ON DENSENESS OF HOROSPHERES IN HIGHER RANK HOMOGENEOUS SPACES

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ABSTRACT. Let $G$ be a connected, semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. Let $N$ denote a maximal horospherical subgroup of $G$, and $P = MAN$ the minimal parabolic subgroup which is the normalizer of $N$. Let $E$ denote the unique $P$-minimal subset of $\Gamma \backslash G$ and let $E_0$ be a $P^\circ$-minimal subset. We consider a notion of a horospherical limit point in the Furstenberg boundary $G/P$ and show that the following are equivalent for any $[g] \in E_0$,

1) $gP \in G/P$ is a horospherical limit point;
2) $[g]NM$ is dense in $E$;
3) $[g]N$ is dense in $E_0$.

The equivalence of (1) and (2) is due to Dal’bo in the rank one case.

We also observe that unlike convex cocompact groups of rank one Lie groups, the $NM$-minimality of $E$ does not hold in a general Anosov homogeneous space.

1. INTRODUCTION

Let $G$ be a connected semisimple real algebraic group. Let $(X = G/K, d)$ denote the associated Riemannian symmetric space where $K$ is a maximal compact subgroup and the metric $d$ is defined by the inner product on $\mathfrak{g}$ induced from the Killing form. We set $o = [K] \in X$. Let $P = MAN$ be a minimal parabolic subgroup of $G$ with fixed Langlands decomposition, where $A$ is a maximal real split torus, $M$ the centralizer of $A$ and $N$ the unipotent radical of $P$. Note that $N$ is a maximal horospherical subgroup of $G$, which is unique up to conjugations.

Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \log A$ so that $\log N$ consists of positive root subspaces, and we set $A^+ = \exp \mathfrak{a}^+$. This means that $N$ is a contracting horospherical subgroup in the sense that for any $a \in \text{int} A^+$,

$$N = \{ g \in G : a^{-n} ga^n \to e \text{ as } n \to +\infty \}.$$

Let $\Gamma < G$ be a Zariski dense discrete subgroup of $G$. In this paper, we are interested in the topological behavior of the action of the horospherical subgroup $N$ on $\Gamma \backslash G$ via the right translations. When $\Gamma < G$ is a cocompact lattice, every $N$-orbit is dense, i.e., the $N$-action is minimal. This is due to Hedlund [8] for $G = \text{PSL}_2(\mathbb{R})$ and to Veech [14] in general. Dani gave a full classification of possible orbit closures of $N$-action for any lattice $\Gamma < G$ [4].
For a general discrete subgroup, \( \Gamma \backslash G \) does not admit a dense \( N \)-orbit, even a dense \( NM \)-orbit, because of the obvious obstruction from the size of the limit set of \( \Gamma \). Denoting by \( \mathcal{F} = G/P \) the Furstenberg boundary. We denote by \( \Lambda \) the limit set of \( \Gamma \),

\[
\Lambda = \{ \lim_{i \to \infty} \gamma_i(o) \in \mathcal{F} : \gamma_i \in \Gamma \}
\]

where the convergence is understood as in Definition 2.2. This is the unique \( \Gamma \)-minimal subset of \( \mathcal{F} \) ([1], [10]). Thus the set

\[
\mathcal{E} = \{ [g] \in \Gamma \backslash G : gP \in \Lambda \}
\]

is the unique \( P \)-minimal subset of \( \Gamma \backslash G \). For a given point \([g] \in \mathcal{E}\), the topological behavior of the horospherical orbit \([g]N\) (or of \([g]NM\)) is closely related to the ways in which the orbit \( \Gamma(o) \) approaches \( gP \) along its limit cone. The limit cone \( \mathcal{L} \) of \( \Gamma \) is defined as the smallest closed cone of \( a^+\) containing the Jordan projection \( \lambda(\Gamma) \). It is always a convex cone with non-empty interior [1]. For rank \( G = 1 \), \( \mathcal{L} = a^+ \) for any non-elementary subgroup \( \Gamma < G \). In higher ranks, the limit cone of \( \Gamma \) depends more subtly on \( \Gamma \).

Recall that in the rank one case, a horoball in \( X \) based at \( \xi \in \mathcal{F} \) is a subset of the form \( gN(\exp a^+)o \) for \( gP = \xi \). Our generalization of the notion of a horospherical limit point in the rank one [3] to higher rank involves the limit cone:

**Definition 1.1.**

- By a \( \Gamma \)-tight horoball based at \( \xi \in \mathcal{F} \), we mean a subset of the form \( \mathcal{H}_\xi = gN(\exp \mathcal{C})o \) where \( g \in G \) is such that \( \xi = gP \) and \( \mathcal{C} \subset \text{int} \mathcal{L} \cup \{0\} \) is a closed cone. For \( T > 0 \), we write
  \[
  \mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o
  \]
  where \( \mathcal{C}_T = \{ u \in \mathcal{C} : \|u\| < T \} \).

- We call a limit point \( \xi \in \mathcal{F} \) a horospherical limit point of \( \Gamma \) if there exists a \( \Gamma \)-tight horoball \( \mathcal{H}_\xi \) based at \( \xi \) such that for any \( T > 1 \), \( \mathcal{H}_\xi(T) \) contains some point of \( \Gamma(o) \).\(^1\)

See also Definition 3.2 for an equivalent definition given in terms of the \( a \)-valued Busemann function.

