \) is the opposition involution of the root system of \( \text{Lie}(H) \) (cf. [11]).

Corollary. Let \( H \) be a linear connected semisimple Lie group and \( \pi \) an irreducible representation with the highest weight \( \lambda \). Suppose that
\[
\lambda - \Lambda(\lambda) > 2\log \delta_H.
\]

Then \( \pi(H) \) is an \((SL_n(\mathbb{R}), SO_n(\mathbb{R}))\)-tempered subgroup.

Proof. Let \( w \in W_0 \). Since \( \lambda \) and \( \Lambda(\lambda) \) are the highest weight and the lowest weight of \( \pi \) respectively, it follows from the definition of \( a_w \) that
\[
w(\beta_1) = \lambda \text{ and } w(\beta_n) = \Lambda(\lambda).
\]
Let \( X \) be any element in \( a_w \). Since \( w(\beta_i)(X) \geq w(\beta_{i+1})(X) \) for each \( 1 \leq i \leq n - 1 \), we have that if \( n \) is even,
\[
2 \sum_{i=1}^{n/2} w(\beta_i)(X) \geq 2w(\beta_1)(X) + \sum_{i=2}^{n-1} w(\beta_i)(X),
\]
and if \( n \) is odd
\[
\sum_{i=1}^{(n-1)/2} w(\beta_i)(X) + w(\beta_{(n+1)/2})(X) \geq 2w(\beta_1)(X) + \sum_{i=2}^{n-1} w(\beta_i)(X).
\]
On the other hand, since $\sum_{i=1}^{n} \beta_i = 0$,

$$2w(\beta_1) + \sum_{i=2}^{n-1} w(\beta_i) = w(\beta_1) - w(\beta_n),$$

which is equal to $\lambda - \Lambda(\lambda)$. Therefore the assumption that $\lambda - \Lambda(\lambda) > 2\log \delta_H$ implies the inequalities in Corollary 5.1, finishing the proof. \(\square\)

Remark. By the remark prior to Corollary 5.2 and the fact that

$$(\log \delta_H, \lambda_j) = (\log \delta_H, i(\lambda_j))$$

for each $1 \leq j \leq k$, we have that if $\lambda > \log \delta_H$, then $\lambda - \Lambda(\lambda) > 2\log \delta_H$; so the hypothesis of the above corollary is satisfied.

Example. If $H = SL_{k+1}(\mathbb{R})$ in Corollary 5.2, then

$$\lambda - \Lambda(\lambda) > 2\log \delta_H$$

is equivalent to the following:

$$c_j + c_{k+1-j} > 2j(k+1-j)$$

for $1 \leq j \leq k$

where $c_j = (\lambda, \lambda_j)$.

5.3. Examples. The following examples are applications of Corollary 5.1.

1. If $\pi$ is an irreducible representation of $SL_2(\mathbb{R})$ into $SL_n(\mathbb{R})$, then it is well known that $(\lambda, \lambda_1) = \frac{n-1}{2}$; whereas $(\log \delta_H, \lambda_1) = 1$. Therefore $\pi(SL_2(\mathbb{R}))$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup if and only if $n \geq 4$.

2. The embedding of $SL_k(\mathbb{R})$ as the first $k$ by $k$ diagonal block matrix in $SL_n(\mathbb{R})$ is not an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup for any positive integers $k$ and $n$.

3. For matrices $A$ of order $m$ and $B$ of order $k$, the Kronecker product $A \otimes B$ of $A$ and $B$ is the matrix of order $mk$ such that the $(ij)$-matrix block of $A \otimes B$ is $a_{ij}B$ where $a_{ij}$ is the $(ij)$-entry of $A$.

The group $SL_m(\mathbb{R}) \otimes I_k$ is an $(SL_{mk}(\mathbb{R}), SO_{mk}(\mathbb{R}))$-tempered subgroup if and only if $k > 2(m-1)$.

