THE RUZIEWICZ PROBLEM AND DISTRIBUTING POINTS ON HOMOGENEOUS SPACES OF A COMPACT LIE GROUP.

HEE OH

Abstract. Let G be any compact non-commutative simple Lie group not locally isomorphic to SO(3). We present a generalization of a theorem of Lubotzky, Phillips and Sarnak on distributing points on the sphere S^2 (or S^3) to any homogeneous space of G, in particular, to all higher dimensional spheres. Our results can also be viewed as a quantitative solution to the generalized Ruziewicz problem for any homogeneous space of G.

1. **Introduction**

The Ruziewicz problem (also known as the Banach-Ruziewicz problem) asks whether every finitely additive rotation invariant measure on the sphere S^n, defined on all Lebesgue measurable sets, is equal to the Lebesgue measure (see [Lu], [Sa]). It is also equivalent to the question whether the Lebesgue integral is the only rotation invariant mean on the space $L^\infty(\mathrm{S}^n)$. For $n = 1$, the answer is negative. For $n \geq 2$, it was answered in the affirmative by Drinfeld ($n = 2, 3$) [Dr] and independently by Margulis [Ma1] and Sullivan [Su] for $n \geq 4$.

By the results of Rosenblatt [Ro] and Del Junco-Rosenblatt [DR] (see Proposition 3.4.1 in [Lu]), the affirmative answer follows if there exists a finitely generated subgroup Γ of $\mathrm{SO}(n + 1)$ such that the trivial representation of Γ is isolated in the restriction to Γ of the quasi-regular representation of $\mathrm{SO}(n+1)$ on $L^2(\mathrm{S}^n)$. The solutions made by Margulis and Sullivan was to present a finitely generated subgroup of $\mathrm{SO}(n + 1)$ with property (T) for $n \geq 4$.

Margulis observed that any connected simple non-commutative compact Lie group G which is not locally isomorphic to SO(3) possesses a finitely generated dense subgroup with property (T), and hence gave the affirmative solution to the generalized Ruziewicz problem: the Haar integral is the unique G-invariant mean on $L^\infty(Y)$ for any homogeneous space Y of G [Ma2, Prop 7, P. 136].

In this paper, for any connected simple non-commutative compact Lie group G not locally isomorphic to SO(3), we present a sequence of finite subsets S with estimates on the spectral gap between the trivial representation of the subgroup $\Gamma := \langle S\rangle$ generated by S and the restriction to Γ of the quasi-regular

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representation of G on the space $L^2_0(G)$ of square integrable functions of zero mean. To measure the spectral gap, we use a Hecke operator.

For a finite subset S of G and a closed subgroup H of G, the Hecke operator T_S on $L^2(G/H)$ is defined by

$$\mathrm{T}_S\, f(x) = \frac{1}{\# S} \sum_{\alpha \in S} f(\alpha^{-1}x), \quad \text{for any } f \in L^2(G/H) \text{ and } x \in G/H.$$

It is easy to check that the operator norm of T_S is precisely 1 and that T_S preserves the space $L^2_0(G/H)$ of square integral functions of zero mean relative to a G-invariant measure on G. Denote by λ_S the operator norm of the restriction $\mathrm{T}_S : L^2_0(G/H) \to L^2_0(G/H)$. It can be shown that $\lambda_S < 1$ if and only if the trivial representation of the subgroup $\langle S \rangle$ is isolated from the restriction to $\langle S \rangle$ of the quasi-regular representation of G on $L^2(G/H)$ (cf. [Lu]).

From this viewpoint, the following theorem presents a quantitative solution to the (generalized) Ruziewicz problem:

Theorem 1.1. *Let G be a connected simple non-commutative compact Lie group which is not locally isomorphic to* $\mathrm{SO}(3)$ *and H a closed subgroup of G. We construct, for infinitely many primes p, a finite subset S_p such that*

$$\lambda_{S_p} \leq C \cdot \left(\frac{1}{\sqrt{p}} \right)$$

where $C > 0$ is a constant independent of p, and

$$\# S_p = \min_{\alpha \in P_p^+} \frac{\#(\mathcal{G}/P_\alpha)(\mathbb{F}_p)}{p^{\dim(\mathcal{G}/P_\alpha)}} \cdot p^{\langle \alpha, 2\rho \rangle}$$

where $\mathcal{G}$ is a connected simple simply connected $\mathbb{Q}$-group with the same Dynkin type as G (see Proposition 2.4 for other notations).

In [LPS1-2], Lubotzky, Phillips and Sarnak showed how to generate evenly distributed sequences on the spheres S^2 and S^3 using Hecke operators. In this respect, the above theorem is a generalization of their theorem to homogeneous spaces of an arbitrary compact non-commutative simple Lie group not locally isomorphic to SO(3), which in particular includes all higher dimensional spheres S^n. In the case of spheres, our result can be made more explicit:

Theorem 1.2. *Let $G = \mathrm{SO}(n+1)$, $H = \mathrm{SO}(n)$ so that $\mathrm{S}^n = G/H$ and let $n \geq 4$. For any finite prime*

$$\begin{cases} p \neq 2, 3 & \text{if } n \neq 1 \pmod 4 \\ p = 1 \pmod 4 & \text{if } n = 1 \pmod 4, \end{cases}$$

we construct a subset S_p *of* $\mathrm{SO}(n+1)$ *such that*

$$\#S_p = \begin{cases} (\sum_{i=0}^{n-1} p^i) & \text{for } n \text{ even} \\ (\sum_{i=0}^{(n-1)/2} p^i)(1+p^{(n-1)/2}) & \text{for } n \text{ odd} \end{cases}$$

and

$$\lambda_{S_p} \le C \cdot \left(\frac{1}{\sqrt{p}}\right),$$

where $C > 0$ *is a constant independent of* p*.*

The finite subsets of SO(3) that Lubotzky, Phillips and Sarnak constructed in [LPS1-2] for S^2 (or S^3) achieves the *optimal* bound for λ_{S_p}. This was possible due to Deligne's theorem on the Ramanujan conjecture for holomorphic cusp forms [De]. Recently, using a theorem of Harris and Taylor in [HT] on automorphic forms for $\mathrm{GL}_n(\mathbb{Q}_p)$, Clozel obtained Hecke operators on the *odd* dimensional spheres with much stronger bounds than Theorem 1.2 [Cl].

The bound for λ_{S_p} we have in Theorem 1.2 is based on the information on the spherical unitary dual of a $\mathbb{Q}_p$-split special orthogonal group of rank ≥ 2 obtained in [Oh1]. Though it is believed that this bound should be improved as the relevant theory in automorphic forms develops, such results are not available yet. Nonetheless our method has an advantage of dealing with more general homogeneous spaces of a compact Lie group simultaneously, thus yielding Theorem 1.1.

We remark that Theorem 1.1 implies that for any $f \in L^2(G/H)$,

$$\Big\| \frac{1}{\#S_p} \sum_{\gamma \in S_p} f(\gamma^{-1}x) - \frac{1}{\mathrm{vol}(G/H)} \int_{G/H} f d\omega \Big\| = O\left(\frac{1}{\sqrt{p}}\right)$$

where $d\omega$ denotes a G-invariant measure on G/H. Moreover a Sobolev type argument applied to G (cf. [B.1, BR], [Cl]) yields that for any smooth function f on G/H and for any $x \in G/H$,

$$\left| \frac{1}{\#S_p} \sum_{\gamma \in S_p} f(\gamma^{-1}x) - \frac{1}{\mathrm{vol}\ (G/H)} \int_{G/H} f d\omega \right| = O_f\left(\frac{1}{\sqrt{p}}\right)$$

where the implied constant depends on a Sobolev norm of f.

Theorems 1.1 and 1.2 were previously announced in [Oh2].

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2. **Construction of S_p**

Let $\mathcal{G}$ be a connected simple linear algebraic group defined over $\mathbb{Q}$ with absolute rank at least 2. Suppose that $\mathcal{G}$ is anisotropic over $\mathbb{R}$, or equivalently $\mathcal{G}(\mathbb{R})$ is compact. This assumption implies that $\mathcal{G}$ is anisotropic over $\mathbb{Q}$ as well.

Since $\mathcal{G}$ splits over a finite extension of $\mathbb{Q}$, it follows from Chebotarev density theorem (cf. [PR, Theorem 1.2]) that the set

$$\mathcal{I} := \{p : \ \mathcal{G} \text{ is split over } \mathbb{Q}_p\}$$

is infinite.

Lemma 2.1. *There exists a $\mathbb{Q}$-embedding $\mathcal{G} \hookrightarrow \mathrm{SL}_N$ so that $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Z}_p)$ is a hyper-special compact subgroup of $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Q}_p)$ for each $p \in \mathcal{I}$.*

Proof. Consider any $\mathbb{Q}$-embedding $\mathcal{G} \hookrightarrow \mathrm{SL}_N$. Then $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Z}_p)$ is a hyper-special compact subgroup of $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Q}_p)$ for every prime p outside a finite subset, say I [Ti]. Let $J \subset I$ be the subset of primes p such that $\mathcal{G}$ is split over $\mathbb{Q}_p$ and $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Z}_p)$ is not a hyper-special compact subgroup of $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Q}_p)$. For each $p \in J$, since $\mathcal{G}$ is split over $\mathbb{Q}_p$, $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Q}_p)$ possesses a hyper-special compact subgroup, say, K_p and there exists a $\mathbb{Z}_p$-lattice, say $L_p \subset \mathbb{Q}_p^N$, such that K_p is the stabilizer of the lattice L_p in $\mathcal{G} \cap \mathrm{SL}_N(\mathbb{Q}_p)$ [Ti]. Now by [PR, Theorem 1.15], one can find a $\mathbb{Z}$-lattice $X \subset \mathbb{Q}^N$ such that

$$X \otimes_{\mathbb{Z}} \mathbb{Z}_p = \begin{cases} \mathbb{Z}_p^N & \text{for } p \notin J \\ L_p & \text{for } p \in J. \end{cases}$$

It remains to conjugate the embedding by an element $g \in \mathrm{GL}_N(\mathbb{Q})$ which changes the basis $\mathbb{Z}^N$ to X, to finish the proof. $\square$

We fix a $\mathbb{Q}$-embedding $\mathcal{G} \hookrightarrow \mathrm{SL}_N$ as in the above lemma and fix a prime $p \neq 3$ belonging to the set $\mathcal{I}$. For any ring J containing $\mathbb{Z}$, the notation $\mathcal{G}(J)$ denotes the subgroup $\mathcal{G} \cap \mathrm{SL}_N(J)$.