Note that unlike the rank one case, a sequence \( \gamma_i(o) \in \mathcal{H}_\xi(T_i) \), with \( T_i \to \infty \), does not necessarily converge to \( \xi \) for a \( \Gamma \)-tight horoball \( \mathcal{H}_\xi \) based at \( \xi \). It is hence plausible that a general discrete group \( \Gamma \) would support a horospherical limit point outside of its limit set.

We denote by \( \Lambda_h \subset \Lambda \) the set of all horospherical limit points of \( \Gamma \) inside \( \Lambda \). The attracting fixed point \( y_\gamma \) of a loxodromic element \( \gamma \in \Gamma \) whose Jordan component belongs to \( \text{int} \mathcal{L} \) is always a horospherical limit point (Lemma 3.4). Moreover, any \( u \)-directional radial limit point (i.e, \( \xi = gP \) such that

\(^1\)We remark that there exists a notion of horospherical limit points in the geometric boundary associated to a symmetric space, see [7]. In rank \( \geq 2 \) this notion and the one considered here are distinct.
\[
\limsup \Gamma g \exp(tu) \neq \emptyset \text{ for some } u \in \text{int } \mathcal{L} \text{ is also a horospherical limit point (Lemma 5.2).}
\]

The following theorem generalizes Dal’bo’s theorem [3] to discrete subgroups in higher rank semisimple Lie groups:

**Theorem 1.2.** Let \( \Gamma < G \) be a Zariski dense discrete subgroup. For any \( [g] \in \mathcal{E} \), the following are equivalent:

1. \( gP \in \Lambda_h \);
2. \( [g]NM \) is dense in \( \mathcal{E} \).

We remark that for Anosov subgroups of \( G \) (with respect to \( P \)), almost all \( NM \)-orbits are dense in \( \mathcal{E} \) with respect to any Patterson-Sullivan measure on \( \Lambda \) [10] and hence the set of all horospherical limit points has full Patterson-Sullivan measures.

In order to extend Theorem 1.2 to \( N \)-orbits, we fix a \( P^o \)-minimal subset \( \mathcal{E}_0 \) where \( P^o \) denotes the identity component of \( P \). Since \( P = P^o M \), any \( P^o \)-minimal subset is a translate of \( \mathcal{E}_0 \) by an element of the finite group \( M^o \setminus M \), where \( M^o \) is the identity component of \( M \). Denote by \( \mathcal{D}_\Gamma = \{ \mathcal{E}_0, ..., \mathcal{E}_p \} \) the finite collection of \( P^o \)-minimal sets in \( \mathcal{E} \). In order to understand \( N \)-orbit closures it hence suffices to restrict to \( \mathcal{E}_0 \).

The following is a refinement of Theorem 1.2:

**Theorem 1.3.** Let \( \Gamma < G \) be a Zariski dense discrete subgroup. For any \( [g] \in \mathcal{E}_0 \), the following are equivalent:

1. \( gP \in \Lambda_h \);
2. \( [g]N \) is dense in \( \mathcal{E}_0 \).

**Remark 1.4.** As mentioned, we may consider horospherical limit points outside the context of \( \Lambda \). In this case our proofs of Theorems 1.2 and 1.3 show that if \( gP \in \mathcal{F} \) is horospherical then the closures of \([g]MN\) and \([g]N\) contain \( \mathcal{E} \) and \( \mathcal{E}_i \), for some \( \mathcal{E}_i \in \mathcal{D}_\Gamma \), respectively.

For \( G = \text{SO}^o(n,1), n \geq 2 \), Theorem 1.3 was proved in [11]. When \( G \) has rank one and \( \Gamma < G \) is convex cocompact, every limit point is horospherical and Winter’s mixing theorem [15] implies the \( N \)-minimality of \( \mathcal{E}_0 \).

As Anosov subgroups are regarded as higher rank generalizations of convex cocompact subgroups, it is a natural question whether the minimality of the \( NM \)-action persists in the higher rank setting. It turns out that is not the case.

Denoting by \( \partial \mathcal{L} \) the boundary of \( \mathcal{L} \), note that \( \lambda(\gamma) \in \partial \mathcal{L} \) if and only if \( \lambda(\gamma^{-1}) \in \partial \mathcal{L} \). We observe the following:

**Proposition 1.5.** If \( \lambda(\gamma) \in \partial \mathcal{L} \) for some \( \gamma \in \Gamma \), then \( \{ y_\gamma, y_{\gamma^{-1}} \} \not\subseteq \Lambda_h \).

Thurston’s work [13] provides many examples satisfying the hypothesis of the above proposition. Let \( \Sigma \) be a surface subgroup and let \( \pi_1, \pi_2 : \Sigma \to \text{PSL}_2(\mathbb{R}) \) be discrete faithful representations.
Let $0 < d_- \leq d_+ < \infty$ be the minimal and maximal geodesic stretching constants of $\pi_2$ relative to $\pi_1$:

\begin{equation}
\begin{aligned}
d_+(\pi_1, \pi_2) &= \sup_{\sigma \in \Sigma^-} \frac{\ell_2(\sigma)}{\ell_1(\sigma)} \quad \text{and} \quad d_-(\pi_1, \pi_2) = \inf_{\sigma \in \Sigma^-} \frac{\ell_2(\sigma)}{\ell_1(\sigma)}
\end{aligned}
\end{equation}

where $\ell_i(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\pi_i(\Sigma) \setminus \mathbb{H}^2$ corresponding to $\pi_i(\sigma)$.

