Théorie des groupes/Group theory

DISCRETE SUBGROUPS OF $SL_n(\mathbf{R})$ GENERATED BY LATTICES IN HOROSPHERICAL SUBGROUPS

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Abstract - We show that a discrete subgroup of $SL_n(\mathbf{R})$ generated by lattices in two opposite minimal horospherical subgroups is an arithmetic subgroup and we determine which arithmetic subgroups can occur in this way.

SOUS-GROUPES DISCRETS DE $SL_n(\mathbf{R})$ ENGENDRÉS PAR DES RÉSEAUX DE SOUS-GROUPES HOROSPHÉRIQUES

Résumé - *Nous montrons qu'un sous goupe discret de $SL_n(\mathbf{R})$ engendré par des réseaux dans deux sous-groupes horosphériques minimaux opposés est*

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un sous-groupe arithmétique et nous déterminons les sous-groupes arithmétiques engendrés de cette façon .

Version française abrégée - Un groupe algébrique $G \subset GL_n(\mathbf{C})$ est défini sur $\mathbf{Q}$ si un ensemble de polynômes sur $M_n(\mathbf{C})$ définissant le groupe $\tilde{G}$ peut être choisi de telle sorte que leurs coefficients soient dans $\mathbf{Q}$. Dans ce cas, soient $G(\mathbf{Z})$ le sous-groupe $\{g \in G \mid g, g^{-1} \in GL_n(\mathbf{Z})\}$ et $G(\mathbf{R}) = G \cap GL_n(\mathbf{R})$. Une $\mathbf{Q}$-forme d'un $\mathbf{R}$-groupe algébrique G est une paire $(\tilde{G}, f)$ où $\tilde{G}$ est groupe algébrique défini sur $\mathbf{Q}$ et f un isomorphisme $\tilde{G} \to G$ défini sur $\mathbf{R}$. Etant donné une $\mathbf{Q}$-forme $(\tilde{G}, f)$, nous dénotons $f(\tilde{G}(\mathbf{Q}))$ par $G(\mathbf{Q})$ et $f(\tilde{G}(\mathbf{Z}))$ par $G(\mathbf{Z})$; un sous-groupe de $G(\mathbf{R})$ commensurable à $G(\mathbf{Z})$ est appelé un sous-groupe arithmétique de $G(\mathbf{R})$. Si un $\mathbf{R}$-sous-groupe

H est tel que $f^{-1}(H)$ est un $\mathbf{Q}$-sous-groupe de $\tilde{G}$, nous disons que H est défini sur $\mathbf{Q}$ et dénotons $H \cap G(\mathbf{Z})$ par $H(\mathbf{Z})$. Un sous-groupe unipotent de $G(\mathbf{R})$ pour un $\mathbf{R}$-groupe algébrique semi-simple G est appelé *horosphérique* s'il est le groupe des $\mathbf{R}$-points du radical unipotent d'un sous-groupe parabolique défini sur $\mathbf{R}$. Deux sous-groupes horosphériques sont appelés *opposés* s'ils sont obtenus à partir des radicaux unipotents de sous-groupes paraboliques opposés.

La conjecture suivante a été proposée par G. Margulis.

Conjecture. *Soient G un $\mathbf{R}$-groupe semi-simple de rang réel au moins égal à 2, U_1 et U_2 deux sous-groupes horosphériques opposés de $G(\mathbf{R})$, et F_i un* *réseau de U_i tel que pour tout sous-goupe normal propre H de $G(\mathbf{R})$, $F_i \cap H$ est fini, $i = 1, 2$. Alors si le sous groupe Γ_{F_1,F_2} engendré par F_1 et F_2 est discret, il existe une $\mathbf{Q}$-forme de G telle que U_i soit défini sur $\mathbf{Q}$ et que F_i soit commensurable à $U_i(\mathbf{Z})$ pour $i = 1, 2$.*

D'après les résultats sur les générateurs des sous-groupes arithmétiques obtenus dans ([9], [12]), la conjecture implique que si Γ_{F_1,F_2} est discret, alors il existe une $\mathbf{Q}$-forme de G telle que le sous groupe Γ_{F_1,F_2} soit commensurable au sous-groupe arithmétique $G(\mathbf{Z})$ et par conséquent il est une réseau de $G(\mathbf{R})$ par un théorème de Borel et Harish-Chandra.