For simplicity, we set

$$G = \mathcal{G}(\mathbb{R}),\ G_p = \mathcal{G}(\mathbb{Q}_p) \text{ and } K_p = \mathcal{G}(\mathbb{Z}_p).$$

We note that G is a connected [Bo2, 24.6] compact simple real Lie group and G_p is a $\mathbb{Q}_p$-split simple Lie group of which K_p is a hyper-special compact subgroup. Then there exists a maximal $\mathbb{Q}_p$-split torus A_p of G_p such that the following Cartan decomposition holds:

$$G_p = \amalg_{\alpha \in P_p^+} K_p \alpha(p) K_p$$

where P_p^+ denotes the set of positive co-characters of A_p (cf. [Gr]). We set $\alpha_p = \alpha(p)$ for simplicity.

For $k \in \mathbb{N}$, the notation $k\mathbb{Z}[\frac{1}{p}]$ denotes the subring of $\mathbb{Q}$ generated by $k\mathbb{Z}$ and $\frac{1}{p}$. We now denote by Γ the image of the congruence subgroup

$$\{g \in \mathcal{G}(\mathbb{Z}[\frac{1}{p}]) \mid g \equiv I_N \pmod{3\mathbb{Z}[\frac{1}{p}]}\}$$

under the diagonal embedding into $G \times G_p$.

Since $\mathcal{G}$ is anisotropic over $\mathbb{Q}$, Γ is a uniform lattice in $G \times G_p$ [Bo1]. In addition, Γ is torsion-free by a lemma of Minkowski (cf. [PR, Lemma 4.19, P. 232]).

Denote by Γ_p the image of Γ under the canonical projection of $G \times G_p$ to G_p. Clearly Γ_p is a torsion-free uniform lattice in G_p as well. It follows that there exists a finite subset Δ_p of G_p such that for any $g \in G_p$, there exist unique elements $\gamma \in \Gamma_p$ and $\delta \in \Delta_p$ such that $g \in \gamma\delta K_p$.

For $a \in G_p$, the notation $\deg(a)$ means the cardinality of the set $K_p a K_p/K_p$. For any $\alpha \in P_p^+$, we may write $K_p \alpha_p K_p$ as $\amalg_{j=1}^{\deg(\alpha_p)} \gamma_j \delta_j K_p$ with uniquely determined elements $\gamma_j \in \Gamma_p$ and $\delta_j \in \Delta_p$, $1 \leq j \leq \deg(\alpha_p)$.

We now set

$$S(\alpha_p) = \{\gamma_j \in \Gamma_p \mid 1 \leq j \leq \deg(\alpha_p)\} \tag{2.2}$$

where γ_j is counted with (possible) multiplicity so that $\#S(\alpha_p) = \deg(\alpha_p)$.

Since $\Gamma = \{(\gamma, \gamma) \in G \times G_p \mid \gamma \in \Gamma_p\}$, $S(\alpha_p)$ may be considered as a subset of G, which then acts on G by left translations.

Definition 2.3. *The Hecke operator* $\mathrm{T}_{S(\alpha_p)}$ *on* $L^2(G)$ *is defined as follows:*

$$\mathrm{T}_{S(\alpha_p)}(f)(x) = \frac{1}{\#S(\alpha_p)} \sum_{\gamma \in S(\alpha_p)} f(\gamma^{-1}x)$$

for $f \in L^2(G)$ *and* $x \in G$.

By the following formula given in [Gr, Prop. 7.4], we can compute the cardinality of the set $S(\alpha_p)$.

Proposition 2.4. *For any* $\alpha \in P_p^+$, *we have*

$$\deg(\alpha_p) = \frac{\#(\mathcal{G}/P_\alpha)(\mathbb{F}_p)}{p^{\dim(\mathcal{G}/P_\alpha)}} \cdot p^{\langle \alpha, 2\rho \rangle}$$

where P_α *denotes the standard (maximal) parabolic subgroup of* $\mathcal{G}$ *defined by* α, 2ρ *denotes the sum of all positive roots of* G_p *with respect to* A_p *and* $\mathbb{F}_p$ *denotes the finite field of order* p.

In order to give an estimate on the cardinality of $\Delta_p = \#\Gamma_p \backslash G_p/K_p$, let $\mathcal{G}(\mathbb{A})$ denote the adele group attached to $\mathcal{G}$. Denote by $h_{\mathcal{G}}$ the cardinality of the double coset space $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/(G \times \prod_q \mathcal{G}(\mathbb{Z}_q))$. It is well known that the number $h_{\mathcal{G}}$, called the *class number* of $\mathcal{G}$, is finite (cf. [PR, Theorem 5.1, P. 251]).