Set $\Gamma := (\pi_1 \times \pi_2)(\Sigma) \subset \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$. Identifying $a = \mathbb{R}^2$, for $\gamma = (\pi_1(\sigma), \pi_2(\sigma))$ with $\sigma \in \Sigma$, the Jordan projection $\lambda(\gamma)$ is given by $(\ell_1(\sigma), \ell_2(\sigma)) \in \mathbb{R}^2$. Hence the limit cone $\mathcal{L}$ of $\Gamma$ is given by

\[ \mathcal{L} := \{(v_1, v_2) \in \mathbb{R}^2_+ : d_- v_1 \leq v_2 \leq d_+ v_1\}. \]

Therefore Proposition 1.5 implies:

**Corollary 1.6.** Whenever $d_+(\pi_1, \pi_2)$ is achieved by an element $\sigma \in \Sigma$, we have $\Lambda \neq \Lambda_h$ and hence there exists a non-dense $NM$-orbit in $\mathcal{E}$.

Thurston [13, Theorem 10.7] showed that $d_+(\pi_1, \pi_2)$ is realized by a simple closed curve in most of cases, which hence provides many examples which satisfy the hypothesis of Corollary 1.6.

We close the introduction by the following question, c.f. [9, 12]:

**Question 1.7.** For rank $G \geq 2$, is a Zariski dense discrete subgroup $\Gamma < G$ with $\Lambda = \Lambda_h$ necessarily a cocompact lattice in $G$?

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## 2. Preliminaries

Let $G$ be a connected, semisimple real algebraic group. We fix, once and for all, a Cartan involution $\theta$ of the Lie algebra $\mathfrak{g}$ of $G$, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of $\theta$, respectively. We denote by $K$ the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$.

Choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of $A$ in $K$ is denoted by $M$, and we set $N$ to be the maximal contracting horospherical subgroup: for $a \in \text{int} A^+$,

\[ N = \{ g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty \}. \]

We set $P = MAN$, which is a minimal parabolic subgroup.

For $u \in \mathfrak{a}$, we write $a_u = \exp u \in A$. We denote by $\| \cdot \|$ the norm on $\mathfrak{g}$ induced by the Killing form. Consider the Riemannian symmetric space $X := G/K$ with the metric induced from the norm $\| \cdot \|$ on $\mathfrak{g}$ and $o = K \times X$.

Let $\mathcal{F} = G/P$ denote the Furstenberg boundary. Since $K$ acts transitively on $\mathcal{F}$ and $K \cap P = M$, we may identify $\mathcal{F} = K/M$. We denote by $\mathcal{F}^{(2)}$ the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$. 


Denote by $w_0 \in K$ the unique element in the Weyl group such that
$\text{Ad}_{w_0} \alpha^+ = -\alpha^+$; it is the longest Weyl element. We then have $P^+ := w_0 P w_0^{-1}$ is an opposite parabolic subgroup of $G$. The map $i = -\text{Ad}_{w_0} : \alpha^+ \to \alpha^+$ is called the opposition involution.

For $g \in G$, we consider the following visual maps

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := gw_0 P \in \mathcal{F}.$$  

Then $\mathcal{F}^{(2)} = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}$.

An element $g \in G$ is called loxodromic if $g = \varphi m \exp u \varphi^{-1}$ for some $u \in \text{int} \, \alpha^+$, $m \in M$ and $\varphi \in G$. The Jordan projection of $g$ is defined as $\lambda(g) = u$, and $\varphi^+$ is called the attracting fixed point of $g$, which we denote by $y_g$. For any loxodromic element $g \in G$ and $\xi \in \mathcal{F}$ with $(\xi, y_g^{-1}) \in \mathcal{F}^{(2)}$, 

$$\lim_{k \to \infty} g^k \xi = y_g$$

and the convergence is uniform on compact subsets.

We also note that for any loxodromic element $g \in G$,

$$\lambda(g^{-1}) = i \lambda(g).$$

Let $\Gamma < G$ be a Zariski dense discrete subgroup of $G$. We define the Jordan projection only for loxodromic elements, and hence the set of all Jordan projections $\lambda(\Gamma)$ of loxodromic elements of $\Gamma$ is contained in $\text{int} \, \alpha^+$ by definition. The limit cone $L = L_{\Gamma}$ of $\Gamma$ is the smallest closed cone of $\alpha^+$ containing $\lambda(\Gamma)$. It is a convex cone with non-empty interior [1].

We will use the following simple lemma.