Dans cette note, nous présentons une preuve de cette conjecture dans le cas où $G(\mathbf{R}) = SL_n(\mathbf{R})$

et les sous-groupes horosphériques sont minimaux. En fait, toute paire composée de deux sous-groupes horosphériques minimaux opposés est conjuguée à une paire U_1, U_2 avec $U_1 = \begin{pmatrix} I_m & M_{m\times k}(\mathbf{R}) \\ 0 & I_k \end{pmatrix}$ et $U_2 = \begin{pmatrix} I_m & 0 \\ M_{k\times m}(\mathbf{R}) & I_k \end{pmatrix}$ pour un couple $m, k \in \mathbf{N}$ tel que $m+k = n$. Dans la suite, nous utiliserons parfois l'identification canonique de U_1 avec $M_{m\times k}(\mathbf{R})$ et de U_2 avec $M_{k\times m}(\mathbf{R})$ par souci de simplicité. Maintenant, soient F_1 et F_2 des réseaux de $M_{m\times k}(\mathbf{R})$ et $M_{k\times m}(\mathbf{R})$ respectivement, et soit Γ_{F_1,F_2} le sous-groupe de $SL_n(\mathbf{R})$ engendré par $\begin{pmatrix} I_m & F_1 \\ 0 & I_k \end{pmatrix}$ et $\begin{pmatrix} I_m & 0 \\ F_2 & I_k \end{pmatrix}$, où I_j dénote la matrice identité $j \times j$, pour $j \in \mathbf{N}$.

Observons que si une $\mathbf{Q}$-forme de $SL_n(\mathbf{R})$ a un sous-groupe parabolique maximal défini sur $\mathbf{Q}$, alors les sous-groupes arithmétiques de $SL_n(\mathbf{R})$ correspondants figurent parmi ceux énumérés dans le théorème suivant, à conjugaison près.

Théorème. *Soit $n \geq 3$. Si le sous-groupe Γ_{F_1,F_2} est discret, il existe deux éléments $g \in GL_m(\mathbf{R})$ et $h \in GL_k(\mathbf{R})$ tels que gF_1h^{-1} et hF_2g^{-1} soient, à commensurabilité près, l'une quelconque des paires suivantes:*

1. la paire formée de $M_{r\times s}(D_{\mathbf{Z}})$ et $M_{s\times r}(D_{\mathbf{Z}})$, où D est une $\mathbf{R}$-algèbre définie sur $\mathbf{Q}$ avec $D_{\mathbf{R}} = M_d(\mathbf{R})$ telle que $D_{\mathbf{Q}}$ soit une algèbre centrale à division sur $\mathbf{Q}$, $d = Deg_{\mathbf{Q}}D_{\mathbf{Q}}$, $rd = m$, $sd = k$, $D_{\mathbf{Z}}$ est un $\mathbf{Z}$-ordre dans $D_{\mathbf{Q}}$ et $M_{r\times s}(D_{\mathbf{Z}})$ représente l'ensemble des matrices $r \times s$ sur l'anneau $D_{\mathbf{Z}}$;

2. la paire constituée pas deux copies du réseau $\{(X_{ij}) \in M_r(D_J) \mid X_{ij} + \sigma(X_{ji}) = 0\}$ où K est *un corps extension quadratique réelle de* $\mathbf{Q}$*,* J *est l'anneau des entiers de* K*, et* D *est une* $\mathbf{R}$*-algèbre définie sur* K *avec* $D_{\mathbf{R}} = M_d(\mathbf{R})$ *telle que* D_K *soit une algèbre centrale à division munie d'une involution de seconde espèce* σ*,* $d = Deg_K D_K$*,* $rd = m = k$ *et* D_J *est un* J*-ordre dans* D_K *compatible avec* σ*.*