The following shows that the cardinality $\#\Delta_p$ is bounded by a constant independent of p:

Lemma 2.5. *We have*

$$\#\Delta_p \le h_{\mathcal{G}} \cdot \# \mathrm{SL}_N(\mathbb{F}_3).$$

Proof. Note that

$$\mathcal{G}(\mathbb{Z}[\frac{1}{p}]) = \mathcal{G}(\mathbb{Q}) \cap (G \times \prod_{q \neq p} \mathcal{G}(\mathbb{Z}_q))$$

where $\mathcal{G}(\mathbb{Q})$ is identified with its image under the diagonal mapping into $\mathcal{G}(\mathbb{A})$. It is then easy to check that the following canonical map deduced from the obvious embedding $G_p \to \mathcal{G}(\mathbb{A})$ is injective:

$$\mathcal{G}(\mathbb{Z}[\frac{1}{p}])\backslash G_p/K_p \to \mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/(G \times \prod_{q} \mathcal{G}(\mathbb{Z}_q)).$$

Therefore

$$\#\mathcal{G}(\mathbb{Z}[\frac{1}{p}])\backslash G_p/K_p \le h_{\mathcal{G}}.$$

If Γ_0 denotes the kernel of the canonical projection map

$$\mathrm{SL}_N(\mathbb{Z}[\frac{1}{p}]) \to \mathrm{SL}_N(\mathbb{Z}[\frac{1}{p}]/3\mathbb{Z}[\frac{1}{p}])$$

it is easy to check that the natural map

$$\mathcal{G}(\mathbb{Z}[\frac{1}{p}])/\Gamma_p \to \mathrm{SL}_N(\mathbb{Z}[\frac{1}{p}])/\Gamma_0$$

is injective. Since $\# \mathrm{SL}_N(\mathbb{Z}[\frac{1}{p}])/\Gamma_0 \le \# \mathrm{SL}_N(\mathbb{F}_3)$, the claim follows. $\square$

Let $\Xi_{\mathrm{PGL}_2(\mathbb{Q}_p)}$ denote the Harish-Chandra function of $\mathrm{PGL}_2(\mathbb{Q}_p)$, and l et ξ_p be the bi-K_p-invariant function of G_p defined as

$$\xi_p(\alpha_p) = \prod_{\beta \in \mathcal{Q}} \Xi_{\mathrm{PGL}_2(\mathbb{Q}_p)} \begin{pmatrix} \beta(\alpha_p) & 0 \\ 0 & 1 \end{pmatrix}$$

where $\mathcal{Q}$ is a strongly orthogonal system, which is maximal in the sense of [Oh1], of the root system of G_p with respect to A_p, with the ordering given by P_p^+.

Since $\mathcal{G}$ has absolute rank at least 2 and G_p is split over $\mathbb{Q}_p$, the $\mathbb{Q}_p$-rank of G_p is at least 2. Hence the following is a special case of [Oh1, Theorem 1.1].

Theorem 2.6. *For any infinite dimensional irreducible unitary representation ρ_p of G_p with K_p-invariant unit vectors v_p and w_p, we have*

$$|\langle \rho_p(\alpha_p)v_p, w_p\rangle| \leq \xi_p(\alpha_p)$$

for any $\alpha \in P_p^+$.

We remark that if β is the highest root in the root system of (G_p, A_p), then

$$\xi_p(\alpha_p) \leq \Xi_{\mathrm{PGL}_2(\mathbb{Q}_p)} \begin{pmatrix} \beta(\alpha_p) & 0 \\ 0 & 1 \end{pmatrix}$$

[Oh1] and in particular, for any non-trivial $\alpha \in P_p^+$,

$$\xi_p(\alpha_p) \leq \frac{2}{\sqrt{p}}$$

since

$$\Xi_{\mathrm{PGL}_2(\mathbb{Q}_p)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \frac{2\sqrt{p}}{p+1}.$$

Denote by $\lambda_{S(\alpha_p)}$ the operator norm of the restriction $\mathrm{T}_{S(\alpha_p)} : L_0^2(G) \to L_0^2(G)$ where $L_0^2(G) = \{f \in L^2(G) : \int f dg = 0\}$.

Theorem 2.7. *Let $p \neq 3$ be any prime contained in $\mathcal{I}$. For any non-trivial $\alpha \in P_p^+$ we have*

$$\lambda_{S(\alpha_p)} \leq \#\Delta_p \cdot \xi_p(\alpha_p)$$

where the cardinality $\#S(\alpha_p)$ is given in Proposition 2.4. In particular, by Lemma 2.5, we have

$$\lambda_{S(\alpha_p)} \leq C \cdot p^{-1/2}$$

where $C > 0$ is a constant independent of p.

3. The norm estimate on the related local Hecke operators

In the next two sections, where we continue the notations defined in section 2, we prove Theorem 2.7. We fix the Haar measure μ_p on G_p so that $\mu_p(K_p) = 1$ and the probability Haar measure μ_∞ on G. Then the product $\mu_\infty \times \mu_p$ induces the $G \times G_p$-right invariant measure on the quotient $\Gamma\backslash G \times G_p$, which we denote by μ. The notation $L^2(\Gamma\backslash G \times G_p)^{K_p}$ denotes the subset of right K_p-invariant functions in $L^2(\Gamma\backslash G \times G_p)$.