5.4. In this section we consider the case when $\pi$ is symplectic or orthogonal. It is worthwhile to state the following fact, which enables us to tell when an irreducible representation $\pi$ with the highest weight $\lambda$ has such a property.

Theorem. ([11], Ch 3, Theorem 2.15) The representation $\pi$ is self-dual if and only if $\lambda = -\Lambda(\lambda)$. In such cases, $\pi$ is orthogonal (resp. symplectic) if $\sum_{j=1}^{k} (\log \delta_H, \lambda_j)(\lambda, \lambda_j)$ is even (resp. odd).

We remark that all finite dimensional irreducible representations of $H$ are self-dual unless $H$ is of type $A_n$, $D_{2k+1}$ or $E_6$.  

16
5.5. We have the following corollary of Theorem D when $G = Sp_n(\mathbb{R})$, which is analogous to Corollaries 5.1 and 5.2.

We use the same realization of $Sp_n(\mathbb{R})$ as in section 4.2 so that a positive Weyl chamber of $Sp_n(\mathbb{R})$ is the following:

$$Sp_n(\mathbb{R}) \cap A^+ = \{ \text{diag}(a_1, \ldots, a_{n/2}, a_{n/2}^{-1}, \ldots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq \frac{n}{2} - 1 \}.$$ 

**Corollary.** Let $H$ be a linear connected semisimple Lie group and $\pi$ a representation such that $\pi(H) \subset Sp_n(\mathbb{R})$.

1. The subgroup $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered subgroup if and only if for each $w \in W_0$,

$$w(\beta_1)(X) + \cdots + w(\beta_{n/2})(X) > \log \delta_H(X) \text{ for all } X \in a_w.$$ 

2. Furthermore assume that $\pi$ is irreducible with the highest weight $\lambda$. Suppose that

$$\lambda > \log \delta_H.$$ 

Then $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered subgroup.

**Proof.** The proof of the first claim is similar to that of Corollary 5.1; so we will omit it. Since $\lambda$ is the highest weight, $w(\beta_1) = \lambda$ for each $w \in W_0$. Since $w(\beta_i)(X) \geq 0$ for any $X \in a_w$ and each $1 \leq i \leq \frac{n}{2}$, we have $\sum_{i=1}^{n/2} w(\beta_i)(X) \geq \lambda(X)$. Now the second claim follows from the first one. $\square$

5.6. We consider a realization of $SO(m, n - m)$, $m = \left\lfloor \frac{2}{5} \right\rfloor$ so that a positive Weyl chamber of $SO(m, n - m)$ is given by $SO(m, n - m) \cap A^+$, that is, if $n$ is even,

$$\{ \text{diag}(a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq m - 1 \}$$

and if $n$ is odd,

$$\{ \text{diag}(a_1, \ldots, a_m, 1, a_m^{-1}, \ldots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq m - 1 \}.$$ 

**Corollary.** Let $H$ be a linear connected semisimple Lie group and $\pi$ an $n$-dimensional irreducible representation with the highest weight $\lambda$ such that $\pi(H) \subset SO(m, n - m)$ where $m = \left\lfloor \frac{2}{5} \right\rfloor$. Suppose that

$$\lambda > \log \delta_H.$$ 

Then $\pi(H)$ is an $(SO(m, n - m), SO(m, n - m) \cap SO_n(\mathbb{R}))$-tempered subgroup.