De plus, par conjugaison par l'élément $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$*, le sous-groupe* Γ_{F_1,F_2} *est commensurable au sous-groupe* $(SL_{r+s}D)_{\mathbf{Z}}$*, ou au sous-groupe* $SU(h_0)_{\mathbf{Z}} = \{Y \in (SL_{2r}D)_J \mid {}^tY^\sigma h_0 Y = h_0\}$ *où* $h_0 = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$*, respectivement.*

An algebraic group $G \subset GL_n(\mathbf{C})$ is defined over $\mathbf{Q}$ if it consists of all invertible matrices whose coefficients annihilate some set of polynomials on $M_n(\mathbf{C})$

with rational coefficients. In this case, let $G(\mathbf{Z})$ be the subgroup $\{g \in G \mid g, g^{-1} \in GL_n(\mathbf{Z})\}$ of G and $G(\mathbf{R}) = G \cap GL_n(\mathbf{R})$. A $\mathbf{Q}$-form of an algebraic $\mathbf{R}$-group G is a pair $(\tilde{G}, f)$ where $\tilde{G}$ is an algebraic group defined over $\mathbf{Q}$ and f an isomorphism $\tilde{G} \to G$ defined over $\mathbf{R}$. For a given $\mathbf{Q}$-form $(\tilde{G}, f)$ of G, we denote $f(\tilde{G}(\mathbf{Q}))$ by $G(\mathbf{Q})$ and $f(\tilde{G}(\mathbf{Z}))$ by $G(\mathbf{Z})$ and a subgroup of $G(\mathbf{R})$ commensurable to $G(\mathbf{Z})$ is said to be an arithmetic subgroup of $G(\mathbf{R})$. If an algebraic $\mathbf{R}$-subgroup H of G is such that $f^{-1}(H)$ is a $\mathbf{Q}$-subgroup of $\tilde{G}$, we may say that H is defined over $\mathbf{Q}$ and denote $H \cap G(\mathbf{Z})$ by $H(\mathbf{Z})$. A unipotent subgroup of $G(\mathbf{R})$ for a semisimple algebraic $\mathbf{R}$-group G is called *horospherical* if it is the group of $\mathbf{R}$-points of the unipotent radical of a parabolic subgroup defined over $\mathbf{R}$. Two horospherical subgroups are called *op-*

posite if they are the unipotent radicals of opposite parabolic subgroups.

The following conjecture was posed by G. Margulis.

Conjecture. *Let G be a semisimple algebraic $\mathbf{R}$-group with $\mathbf{R}$-rank greater than 1, U_1 and U_2 two opposite non-trivial horospherical subgroups of $G(\mathbf{R})$ and F_i a lattice in U_i such that for any proper normal subgroup H of $G(\mathbf{R})$, $H \cap F_i$ is finite for $i = 1, 2$. Then if the subgroup Γ_{F_1,F_2} generated by F_1 and F_2 is discrete, there exists a $\mathbf{Q}$-form of G such that U_i is defined over $\mathbf{Q}$ and F_i is commensurable to $U_i(\mathbf{Z})$ for each $i = 1, 2$.*

According to the result on the generators of arithmetic subgroups in ([9], [12]), the conjecture implies that if Γ_{F_1,F_2} is discrete, then there exists a $\mathbf{Q}$-form of G such that the subgroup Γ_{F_1,F_2} is commensurable to the arithmetic subgroup $G(\mathbf{Z})$ and hence it is a lattice in $G(\mathbf{R})$ by a theorem of Borel and Harish-Chandra.