Definition 3.1. *For any $a \in G_p$, we consider the local Hecke operator $\tilde{\mathrm{T}}_a$ acting on $L^2(\Gamma\backslash G \times G_p)^{K_p}$ defined by*

$$\tilde{\mathrm{T}}_a(\psi)(g, g_p) = \frac{1}{\deg(a)} \sum_{x \in K_p a K_p / K_p} \psi(g, g_p x)$$

for $\psi \in L^2(\Gamma\backslash G \times G_p)^{K_p}$, $g \in G$ and $g_p \in G_p$.

It is easy to check that it is well defined.

Note that G_p acts on $L^2(\Gamma\backslash G \times G_p)$ in the following way: for any $a \in G_p$,

$$a.f(g, g_p) = f(g, g_p a)$$

for any $g \in G$ and $g_p \in G_p$. In particular, this action is a unitary action on the Hilbert space $L^2(\Gamma\backslash G \times G_p)$. Since $\Gamma\backslash G \times G_p$ is compact, $L^2(\Gamma\backslash G \times G_p)$ can be written as a direct sum of irreducible unitary G_p-representations. We write

$$L^2(\Gamma\backslash G \times G_p) = W_f \oplus W_f^{\perp}$$

where W_f is the direct sum of finite dimensional irreducible G_p-sub-representations occurring in $L^2(\Gamma\backslash G \times G_p)$ and $W_f^{\perp}$ its orthogonal complement.

The main theorem of this section is the following:

Theorem 3.2. *For any $\phi \in L^2(\Gamma\backslash G \times G_p)^{K_p} \cap W_f^{\perp}$ and any $\alpha \in P_p^+$,*

$$\|\tilde{\mathrm{T}}_{\alpha_p}(\phi)\| \le \xi_p(\alpha_p)\|\phi\|.$$

A crucial but elementary observation for a proof of Theorem 3.2 is the relation between the local Hecke operator and the matrix coefficients of the G_p-representation on $L^2(\Gamma\backslash G \times G_p)$:

Lemma 3.3. *For any $a \in G_p$, we have*

$$\langle \tilde{\mathrm{T}}_a f_1, f_2 \rangle = \langle a.f_1, f_2 \rangle$$

where $f_1, f_2 \in L^2(\Gamma\backslash G \times G_p)^{K_p}$.

Proof. Let $K_p a K_p = \amalg_{i=1}^{\deg(a)} a_i K_p$. By definition,

$$\deg(a) \cdot \langle \tilde{\mathrm{T}}_a f_1, f_2 \rangle = \int_{\Gamma\backslash G \times G_p} \sum_{i=1}^{\deg(a)} f_1(x, x_p a_i) \bar{f}_2(x, x_p)\, d\mu(x, x_p).$$

Write for each $1 \le i \le \deg(a)$, $a_i = k_i a k_i'$ for some $k_i, k_i' \in K_p$. Since f_1 is right K_p-invariant,

$$f_1(x, x_p a_i) = f_1(x, x_p k_i a)$$

for each $1 \le i \le \deg(a)$. By change of variables, the above is now equal to

$$\int \sum_{i=1}^{\deg(a)} f_1(x, x_p a) \bar{f}_2(x, x_p k_i^{-1})\, d\mu = \int \sum_{i=1}^{\deg(a)} f_1(x, x_p a) \bar{f}_2(x, x_p)\, d\mu$$

since f_2 is right K_p-invariant as well. Hence

$$\deg(a) \cdot \langle \tilde{\mathrm{T}}_a f_1, f_2 \rangle = \deg(a) \cdot \langle a.f_1, f_2 \rangle,$$

proving the claim. $\square$

Theorem 3.2 now follows from Lemma 3.3 and Theorem 2.6 using a general theory of direct integral decomposition of a unitary representation (this process is almost identical to the proof of [COU, Prop. 2.4], which we won't repeat here).

4. **Relation between local and global Hecke operators**

The following theorem, together with Theorem 3.2, implies Theorem 2.7. Set

$$L_0 = L^2(\Gamma\backslash G \times G_p)^{K_p} \cap W_f^{\perp}$$

and

$$\tilde{\lambda}_{\alpha_p} = \sup_{\phi \in L_0, \|\phi\|=1} \|\tilde{\mathrm{T}}_{\alpha_p}(\phi)\|.$$

Theorem 4.1. *For any* $\alpha \in P_p^+$,

$$\lambda_{S(\alpha_p)} \leq \tilde{\lambda}_{\alpha_p} \cdot \#\Delta_p.$$

In order to prove Theorem 4.1, we consider the G-equivariant embedding $i : L^2(G) \to L^2(G \times \Delta_p)$ defined by

$$i(f)(g, \delta) = f(g)$$

for $f \in L^2(G)$, $g \in G$ and $\delta \in \Delta_p$. Note that

$$\|i(f)\| = \#\Delta_p \cdot \|f\|.$$

Now the natural embedding $G \times \Delta_p \hookrightarrow G \times G_p$ induces a G-equivariant bijection between $(\Gamma \cap (G \times \{e\}))\backslash G \times \Delta_p$ and $\Gamma\backslash(G \times G_p)/K_p$. Since G is compact, $\Gamma \cap (G \times \{e\})$ is a finite subgroup. Hence it is trivial, as Γ being torsion free. We therefore obtain a G-equivariant bijection

$$G \times \Delta_p \simeq \Gamma\backslash(G \times G_p)/K_p.$$

Define the map $\Phi : L^2(G \times \Delta_p) \to L^2(\Gamma\backslash G \times G_p)^{K_p}$ by

$$\Phi(\psi)(g, g_p) = \psi(\gamma_p^{-1} g, \delta_p)$$

where $\psi \in L^2(G \times \Delta_p)$, $g \in G$ and $g_p \in \gamma_p \delta_p K_p$ for some $\gamma_p \in \Gamma_p$ and $\delta_p \in \Delta_p$.