**Proof.** Consider the case when $n$ is even. Let $p = \left\lfloor \frac{n}{4} \right\rfloor$. Then for any $w \in W_0$ and any $X \in a_w$, the function $F$ in Theorem A is such that

$$- \log F \circ \pi(X) = w(\beta_1)(X) + \cdots + w(\beta_p)(X).$$

Therefore by the same argument as in the previous corollary, it is enough to show that

$$w(\beta_1)(X) + \cdots + w(\beta_p)(X) > \log \delta_H(X).$$

This is true since $w(\beta_i)(X) \geq 0$ for all $1 \leq i \leq p$ and $w(\beta_1) = \lambda$. The proof in the case when $n$ is odd is similar. $\square$
Example. If $H = SL_{k+1}(\mathbb{R})$ and $c_j = (\lambda, \lambda_j)$ for $1 \leq j \leq k$, then $\pi$ is self-dual if and only if $c_j = c_{k+1-j}$ for $1 \leq j \leq k$, and the condition $\lambda > \log \delta_H$ is equivalent to the condition $c_j > 2j(k+1-j)$ for each $j = 1, \cdots, k$. Therefore with these two conditions satisfied, if $\sum_{i=1}^{k} i(k+1-i)c_i$ is even, then $\pi(SL_{k+1}(\mathbb{R}))$ is an $(SO(m, n-m), SO(m, n-m) \cap SO_n(\mathbb{R}))$-tempered subgroup where $m = \lfloor n/2 \rfloor$, and if $\sum_{i=1}^{k} i(k+1-i)c_i$ is odd, then $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered subgroup.

Moreover in the case when $H = SL_2(\mathbb{R})$ and $\pi$ is an $n$-dimensional irreducible representation with $n \geq 4$ (cf. Example 5.3), the subgroup $\pi(SL_2(\mathbb{R}))$ is $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$-tempered if $n$ is even; otherwise it is $(SO(m, n-m), SO(m, n-m) \cap SO_n(\mathbb{R}))$-tempered.

5.7. Unipotent tempered subgroups. Lastly we give examples of some unipotent tempered subgroups of $G = SL_n(\mathbb{R})$. In order to apply Theorem D when $H$ is not semisimple, we need to know how each element of $H$ decomposes under the Cartan decomposition of $G$.

Consider the decomposition of the element $v_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ as $k_1a_k^2$ under the Cartan decomposition of $SL_2(\mathbb{R})$ with $K = SO_2(\mathbb{R})$ and $A$ the torus consisting of all the diagonal elements. Since $v_s(a)^t = k_1a^2k_1^{-1}$, the eigenvalues of $a^2$ coincide with those of $v_s(v_s)^t$. If $a = \text{diag}(b, b^{-1})$, then $b = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2 + 4})}$.

Consider the one parameter unipotent subgroup $U_{ij}$ of $SL_n(\mathbb{R})$ consisting of the elements $u_{ij}(s) = I + sE_{ij}$, $s \in \mathbb{R}$, where $i \neq j$ and $E_{ij}$ is the elementary matrix whose non-zero entry is 1 only at $(i, j)$. We keep the same notation as in section 5.1. Then the $A^+$-component of $u_{ij}(s)$ under the Cartan decomposition of $SL_n(\mathbb{R})$ is $\text{diag}(b, 1, \cdots, 1, b^{-1})$ where $b = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2 + 4})}$ by the previous argument.

Therefore $F(u_{ij}(s))$ is equal to $(\sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2 + 4})})^{-1}$.

Proposition. Let $n \geq 2$ and $i \neq j$.

(1) For any $\epsilon > 0$, the restriction $F|_{U_{ij}}$ is $L^{1+\epsilon}(U_{ij})$-integrable; hence $U_{ij}$ is not an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup.

(2) The diagonal embedding $\delta(U_{ij}) = \{(g, g) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \mid g \in U_{ij}\}$ is an $(SL_n(\mathbb{R}) \times SL_n(\mathbb{R}), SO_n(\mathbb{R}) \times SO_n(\mathbb{R}))$-tempered subgroup.

Proof. The part (1) is clear. For the second claim, see the remark following Corollary B. □

Now consider the unipotent one-parameter subgroup $U$ of $SL_4(\mathbb{R})$ consisting of the elements $U(s) = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$. It is easy to see that the following proposition holds.

5.8. Proposition. The subgroup $U$ is an $(SL_4(\mathbb{R}), SO_4(\mathbb{R}))$-tempered subgroup.

References


18


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