In this paper, we present a sketch of proof of the conjecture in the case when $G(\mathbf{R}) = SL_n(\mathbf{R})$ and the opposite horospherical subgroups are minimal. In fact, any pair of two opposite minimal horospherical subgroups in $SL_n(\mathbf{R})$ is conjugate to a pair $U_1,\ U_2$ where $U_1 = \begin{pmatrix} I_m & M_{m\times k}(\mathbf{R}) \\ 0 & I_k \end{pmatrix}$ and $U_2 = \begin{pmatrix} I_m & 0 \\ M_{k\times m}(\mathbf{R}) & I_k \end{pmatrix}$ for some $m, k \in \mathbf{N}$ such that $m + k = n$. Now let F_1 and F_2 be lattices in $M_{m\times k}(\mathbf{R})$ and $M_{k\times m}(\mathbf{R})$ respectively and Γ_{F_1,F_2} be the subgroup of $SL_n(\mathbf{R})$ which is generated by $\begin{pmatrix} I_m & F_1 \\ 0 & I_k \end{pmatrix}$

and $\begin{pmatrix} I_m & 0 \\ F_2 & I_k \end{pmatrix}$ where I_j denotes the $j \times j$ identity matrix for $j \in \mathbf{N}$.

We observe that if a $\mathbf{Q}$-form of $SL_n(\mathbf{R})$ has a maximal parabolic subgroup defined over $\mathbf{Q}$ then the arithmetic subgroups of $SL_n(\mathbf{R})$ with respect to that $\mathbf{Q}$-form must be among those arising in the last part of the following theorem up to conjugation.

Theorem. *Let $n \geq 3$ and $G = SL_n(\mathbf{R})$. Then if the subgroup Γ_{F_1,F_2} is discrete, there exist elements $g \in GL_m(\mathbf{R})$ and $h \in GL_k(\mathbf{R})$ such that gF_1h^{-1} and hF_2g^{-1} are, up to commensurability, one of the following pairs:*

1. the pair consisting of $M_{r\times s}(D_{\mathbf{Z}})$ and $M_{s\times r}(D_{\mathbf{Z}})$ where D is an $\mathbf{R}$-algebra defined over $\mathbf{Q}$ with $D_{\mathbf{R}} = M_d(\mathbf{R})$ such that $D_{\mathbf{Q}}$ is a central division algebra *over* $\mathbf{Q}$, $d = Deg_{\mathbf{Q}} D_{\mathbf{Q}}$, $rd = m$, $sd = k$, $D_{\mathbf{Z}}$ *is a* $\mathbf{Z}$-*order of the algebra* $D_{\mathbf{Q}}$ *and* $M_{r\times s}(D_{\mathbf{Z}})$ *denotes the set of* $r \times s$ *matrices over the ring* $D_{\mathbf{Z}}$;

2. *the pair consisting of* $\{(X_{ij}) \in M_r(D_J) \mid X_{ij} + \sigma(X_{ji}) = 0\}$ *repeated twice, where* K *is a real quadratic extension field of* $\mathbf{Q}$, J *is the ring of integers of* K *and* D *is an* $\mathbf{R}$-*algebra defined over* K *with* $D_{\mathbf{R}} = M_d(\mathbf{R})$ *such that* D_K *is a central division algebra with an involution of the second kind* σ, $d = Deg_K D_K$, $rd = m = k$ *and* D_J *is a* J-*order of the algebra* D_K *compatible with* σ.

Moreover by conjugation by the element $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$, *the subgroup* Γ_{F_1,F_2} *is commensurable to either the subgroup* $(SL_{r+s}D)_{\mathbf{Z}}$ *or the subgroup* $SU(h_0)_{\mathbf{Z}} = \{Y \in (SL_{2r}D)_J \mid {}^tY^\sigma h_0 Y = h_0\}$ *where* $h_0 = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, *respectively.*