It follows from the above discussion that Φ is a well defined G-equivariant bijection, preserving the inner products.

Lemma 4.2. *For any* $f \in L^2(G)$ *and* $g \in G$,

$$\tilde{\mathrm{T}}_{\alpha_p} \Phi(i(f))(g, e) = \Phi(i(\mathrm{T}_{S(\alpha_p)} f))(g, e).$$

Proof. Write $K_p\alpha_pK_p = \amalg_{j=1}^{\deg(\alpha_p)} a_jK_p$. Then $a_j = \gamma_j\delta_jk_j \in \Gamma_p\Delta_pK_p$ where $\gamma_j \in \Gamma_p$ and $\delta_j \in \Delta_p$ are uniquely determined. Note that

$$(\tilde{\mathrm{T}}_{\alpha_p}\Phi)(i(f))(g,e) = \frac{1}{\deg(\alpha_p)}\sum_j \Phi(i(f))(g,a_j) = \frac{1}{\deg(\alpha_p)}\sum_{j=1}^{\deg(\alpha_p)} i(f)(\gamma_j^{-1}g,\delta_j).$$

On the other hand,

$$\Phi(i(\mathrm{T}_{S(\alpha_p)}f))(g,e) = i(\mathrm{T}_{S(\alpha_p)}f)(g,e) = (\mathrm{T}_{S(\alpha_p)}f)(g).$$

By the definition of $\mathrm{T}_{S(\alpha_p)}$, this is equal to

$$\frac{1}{\deg(\alpha_p)}\sum_{j=1}^{\deg(\alpha_p)} f(\gamma_j^{-1}g) = \frac{1}{\deg(\alpha_p)}\sum_{j=1}^{\deg(\alpha_p)} i(f)(\gamma_j^{-1}g,\delta_j).$$

This proves the claim. $\square$

To be able to use Theorem 3.2 in estimating the norm of $\mathrm{T}_{S(\alpha_p)}$ via the previous lemma, we need to show that the image of $L^2_0(G)$ under $\Phi\circ i$ is orthogonal to any finite dimensional representation of G_p occurring in $L^2(\Gamma\backslash G\times G_p)$ with a non-trivial K_p-invariant vector.

We first show:

Lemma 4.3. *Let Γ_∞ denote the image of Γ under the natural projection of $G\times G_p$ onto G. Then Γ_∞ is dense in G.*

Proof. Denote by H the closure of Γ_∞ in G and by $\mathfrak{h}$ the Lie algebra of H. Since Γ is an infinite subgroup of a compact group G, $\mathfrak{h}$ is not trivial. On the other hand $\mathfrak{h}$ is invariant by Γ_∞ under the adjoint representation of G. Since Γ_∞ is Zariski dense in G and the adjoint representation is algebraic, this implies that $\mathfrak{h}$ should be an ideal of the Lie algebra of G. Since G is simple, it follows that H is a subgroup of finite index in G. Since G is connected and hence has no finite index subgroup, it follows that Γ is dense in G. $\square$

Proposition 4.4. *We have $\Phi(i(L^2_0(G)))\subset L^2(\Gamma\backslash G\times G_p)^{K_p}\cap W_f^{\perp}$.*

Proof. Note that any finite dimensional irreducible unitary representation ρ of G_p is a finite character representation. In fact, the kernel of ρ contains the subgroup G_p^+ generated by one-parameter unipotent subgroups of G_p. Hence ρ is essentially an irreducible representation of G_p/G_p^+, which is a finite abelian group. Since any irreducible unitary representation of a finite abelian group is a finite character, the claim is shown.

Since $\Phi(i(L^2_0(G)))\subset L^2(\Gamma\backslash G\times G_p)^{K_p}$, it is enough to show that the image $\Phi(i(L^2_0(G)))$ is orthogonal to each one-dimensional representation of G_p with a non-zero K_p-invariant vector. Write $L^2(\Gamma\backslash G\times G_p)$ as $\oplus(V_\infty\otimes V_p)$ where V_∞ and V_p are irreducible representations of G and G_p respectively. Let V_p be any one-dimensional representation of G_p with a non-zero K_p-invariant vector; hence

$V_p = V_p^{K_p}$. We first claim that for any function $f \in V_\infty \otimes V_p$ and for a fixed element $g_p \in G_p$, $f(\cdot, g_p)$ is a constant function on G. Using the convolution operators on the G-action, we may assume that $f(\cdot, g_p)$ is a continuous function on G. Since V_p is of one-dimension,

$$f(g, g_p) = \tau(g_p) f(g, e)$$

for any $g \in G$ and $g_p \in G_p$ where τ is a finite character on G_p. Observe that fixing $g_p \in G_p$, for any $\gamma \in \Gamma$, we have

$$f(\gamma, g_p) = f(1, \gamma^{-1} g_p) = \tau(\gamma^{-1} g_p) f(1, 1).$$

Since Γ is dense in G by the previous lemma,

$$f(g, g_p) \in f(1, 1) \cdot \mathrm{Image}(\tau)$$

for any $g \in G$. Since the map $g \mapsto f(g, g_p)$ is continuous, G is connected and Image(τ) is finite, it follows that for each $g_p \in G_p$, $f(\cdot, g_p)$ is a constant function as a function on G.