**Lemma 2.1.** For any $v \in \lambda(\Gamma)$ and $\zeta \in \mathcal{F}$, there exists a loxodromic element $\gamma \in \Gamma$ with $\lambda(\gamma) = v$ and a neighborhood $U$ of $\zeta$ in $\mathcal{F}$ such that $\{y_\gamma\} \times U$ is a relatively compact subset of $\mathcal{F}^{(2)}$ and

$$\gamma^{-k} U \to y_{\gamma^{-1}} \quad \text{uniformly.}$$

**Proof.** Let $\zeta \in \mathcal{F}$. Choose $\gamma_1 \in \Gamma$ such that $\lambda(\gamma_1) = v$. Since the set of all loxodromic elements of $\Gamma$ is Zariski dense in $G$ and $\mathcal{F}^{(2)}$ is Zariski open in $\mathcal{F} \times \mathcal{F}$, there exists $\gamma_2 \in \Gamma$ such that $(\zeta, \gamma_2 y_{\gamma_1}) \in \mathcal{F}^{(2)}$. Let $\gamma = \gamma_2 \gamma_1^{-1}$, so that $y_\gamma = \gamma_2 y_{\gamma_1}$. It now suffices to take any neighborhood $U$ of $\zeta$ so that $U \times \{\gamma_2 y_{\gamma_1}\}$ is a relatively compact subset of $\mathcal{F}^{(2)}$. \hfill $\square$

**Convergence of a sequence in $X$ to $\mathcal{F}$.** Let $\Pi$ be the set of simple roots for $(\mathfrak{g}, \mathfrak{a})$. For a sequence $g_i \to G$, we write $g_i \to \infty$ regularly if $\alpha(\mu(g_i)) \to \infty$ for all $\alpha \in \Pi$. By the Cartan decomposition $G = KA^+K$, for $g \in G$, we may write $g = \kappa_1(g) \exp(\mu(g)) \kappa_2(g) \in KA^+K$ where $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of $g$, is uniquely determined, and $\kappa_1(g), \kappa_2(g) \in K$. If $\mu(g) \in \text{int} \, \mathfrak{a}^+$, then $[\kappa_1(g)] \in K/M = \mathcal{F}$ is uniquely determined. Note that if $g_i \to \infty$ regularly, then for all sufficiently large $i$, $\mu(g_i) \in \text{int} \, \mathfrak{a}^+$ and hence $[\kappa_1(g_i)]$ is well-defined.

**Definition 2.2.** A sequence $p_i \in X$ is said to converge to $\xi \in \mathcal{F}$ if there exists $g_i \to \infty$ regularly in $G$ with $p_i = g_i(o)$ and $\lim_{i \to \infty} [\kappa_1(g_i)] = \xi$. 


\textbf{P*-minimal subsets.} We denote by $\Lambda \subset F$ the limit set of $\Gamma$, which is defined as

\begin{equation}
\Lambda = \{ \lim \gamma_i(o) : \gamma_i \in \Gamma \}.
\end{equation}

For a non-Zariski dense subgroup, $\Lambda$ may be an empty set. For $\Gamma < G$ Zariski dense, this is the unique $\Gamma$-minimal subset of $F$ ([1], [10]).

It follows that the following set $E$ is the unique $P$-minimal subset of $\Gamma \backslash G$:

\[ E = \{ [g] \in \Gamma \backslash G : g^+ \in \Lambda \}. \]

Let $P^o$ denote the identity component of $P$. Then $E$ is a disjoint union of at most $|P^o|$-number of $P^o$-minimal subsets. We fix one $P^o$-minimal subset $E_0$ once and for all. Note that any $P^o$-minimal subset is then of the form $E_0^m$ for some $m \in M$. We set

\[ \Omega := \{ [g] \in \Gamma \backslash G : g^+, g^- \in \Lambda \} \quad \text{and} \quad \Omega_0 := \Omega \cap E_0. \]

\textbf{Busemann map.} The Iwasawa cocycle $\sigma : G \times F \to a$ is defined as follows: for $(g, \xi) \in G \times F$ with $\xi = [k]$ for $k \in K$, $\exp(\sigma(g, \xi))$ is the $A$-component of $gk$ in the $KAN$ decomposition, that is,

\[ gk \in K \exp(-\beta(0, go))N, \]

and hence $\beta_P(o, a_u o) = u$ for any $u \in a$. For $h, g \in G$, we set $\beta(0, g) := \beta(h, go)$. We note that for any $g \in G$, $\xi \in F$, and $x, y, z \in X$,

\[ \beta(x, y) = \beta(gx, gy), \quad \text{and} \quad \beta(x, y) = \beta(x, z) + \beta(z, y). \]

In particular, $\beta(o, go) \in a$ is defined by

\begin{equation}
(2.2) \quad g^{-1}k \in K \exp(-\beta(o, go))N,
\end{equation}

and hence $\beta_P(o, a_u o) = u$ for any $u \in a$. For $h, g \in G$, we set $\beta(h, g) := \beta(h, go)$.

\textbf{Shadows.} For $q \in X$ and $r > 0$, we set $B(q, r) = \{ x \in X : d(x, q) < r \}$. For $p = g(o) \in X$, the shadow of the ball $B(q, r)$ viewed from $p$ is defined as

\[ O_r(p, q) := \{ (gk)^+ \in F : k \in K, \, gk \text{ int } A^+ o \cap B(q, r) \neq \emptyset \}. \]

Similarly, for $\xi \in F$, the shadow of the ball $B(q, r)$ as viewed from $\xi$ is

\[ O_r(\xi, q) := \{ h^+ \in F : h^+ \text{ satisfies } h^+ = \xi, \, ho \in B(q, r) \}. \]

\textbf{Lemma 2.3.} [10, Lemma 5.6 and 5.7]

1. There exists $\kappa > 0$ such that for any $g \in G$ and $r > 0$,

\[ \sup_{\xi \in O_r(g(o), o)} \| \beta(\xi, g(o), o) - \mu(g^{-1}) \| \leq \kappa r. \]
(2) If \( p_i \in X \) converges to \( \xi \in \mathcal{F} \), then for any \( 0 < \varepsilon < r \), we have
\[
O_{r-\varepsilon}(p_i, o) \subset O_r(\xi, o) \subset O_{r+\varepsilon}(p_i, o)
\]
for all sufficiently large \( i \).