The main idea of the proof is as follows: Let $N(U_i)$ be the normalizer of U_i, H the commutator subgroup of the subgroup $N(U_1) \cap N(U_2)$ and Ω_i the space of lattices in U_i of determinant 1 for each $i = 1, 2$. By the natural identification of each space Ω_i with the homogeneous space $SL_l(\mathbf{R})/SL_l(\mathbf{Z})$ for $l = mk$, we can embed the conjugation action of H on the space Ω_i as a sub-action of the left translation action of $SL_l(\mathbf{R})$ on the homogeneous space $SL_l(\mathbf{R})/SL_l(\mathbf{Z})$. After obtaining that the closures of the orbits $H.F_1$ and $H.F_2$ are homogeneous by Raghunathan's conjecture proved by Ratner [10], we show that the discreteness assumption on the subgroup Γ_{F_1,F_2} implies that the orbits $H.F_1$ and $H.F_2$ are closed and this provides a $\mathbf{Q}$-structure for $N(U_1)$ and for $N(U_2)$. Finally the proof of the closedness of the orbit $H.(F_1, F_2) = \{(h.F_1, h.F_2) \mid h \in H\}$ in the space $SL_l(\mathbf{R})/SL_l(\mathbf{Z}) \times SL_l(\mathbf{R})/SL_l(\mathbf{Z})$ implies that the **Q**-structures come from a **Q**-structure of G.

As long as the horospherical subgroups are abelian, it seems that each step of the proof can be carried over without serious modification to the case of an arbitrary semisimple algebraic **R**-group.

In the general case, the horospherical subgroups U_1 and U_2 are not necessarily isomorphic to Euclidean spaces. However, using the fact that they are connected, simply connected and nilpotent Lie groups, we can still realize the conjugation action of H on the space of lattices in each U_i as a sub-action of the left translation action of $SL_l(\mathbf{R})$ on the space $SL_l(\mathbf{R})/SL_l(\mathbf{Z})$ where l is the dimension of U_i.

The most difficult case seems to be when the horospherical subgroups are the unipotent radicals of Borel subgroups. Since H becomes trivial in that case, we can not directly apply the same method as in the proof of the theorem.

In what follows, we give a sketch of the proof of the theorem. Denote by H the subgroup $\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in SL_m(\mathbf{R}),\ B \in SL_k(\mathbf{R})\}$. The group H acts by conjugation on the space of lattices in each U_i for $i = 1, 2$ and on the space of pairs of lattices in U_1 and U_2. In fact, the conjugation by an element $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in H$ sends F_1 to AF_1B^{-1}, F_2 to BF_2A^{-1} and (F_1, F_2) to (AF_1B^{-1}, BF_2A^{-1}) where F_i is a lattice in U_i for $i = 1, 2$. Through the identification of the subgroup U_i with the Euclidean space $\mathbf{R}^l$ for $l = mk$, the space of lattices in U_i of determinant 1 is naturally identified with the homogeneous space $SL_l(\mathbf{R})/SL_l(\mathbf{Z})$

for each $i = 1, 2$. For example, if a lattice F_1 in U_1 has determinant 1 and $g_1 \in SL_l(\mathbf{R})$ is a representative of the lattice F_1 in the space $SL_l(\mathbf{R})/SL_l(\mathbf{Z})$, then the lattice AF_1B^{-1} corresponds to the element $(A \otimes {}^tB^{-1})g_1 SL_l(\mathbf{Z})$ in $SL_l(\mathbf{R})/SL_l(\mathbf{Z})$ for each $A \in SL_m(\mathbf{R})$ and $B \in SL_k(\mathbf{R})$. Thus the orbits $H.F_1$, $H.F_2$ and $H.(F_1, F_2)$ are identified with $(SL_m(\mathbf{R}) \otimes SL_k(\mathbf{R})).F_1$, $(SL_k(\mathbf{R}) \otimes SL_m(\mathbf{R})).F_2$ and $\{((A \otimes {}^tB^{-1}).F_1, (B \otimes {}^tA^{-1}).F_2) \mid A \in SL_m(\mathbf{R}),\ B \in SL_k(\mathbf{R})\}$, respectively. For a lattice F in $\mathbf{R}^l$ whose determinant is not necessarily one, there is a lattice $\tilde{F}$ in $\mathbf{R}^l$ having determinant 1 such that $F = \alpha\tilde{F}$ for some $\alpha \in \mathbf{R}$. Therefore we can naturally extend the action to the space of lattices of arbitrary determinants.