We will now show that $\Phi(i(L_0^2(G)))$ is orthogonal to $V_\infty \otimes V_p$, which implies our claim, since as a G_p-representation,

$$L^2(\Gamma\backslash G \times G_p) = \oplus(mV_p)$$

where $m = \mathrm{dim} V_\infty$. Let $h \in L_0^2(G)$ and $f \in V_\infty \otimes V_p$. Since

$$f(g, g_p) = \tau(\delta(g_p)) f(1, 1)$$

where $\delta(g_p)$ denotes the Δ_p-component of g_p in the decomposition into $\Gamma_p \Delta_p K_p$,

$$\begin{aligned} (4.5) \quad \langle \Phi(i(h)), f\rangle &= \int_{\Gamma\backslash G\times G_p} \Phi(i(h))(g, g_p) \bar{f}(g, g_p)\, d\mu(g, g_p) \\ &= \bar{f}(1,1) \cdot \int_{G\times \Delta_p K_p} \Phi(i(h))(g, \delta) \bar{\tau}(\delta)\, d\mu_\infty(g) \times d\mu_p(\delta k). \end{aligned}$$

Since $\Phi(i(h))(g, \delta) = i(h)(g, \delta) = h(g)$ and $\mu_p(K_p) = 1$, the above is equal to

$$\bar{f}(1,1) \cdot \left(\sum_{\delta \in \Delta_p} \bar{\tau}(\delta) \cdot \int_G h(g)\, d\mu_\infty(g) \right).$$

Since $h \in L_0^2(G)$, we have $\int_G h(g)\, d\mu_\infty(g) = 0$ and hence $\langle \Phi(i(h)), f\rangle = 0$. Therefore the claim follows. $\square$

Proof of Theorem 4.1 Let $\phi \in L_0^2(G)$. By Lemma 4.2,

$$\mathrm{T}_{S(\alpha_p)}\, \phi(g) = (\Phi^{-1} \tilde{T}_{\alpha_p} \Phi)(i(\phi))(g, e)$$

for any $g \in G$. Therefore

$$\| \mathrm{T}_{S(\alpha_p)}\, \phi \| \le \|(\Phi^{-1} \tilde{\mathrm{T}}_{\alpha_p} \Phi)(i(\phi))\| = \|\tilde{T}_{\alpha_p}(\Phi(i(\phi)))\|.$$

Since $\Phi(i(\phi)) \in L_0$ by Proposition 4.4, we deduce from above that

$$\| \mathrm{T}_{S(\alpha_p)}\, \phi \| \leq \tilde{\lambda}_{\alpha_p} \cdot \| i(\phi) \|.$$

Since $\|i(\phi)\| = \#\Delta_p \cdot \|\phi\|$, the claim follows.

5. Proof of Theorems 1.1 and 1.2

5.1. **Proof of Theorem 1.1.** Let G be a connected compact non-commutative simple Lie group which is not locally isomorphic to $\mathrm{SO}(3)$. As is well known, G is algebraic, i.e., there exists a linear connected simple real algebraic group G_0 such that $G_0(\mathbb{R})$ is isomorphic to G (note that $G_0(\mathbb{R})$ is always connected [Bo2, 24.6]). If $\tilde{G_0}$ denotes the simply connected covering of G_0 over $\mathbb{R}$, there exists a central $\mathbb{R}$-isogeny $\pi : \tilde{G_0} \to G_0$. Then $\pi(\tilde{G_0}(\mathbb{R})) = G_0(\mathbb{R})$ and $G/H \simeq \tilde{G_0}(\mathbb{R})/\pi^{-1}(H)$. Therefore it suffices to prove Theorem 1.1 for the case when G_0 is simply connected and $H = \{e\}$, which we assume.

It then follows from a theorem of Borel and Harder [BH, Theorem B] there exists a a connected simple (simply connected) algebraic group $\mathcal{G}$ *defined over* $\mathbb{Q}$ such that G_0 is isomorphic to $\mathcal{G}$ over $\mathbb{R}$. Since G is not locally isomorphic to $\mathrm{SO}(3)$, the absolute rank of $\mathcal{G}$ is at least 2. We may therefore assume that $\mathcal{G}$ and G are as in section 2 and hence we may apply Theorem 2.7 to G. It now remains to set $S_p = S(\alpha_p)$ for each $p \neq 3$ in $\mathcal{I}$ where $\alpha_p \in P_p^+$ is such that $\deg(\alpha_p)$ is the minimum among all elements in P_p^+.

5.2. **Proof of Theorem 1.2.** Let $n \geq 4$. Since S^n can be identified with $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, and $L^2(\mathrm{S}^n)$ can be considered as the set of L^2-functions on $\mathrm{SO}(n+1)$ fixed by $\mathrm{SO}(n)$, it suffices to show Theorem 1.2 considering T_S as an operator on $L^2(\mathrm{SO}(n+1))$.