3. Horospherical limit points

Let \( \Gamma < G \) be a Zariski dense discrete subgroup. Recall the definition of a horospherical limit point from Definition 1.1. In this section we provide a mostly self-contained proof of the following theorem:

**Theorem 3.1.** Let \([g] \in \mathcal{E}\). The following are equivalent:

1. \( g^+ \in \Lambda \) is a horospherical limit point;
2. \([g]NM \) is dense in \( \mathcal{E} \).

The main external ingredient used in this proof is the density of the group generated by the Jordan projection of \( \Gamma \), due to Benoist [1], that is,
\[
a = \langle \lambda(\Gamma) \rangle.
\]
for every Zariski dense discrete subgroup \( \Gamma < G \). In fact, for every cone \( C \subset \mathcal{L} \) with non-empty interior, there exists a Zariski dense subgroup \( \Gamma' < \Gamma \) with \( \mathcal{L}_{\Gamma'} \subset C \) (see [1]); therefore we have
\[
a = \langle \lambda(\Gamma) \cap \text{int} \mathcal{L} \rangle.
\]

It is convenient to use a characterization of horospherical limit points in terms of the Busemann function.

**Lemma 3.2.** For \( \xi \in \Lambda \), \( \xi \in \Lambda_h \) if and only if there exists a closed cone \( C \) such that \( C \setminus \{0\} \subset \text{int} \mathcal{L} \) and elements \( \gamma_j \in \Gamma \) satisfying
\[
(3.1) \quad \beta_\xi(o, \gamma_j o) \to \infty \quad \text{and} \quad \beta_\xi(o, \gamma_j o) \in C \text{ for all large } j.
\]

**Proof.** Let \( \xi = gP \in \Lambda_h \) as defined in Definition 1.1. Then there exists \( \gamma_j = g_{n_j}a_{u_j}k_j \in \Gamma \) for some \( p \in P, n_j \in N, k_j \in K \) and \( u_j \to \infty \) in some closed cone \( \mathcal{C} \) contained in \( \text{int} \mathcal{L} \cup \{0\} \). Fix some closed cone \( C' \subset \text{int} \mathcal{L} \cup \{0\} \) whose interior contains \( \mathcal{C} \). Note that
\[
\beta_\xi(o, \gamma_j o) = \beta_gP(e, g) + \beta_gP(g, g_{n_j}a_{u_j})
\]
\[
= \beta_P(g^{-1}, e) + \beta_P(e, p) + \beta_P(e, n_j) + \beta_P(e, a_{u_j})
\]
\[
= \beta_P(g^{-1}, p) + u_j.
\]

Therefore the sequence \( \beta_\xi(o, \gamma_j) - u_j \) is uniformly bounded. Since \( u_j \in \mathcal{C} \), \( \beta_\xi(o, \gamma_j o) \in C' \) for all large \( j \). Therefore (3.1) holds. For the other direction, let \( \gamma_j \) and \( \mathcal{C} \) satisfy (3.1) for \( \xi = gP \) for \( g \in G \). Since \( G = gNAK \), we may write \( \gamma_j = g_{n_j}a_{u_j}k_j \) for some \( n_j \in N, u_j \in a \) and \( k_j \in K \). By a similar computation as above, the sequence \( \beta_\xi(o, \gamma_j o) - u_j \) is uniformly bounded. It follows that \( u_j \in C' \) for all large \( j \) and \( u_j \to \infty \). Therefore for any \( T > 1 \), there exists \( j > 1 \) such that \( \gamma_j(o) \in gN \exp(C' - C_T')(o) \). This proves \( \xi \in \Lambda_h \).
For any \( g \in G \) and \( \xi \in \mathcal{F} \), we have
\[
\|\beta_{\xi}(o, \gamma g o) - \beta_{\xi}(o, \gamma o)\| \leq 2\|\mu(g)\|.
\]

Therefore (3.1) implies that for any \( p \in X \),
\[
(3.2) \quad \beta_{\xi}(p, \gamma j p) \to \infty \quad \text{and} \quad \beta_{\xi}(p, \gamma j p) \in C \text{ for all large } j.
\]

We consider the following seemingly stronger condition:

**Definition 3.3.** For \( u \in a^+ \), a point \( \xi \in \mathcal{F} \) is called \( u \)-horospherical if for some \( p \in X \) (and hence for any \( p \in X \)), there exists a constant \( R > 0 \) and a sequence \( \gamma \in \Gamma \) satisfying
\[
\beta_{\xi}(p, \gamma j p) \to \infty \quad \text{and} \quad \|\beta_{\xi}(p, \gamma j p) - \Re u\| < R \text{ for all } j.
\]

It is easy to see that the set of horospherical (resp. \( u \)-horospherical) limit points is \( \Gamma \)-invariant. Therefore for \( x = [g] \in \Gamma \backslash G \), we may say \( x^+ := \Gamma gP \) horospherical (resp. \( u \)-horospherical) if \( g^+ \) is.