By Raghunathan's conjecture proved by M. Ratner, we obtain that the orbit $H.F$ of a lattice F in

$\mathbf{R}^l$ is either closed or dense, since $SL_m(\mathbf{R}) \otimes SL_k(\mathbf{R})$ is a maximal closed subgroup of $SL_l(\mathbf{R})$.

The following lemma demonstrates a relationship between the discreteness of the subgroup Γ_{F_1,F_2} and the closures of the orbits $H.F_1$ and $H.F_2$.

Lemma 1. *If Γ_{F_1,F_2} is discrete, then for any lattice E_1 in U_1 in the closure of the orbit $H.F_1$, there exists a lattice E_2 in U_2 such that the pair (E_1, E_2) lies in the closure of the orbit $H.(F_1, F_2)$ and Γ_{E_1,E_2} is discrete.*

We shall use the norms $\|x\|$ and $\|g\|$ for $x \in \mathbf{R}^l$ and $g \in SL_n(\mathbf{R})$ coming from the standard inner products of the space $\mathbf{R}^l$ and $\mathbf{R}^{n^2}$ respectively. In what follows, we assume that the subgroup Γ_{F_1,F_2} is discrete.

Proposition 1. *The orbits $H.F_1$ and $H.F_2$ are closed.*

Sketch of proof. Assume that the orbit $H.F_1$ is not closed and hence its closure is $SL_l(\mathbf{R}).F_1$. It also follows that $m \geq 2$ and $k \geq 2$. It is known ([3], [8]) that there exist $\epsilon > 0$ and c such that for any discrete subgroup Λ of $SL_n(\mathbf{R})$, the subgroup generated by elements in Λ of norms less than ϵ is nilpotent and any lattice F in $\mathbf{R}^l$ has a non-zero element of norm less than $c \cdot d(F)$ where $d(F)$ denotes the determinant of the lattice F. Then there is an $\epsilon_0 > 0$ such that $\epsilon_0 < \epsilon$ and $\|ghg^{-1}h^{-1} - e\| < \epsilon$ for all $g \in SL_n(\mathbf{R})$ with $\|g - e\| < \epsilon_0$ and $h = \begin{pmatrix} I_m & 0 \\ y & I_k \end{pmatrix}$ with $\|y\| < c \cdot d(F_2)$. Choose a positive number ϵ' such that $\sqrt{2}\epsilon' < \epsilon_0$. For each $1 \leq i \leq k-1$ and $1 \leq j \leq m-1$, we set e_{ij} to be $\epsilon'(E_{i,j} + E_{i+1,j+1})$,

$\bar{e_j}$ to be $\epsilon'(E_{j,k} + E_{j+1,1})$ and e_1 to be $\epsilon' E_{1,1}$ where $E_{i,j}$ is the elementary matrix whose only non-zero entry is 1 at (i,j). Denote by S the set consisting of all these elements. Since the number of elements in S is $mk - k + 1$ which is less than mk, there exists a lattice F_1' in the orbit $SL_l(\mathbf{R}).F_1$ which contains S and has the same determinant as F_1. There exists a lattice F_2' of U_2 such that $\Gamma_{F_1',F_2'}$ is discrete by Lemma 1 and F_2' contains a non-zero element y of norm less than $c \cdot d(F_2)$, since $d(F_2) = d(F_2')$. Now, set $g_{ij} = \begin{pmatrix} I_m & e_{ij} \\ 0 & I_k \end{pmatrix}$, $\bar{g}_j = \begin{pmatrix} I_m & \bar{e_j} \\ 0 & I_k \end{pmatrix}$ and $g_1 = \begin{pmatrix} I_m & e_1 \\ 0 & I_k \end{pmatrix}$. Denote the set of these elements by $\bar{S}$. Then for $h = \begin{pmatrix} I_m & 0 \\ y & I_k \end{pmatrix}$, each element in the set $\{x, xhx^{-1}h^{-1} \mid x \in \bar{S}\}$ has norm less than ϵ. It follows from the choice of ϵ that the subgroup generated by this set is nilpotent and therefore the subgroup generated by this set together with h is unipotent. One can show that this forces the element y to be trivial. This contradiction establishes the claim that $H.F_1$ is closed. Since the argument is symmetric in F_1 and F_2, the orbit $H.F_2$ is also closed. $\square$