It is well known (cf. [Se]) that quadratic forms of $n+1$-variables over $\mathbb{Q}_p$ are, up to $\mathbb{Q}_p$-equivalence, completely classified by their local discriminant d_p and local Hasse invariants ϵ_p. Over $\mathbb{R}$, any quadratic form of $n+1$ variables is equivalent to the one defined by

$$(x_1, \cdots, x_{n+1}) \mapsto \left(\sum_{i=1}^{r} x_i^2\right) - \left(\sum_{i=r+1}^{n+1} x_i^2\right), \quad \text{for some } 1 \leq r \leq n+1$$

in which case its signature is $(r, n+1-r)$ and the invariant $\epsilon_\infty(Q)$ is defined to be $(-1)^{(n-r+1)(n-r)/2}$.

For a non-degenerate quadratic form Q over $\mathbb{Q}$, $d(Q)$ denotes the discriminant of Q, which is well defined as an element of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, and $\mathrm{sign}(Q)$ denotes the signature of Q over $\mathbb{R}$.

It can easily be deduced from [Se, Prop 7 in Sec 3.3] that there exists a quadratic form Q of $n+1$ variables over $\mathbb{Q}$ satisfying the following conditions: if $n \neq 1 \pmod 4$,

- $d(Q) = 1$
- $\mathrm{sign}(Q) = (n+1, 0)$
- $\epsilon_p(Q) = 1$ for all prime p and $\epsilon_\infty(Q) = 1$.

and if $n = 1 \pmod 4$,

- $d(Q) = -1$
- $\mathrm{sign}(Q) = (0, n+1)$
- $\epsilon_p(Q) = 1$ for all prime $p \neq 2$ and $\epsilon_\infty(Q) = \epsilon_2(Q) = -1$.

We fix this quadratic form Q for each $n \geq 4$ and set $\mathcal{G} = \mathrm{SO}(Q)$, which we may assume that $\mathcal{G} \subset \mathrm{SL}_{n+1}$ with $\mathcal{G} \cap \mathrm{SL}_{n+1}(\mathbb{Z}_p)$ is a hyper-special compact subgroup whenever $\mathcal{G}$ is split over $\mathbb{Q}_p$ by Lemma 2.1. We have that Q is equivalent over $\mathbb{Q}_p$ to the quadratic form defined by the matrix $\bar{I}_{n+1}$, whose only non-zero entries are 1 on the skew diagonal, for each finite prime

$$\begin{cases} p \neq 2 & \text{if } n \neq 1 \pmod 4 \\ p = 1 \pmod 4 & \text{if } n = 1 \pmod 4, \end{cases} \tag{5.1}$$

This can be checked by comparing local discriminants and Hasse invariants using the fact that $-1 = 1 \pmod{(\mathbb{Q}_p^*)^2}$ for primes $p = 1 \pmod 4$. In particular, $\mathcal{G}$ is split over $\mathbb{Q}_p$ for each finite prime p satisfying (5.1) and $\mathcal{G}(\mathbb{R})$ is compact, since Q is definite.

Note that the groups $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}(\mathbb{Q}_p)$, p as in (5.1), are conjugate to

$$\mathrm{SO}(n+1) := \{g \in \mathrm{SL}_{n+1}(\mathbb{R}) \mid {}^t g I_{n+1} g = I_{n+1}\} \text{ and}$$

$$\mathrm{SO}(\bar{I}_{n+1})(\mathbb{Q}_p) := \{g \in \mathrm{SL}_{n+1}(\mathbb{Q}_p) \mid {}^t g \bar{I}_{n+1} g = \bar{I}_{n+1}\}$$

by elements of $\mathrm{GL}_{n+1}(\mathbb{R})$ and $\mathrm{GL}_{n+1}(\mathbb{Q}_p)$.

Observing that $\begin{pmatrix} p & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1/p \end{pmatrix} \in \mathrm{SO}(\bar{I}_{n+1})(\mathbb{Q}_p)$, we denote by α_p the corresponding element in $\mathcal{G}(\mathbb{Q}_p)$ under the conjugation of $\mathrm{SO}(\bar{I}_{n+1})(\mathbb{Q}_p)$ to $\mathcal{G}(\mathbb{Q}_p)$.

Noting that the primes satisfying (5.1) are contained in

$$\mathcal{I} := \{p : \mathcal{G} \text{ is split over } \mathbb{Q}_p\}$$

and the absolute rank of $\mathcal{G}$ is at least 2 (since $n \geq 4$), Theorem 1.2 follows from Theorem 2.7 with the following computation, based on some known facts about the order of the special orthogonal groups over finite fields (cf. [Ca]).

Lemma 5.2. *For each $p \in \mathcal{I}$, set $S_p = S(\alpha_p)$ (see 2.2). Then*

$$\# S_p = \begin{cases} (\sum_{i=0}^{n-1} p^i) & \textit{for } n \textit{ even} \\ (\sum_{i=0}^{(n-1)/2} p^i)(1 + p^{(n-1)/2}) & \textit{for } n \textit{ odd.} \end{cases}$$

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