For \( u \in \mathfrak{a} \), we call \( x \in \Gamma \backslash G \) a \( u \)-periodic point if \( xa_u = x_m \) for some \( m \in M \); note that \( xa_uM \) is then compact. Note that for \( u \in \text{int} \mathfrak{a}^+ \), the existence of a \( u \)-periodic point is equivalent to the condition that \( u \in \lambda(\Gamma) \).

**Lemma 3.4.** If \( x \in \Gamma \backslash G \) is \( u \)-periodic for \( u \in a^+ \) then \( x^+ \) is \( u \)-horospherical.

**Proof.** Since \( x \) is \( u \)-periodic, there exist \( g \in G \) with \( x = [g] \) and \( \gamma \in \Gamma \) such that \( \gamma = ga_o mg^{-1} \) for some \( m \in M \), and \( y_\gamma = g^+ \in \Lambda \). Moreover, for any \( k \geq 1 \)
\[
\beta_{g\gamma}(o, \gamma k o) = \beta_{p}(o, a_{u}^k o) = ku.
\]
This implies \( gP \) is \( u \)-horospherical. \( \square \)

**Proposition 3.5.** Let \( x \in \Gamma \backslash G \). If \( x^+ \) is \( u \)-horospherical for some \( u \in \lambda(\Gamma) \) then \( XN \cap x_0 AM \neq \emptyset \)

**Proof.** Choose \( g \in G \) so that \( x = [g] \). We may assume without loss of generality that \( g = k \in K \), since \( kanN = kNa \), and a translate of a \( u \)-periodic point by an element of \( A \) is again a \( u \)-periodic point. Since \( u \in \lambda(\Gamma) \), there exists a \( u \)-periodic point, say, \( x_0 \in \Gamma \backslash G \). It suffices to show that
\[
(3.3) \quad [k]N \cap x_0 AM \neq \emptyset
\]
as every point in \( x_0 AM \) is \( u \)-periodic.

Since \( k^+ \) is \( u \)-horospherical and using (2.2), there exists \( R > 0 \) and
\[
(3.4) \quad k_j = \gamma^{-1}_j k n_j^{-1} a_{u_j},
\]
with \( ||\Re u - u_j|| < R \) for all \( j \). Let \( \ell_j \to \infty \) be a sequence of integers satisfying
\[
(3.5) \quad ||\ell_j u - u_j|| < R + ||u|| \quad \text{for all } j \geq 1.
\]
By passing to a subsequence, we may assume without loss of generality that \( \gamma_j^{-1}kP \) converges to some \( \xi_0 \in \mathcal{F} \). Since \( N^+P \) is Zariski open and \( \Gamma \) is Zariski dense, we may choose \( g_0 \in G \) such that \( x_0 = [g_0] \) and \( g_0^{-1}\xi_0 \in N^+P \). Let \( h_0 \in N^+ \) be such that \( \xi_0 = g_0h_0P \). Since \( g_0N^+P \) is open and \( \gamma_j^{-1}kP \to g_0h_0P \), we may assume that for all \( j \), there exists \( h_j \in N^+ \) satisfying \( g_0h_jP = \gamma_j^{-1}kP = k_jP \) with \( h_j \to h_0 \). Let \( p_j = a_{v_j}m_j\tilde{n}_j \in P = AMN \) be such that \( g_0h_jp_j = k_j \); since \( h_j \to h_0 \) and the product map \( N^+ \times P \to N^+P \) is a diffeomorphism, the sequence \( p_j \), as well as \( v_j \in a \), are bounded.

Therefore by (3.4), we get for all \( j \),
\[
g_0 = k_jp_j^{-1}h_j^{-1}
\]
\[
  = \gamma_j^{-1}kn_j^{-1}a_{u_j}(\tilde{n}_j^{-1}m_j^{-1}a_{-v_j})h_j^{-1}
\]
\[
  = \gamma_j^{-1}kn_j^{-1}(a_{u_j}\tilde{n}_j^{-1}a_{-u_j})a_{u_j}m_j^{-1}a_{-v_j}h_j^{-1}
\]
\[
  = \gamma_j^{-1}kn_j^{-1}(a_{u_j}\tilde{n}_j^{-1}a_{-u_j})m_j^{-1}(a_{u_j}m_j^{-1}a_{-u_j}h_j^{-1}a_{-u_j+v_j})a_{u_j}a_{-v_j}.
\]

Since \( h_j^{-1} \in N^+ \) and \( v_j \in a \) are uniformly bounded and since \( u_j \to \infty \) within a bounded neighborhood of the ray \( \mathbb{R}^+u \in \text{int } a^+ \), we have
\[
h_j = a_{u_j-v_j}h_j^{-1}a_{-u_j+v_j} \to e \text{ in } N^+.
\]
By setting \( n_j' = n_j^{-1}(a_{u_j}n_j^{-1}a_{-u_j}) \in N \), we may now write
\[
g_0 = \gamma_j^{-1}kn_j'm_j^{-1}\tilde{h}_ja_{u_j}a_{-v_j}.
\]