The closedness of the orbit $H.F_i$ implies that $H \cap \Lambda_{F_i}$ is a lattice in H for $i = 1, 2$ where $\Lambda_{F_1} := \{(A, B) \in SL_m(\mathbf{R}) \times SL_k(\mathbf{R}) \mid AF_1B^{-1} = F_1\}$ and $\Lambda_{F_2} := \{(A, B) \in SL_m(\mathbf{R}) \times SL_k(\mathbf{R}) \mid BF_2A^{-1} = F_2\}$. It follows from Borel density theorem that for each $i = 1, 2$, there exists a $\mathbf{Q}$-form of H such that $H \cap \Lambda_{F_i}$ is commensurable to the arithmetic subgroup $H_{\mathbf{Z}}$.

If we denote by ξ_i, $i = 1, 2, 3, 4$, the representation of $SL_m(\mathbf{R}) \times SL_k(\mathbf{R})$ defined by $\xi_1(A, B) = A \otimes B$, $\xi_2(A, B) = A \otimes {}^tB^{-1}$, $\xi_3(A, B) = {}^tA^{-1} \otimes B$ and $\xi_4(A, B) = {}^tA^{-1} \otimes {}^tB^{-1}$ for $A \in SL_m(\mathbf{R})$ and $B \in SL_k(\mathbf{R})$, we observe that the arithmetic subgroups $H_{\mathbf{Z}}$ which appear here come from $\mathbf{Q}$-forms of H with respect to which one of the representations ξ_i is $\mathbf{Q}$-rational.[1] The notation D, K, σ, … in the following proposition are as in the theorem.

Proposition 2. *(a) If either ξ_1 and ξ_4 is rational over $\mathbf{Q}$, then $H(\mathbf{Q})$ is either $SL_r(D_{\mathbf{Q}}) \times SL_s({}^tD_{\mathbf{Q}})$ or $\{(A, {}^tA^I) \mid A \in SL_r(D_K)\}$ where I is an involution of the second kind of $M_r(D_K)$ defined by*

[1] An absolutely irreducible linear representation ξ of a connected semisimple algebraic group defined over a field k is called rational over k if it is equivalent to a representation defined over k.

$X^I = (\sigma(X_{ji}))$ for $X = (X_{ij})_{1\le i,j\le r}$, $X_{ij} \in D_K$ and ${}^tD_{\mathbf{Q}}$ is a central division algebra defined by $\{X \in M_d(\mathbf{R}) \mid {}^tX \in D_{\mathbf{Q}}\}$, up to conjugation by an element of $GL_m(\mathbf{R}) \times GL_k(\mathbf{R})$.

(b) If either ξ_2 or ξ_3 is rational over $\mathbf{Q}$, then $H(\mathbf{Q})$ is $SL_r(D_{\mathbf{Q}}) \times SL_s(D_{\mathbf{Q}})$, up to conjugation by an element of $GL_m(\mathbf{R}) \times GL_k(\mathbf{R})$.

Therefore the subgroup $H \cap \Lambda_{F_i}$ is, up to commensurability, conjugate to either $(SL_rD)_{\mathbf{Z}} \times (SL_sD)_{\mathbf{Z}}$ or $\{(A, A^I) \mid A \in (SL_rD)_J\}$ where J is the ring of integers of K. It follows that there exist $g_1, h_2 \in GL_m(\mathbf{R})$, $g_2, h_1 \in GL_k(\mathbf{R})$ such that for each $i = 1, 2$, the lattice $g_iF_i{h_i}^{-1}$ is commensurable to either $M_{r\times s}(D_{\mathbf{Z}})$ or $\{(X_{ij}) \in M_r(D_J) \mid \sigma(X_{ij}) + X_{ji} = 0\}$ up to a constant multiple.

We now observe that the orbit $H.(F_1, F_2)$ is either closed or dense in $H.F_1 \times H.F_2$. The following lemma can be shown, using the fact that an irreducible lattice in $G(\mathbf{R})$ for a semisimple $\mathbf{R}$-algebraic group G of $\mathbf{R}$-rank greater than 1 is finitely presentable and locally rigid.

Lemma 2. *If the closure of the orbit $H.(F_1, F_2)$ contains a pair of lattices (E_1, E_2) such that the subgroup Γ_{E_1,E_2} is an irreducible lattice, the orbit $H.(F_1, F_2)$ is closed.*

Lemma 3. *Let G be endowed with a $\mathbf{Q}$-form with respect to which U_1 and U_2 are defined over $\mathbf{Q}$ and E_i be a lattice in U_i commensurable to $U_i(\mathbf{Z})$ for each $i = 1, 2$. Then, if $\Gamma_{E_1,\alpha E_2}$ is discrete for some non-zero $\alpha \in \mathbf{R}$, we have $\alpha \in \mathbf{Q}$.*

Now Lemma 3 and the following proposition imply the theorem. We mention that we make use of the theorem ([9], [12]) that if G is defined over $\mathbf{Q}$ so that U_1 and U_2 are $\mathbf{Q}$-subgroups, then a discrete subgroup generated by two lattices commensurable to $U_1(\mathbf{Z})$ and $U_2(\mathbf{Z})$ respectively is an arithmetic subgroup of $G(\mathbf{R})$.

Proposition 3. *The orbit $H.(F_1, F_2)$ is closed.*

Sketch of proof. By Lemma 2, it is enough to find a pair (E_1, E_2) in the $(H \times H)$-orbit $H.F_1 \times H.F_2$ such that the subgroup Γ_{E_1,E_2} is an irreducible lattice, assuming that the orbit $H.(F_1, F_2)$ is not closed. When $m \neq k$, the product $H.F_1 \times H.F_2$ contains a pair of lattices (E_1, E_2) commensurable to a lattice of the form $(M_{r\times s}(D_1)_{\mathbf{Z}}, \alpha M_{u\times v}(D_2)_{\mathbf{Z}})$. Making use of block elementary matrices in the subgroup Γ_{E_1,E_2}, we can prove that $(D_1)_{\mathbf{Z}} = (D_2)_{\mathbf{Z}}$ and $\alpha \in \mathbf{Q}$. Letting $m = k$, there exist $u \in U_1$ and $C, B \in GL_m(\mathbf{R})$ such that $u\Gamma_{F_1,F_2}u^{-1}$ contains $\Gamma_{F_1,\lambda CF_1B^{-1}}$ for some $\lambda \in \mathbf{R}$. We claim that the orbit $H.(F_1, \lambda CF_1B^{-1})$ is closed and from this it follows that the orbit $H.(F_1, F_2)$ is closed. To see this, note that the product $H.F_1 \times H.(\lambda CF_1B^{-1})$ contains $(E, \alpha E)$ for some $\alpha \in \mathbf{R}$ and some lattice E commensurable to either $M_r(D_{\mathbf{Z}})$ or $\{X \in M_r(D_J) \mid X_{ij} + \sigma(X_{ji}) = 0\}$. It then follows from Lemma 2 that $\alpha \in \mathbf{Q}$ and this proves the claim.

□

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