Since \( x_0 \) is \( u \)-periodic, there exists \( \gamma_0 \in \Gamma \) such that \( \gamma_0 = g_0a_um_0g_0^{-1} \) for some \( m_0 \in M \). Hence for all \( j \geq 1 \),
\[
\gamma^{-\ell_j}_0 = g_0a_{-\ell_j}u_{m_0}^{-\ell_j}g_0^{-1} = (\gamma_j^{-1}kn_j'm_j^{-1}\tilde{h}_ja_{u_j}a_{-v_j})(a_{-\ell_j}u_{m_0}^{-\ell_j})g_0^{-1}.
\]
In other words,
\[
\gamma_j^{-1}kn_j' = \gamma^{-\ell_j}_0 g_0m_0^\ell_ja_{-u_j+\ell_ju+v_j}\tilde{h}_j^{-1}m_j.
\]

Since the sequence \(-u_j + \ell_ju + v_j \in a \) is uniformly bounded by (3.5) and \( \tilde{h}_j \to e \) in \( N^+ \), we conclude that the sequence \( \Gamma kn_j' \) has an accumulation point in \( \Gamma g_0AM \). This proves (3.3).

It turns out that a horospherical limit point is also \( u \)-horospherical for any \( u \in \text{int } \mathcal{L} \):

**Proposition 3.6.** Any \( \xi \in \Lambda_h \) is \( u \)-horospherical for all \( u \in \text{int } \mathcal{L} \).

**Proof.** Let \( \xi \in \Lambda_h \). We then have a sequence \( \gamma_j \in \Gamma \) satisfying \( v_j := \beta_{\xi}(e, \gamma_j) \to \infty \) and the sequence \( \|v_j\|^{-1}v_j \) converges to some point \( v_0 \in \text{int } \mathcal{L} \). By passing to a subsequence, we may assume that \( \gamma_j^{-1}\xi \) converges to some \( \xi_0 \in \mathcal{F} \).

Let \( u \in \text{int } \mathcal{L} \). We first consider the case \( u \notin \mathbb{R}^+v_0 \). Let \( r := \text{rank } G - 1 \geq 0 \). Since \( \bigcup_{\gamma \in \Gamma} \mathbb{R}^+\lambda(\gamma) \) is dense in \( \mathcal{L} \), there exist \( w_1, \cdots, w_r \in \lambda(\Gamma) \) such that
$v_0$ belongs to the interior of the convex cone spanned by $u, w_1, \ldots, w_r$, so that
\[
v_0 = c_0 u + \sum_{\ell=1}^r c_{\ell} w_\ell
\]
for some positive constants $c_0, \ldots, c_\ell$.

Since $\|v_j\|^{-1} v_j \rightarrow v_0$, we may assume, by passing to a subsequence, that for each $j \geq 1$, we have
\[
\|v_j\|^{-1} v_j = c_{0,j} u + \sum_{\ell=1}^r c_{\ell,j} w_\ell
\]
for some positive $c_{\ell,j}$, $\ell = 0, \ldots, r$. Note that for each $0 \leq \ell \leq r$, $c_{\ell,j} \rightarrow c_\ell$ as $j \rightarrow \infty$.

By Lemma 2.1, we can find a loxodromic element $g_1 \in \Gamma$ and a neighborhood $U_1$ of $v_0$ such that $\lambda(g_1^{-1}) = w_1$, $\{y_{g_1}\} \times U_1 \subset F(2)$ and $g_1^{-k} U_1 \rightarrow y_{g_1}^{-1}$ uniformly. Applying Lemma 2.1 once more, we can find $g_2 \in \Gamma$ satisfying $\lambda(g_2^{-1}) = w_2$ and a neighborhood $U_2 \subset F$ of $y_{g_2}$ satisfying $\{y_{g_2}\} \times U_2 \subset F(2)$ and that $g_2^{-k} U_2 \rightarrow y_{g_2}^{-1}$ uniformly.

Continuing inductively, we get elements $g_1, \ldots, g_r \in \Gamma$ and open sets $U_1, \ldots, U_r \subset F$ satisfying that for all $\ell = 1, \ldots, r$,
\begin{enumerate}
  \item $w_\ell = \lambda(g_\ell^{-1})$;
  \item $y_{g_\ell^{-1}} \in U_\ell$;
  \item $g_\ell^{-k} U_\ell \rightarrow y_{g_\ell}^{-1}$ uniformly; and
  \item $\{y_{g_\ell}\} \times U_\ell$ is a relatively compact subset of $F(2)$.
\end{enumerate}

We set $\xi_\ell := y_{g_\ell^{-1}}$ for each $1 \leq \ell \leq r$; so $U_\ell$ is a neighborhood of $\xi_{\ell-1}$ for each $1 \leq \ell \leq r$.

Since $Q_{\eta_0} := \{\eta \in F : (\eta_0, \eta) \in F(2)\} = \bigcup_{R>0} O_R(\eta_0, o)$ for any $\eta_0 \in F$ and $U_\ell \subset Q_{\eta_{g_\ell}}$ is a relatively compact subset of $F(2)$, there exists $R_\ell > 0$ such that $U_\ell \subset O_{R_\ell}(y_{g_\ell}, o)$. Since $g_\ell^k o$ converges to $y_{g_\ell}$ as $k \rightarrow +\infty$, by Lemma 2.3(2),
\[
O_{R_\ell}(y_{g_\ell} o, o) \subset O_{R_{\ell+1}}(g_\ell^k o, o)
\]
for all sufficiently large $k > 1$.

For each $1 \leq \ell \leq r$ and $j \geq 1$, let $k_{\ell,j}$ be the largest integer smaller than $c_{\ell,j} \|v_j\|$. As $\|v_j\| \rightarrow \infty$, and $c_{\ell,j} \rightarrow c_\ell$, we have $k_{\ell,j} \rightarrow \infty$ as $j \rightarrow \infty$. By the uniform contraction $g_\ell^{-k} U_\ell \rightarrow \xi_\ell$, there exists $j_0 > 1$ such that for all $j \geq j_0$,
\[
\gamma_j^{-1} \xi \in U_1, \quad g_\ell^{-k_{\ell,j}} U_\ell \subset U_{\ell+1}, \quad \text{and} \quad U_\ell \subset O_{R_{\ell+1}}(g_\ell^{k_{\ell,j}} o, o)
\]
for all $\ell = 1, \ldots, r$.

For each $j \geq j_0$, we now set
\[
\tilde{\gamma}_j := \gamma_j g_1^{k_{1,j}} g_2^{k_{2,j}} \cdots g_r^{k_{r,j}} \in \Gamma.
\]
We claim that $\beta_{\xi}(e, \tilde{\gamma}_j) \to \infty$ as $j \to \infty$ and that

\[
(3.9) \quad \sup_{j \geq j_0} \| \beta_{\xi}(e, \tilde{\gamma}_j) - R_+u \| < \infty;
\]

this proves that $\xi$ is $u$-horospherical.

Fix $j \geq j_0$ and for each $1 \leq \ell \leq r$, let $k_{\ell} := k_{\ell,j}$, $b_{\ell} := c_{\ell,j}\|v_j\|$, and set

\[
h_{\ell} = g_1^{k_1} g_2^{k_2} \cdots g_{\ell}^{k_{\ell}},
\]

and $g_0 = e$. The cocycle property of the Busemann function gives that

\[
(3.10) \quad \beta_{\xi}(e, \tilde{\gamma}_j) = \beta_{\xi}(e, \gamma_j) - \sum_{\ell=1}^{r} \beta_{h_{\ell-1}^{-1} \gamma_j^{-1} \xi}(g_{\ell}^{k_{\ell}}, e).
\]

By (3.8), $\gamma_j^{-1} \xi \in U_1$ and for each $1 \leq \ell \leq r$,

\[
h_{\ell-1}^{-1} \gamma_j^{-1} \xi \in g_1^{-k_1} \cdots g_{\ell}^{-k_{\ell}} U_1 \subset U_{\ell+1} \subset O_{R_{\ell}+1}(g_{\ell}^{k_{\ell}} o, o).
\]

Hence by Lemma 2.3(1), there exists $\kappa \geq 1$ such that for each $1 \leq \ell \leq r$

\[
\| \beta_{h_{\ell-1}^{-1} \gamma_j^{-1} \xi}(g_{\ell}^{k_{\ell}}, e) - \mu(g_{\ell}^{-k_{\ell}}) \| \leq \kappa(R_{\ell} + 1).
\]

Note that for some $C_{\ell} > 0$, $\| \mu(g_{\ell}^{-k_{\ell}}) - k \lambda(g_{\ell}^{k_{\ell}}) \| \leq C_{\ell}$ for all $k \geq 1$. Since $\lambda(g_{\ell}^{k_{\ell}}) = w_{\ell}$, we get

\[
\| \beta_{h_{\ell-1}^{-1} \gamma_j^{-1} \xi}(g_{\ell}^{k_{\ell}}, e) - k_{\ell} w_{\ell} \| \leq \kappa(R_{\ell} + 1) + C_{\ell}.
\]

Therefore by (3.10), we obtain

\[
\| \beta_{\xi}(e, \tilde{\gamma}_j) - (v_j - \sum_{\ell=1}^{r} k_{\ell} w_{\ell}) \| \leq \kappa \sum_{\ell=1}^{r} (R_{\ell} + C_{\ell} + 1).
\]

By (3.6),

\[
c_{0,j}\|v_j\| u = v_j - \sum_{\ell=1}^{r} b_{\ell} w_{\ell}.
\]

Since $|b_{\ell} - k_{\ell}| \leq 1$ and $c_{0,j} > 0$, we deduce that for all $j \geq j_0$,

\[
\| \beta_{\xi}(e, \tilde{\gamma}_j) - R_+u \| \leq \| \beta_{\xi}(e, \tilde{\gamma}_j) - c_{0,j}\|v_j\| \cdot u \|
\]

\[
\leq \| \beta_{\xi}(e, \tilde{\gamma}_j) - (v_j - \sum_{\ell=1}^{r} k_{\ell} w_{\ell}) \| + \sum_{\ell=1}^{r} \| k_{\ell} w_{\ell} - b_{\ell} w_{\ell} \|
\]

\[
\leq \kappa \sum_{\ell=1}^{r} (R_{\ell} + C_{\ell} + \| w_{\ell} \| + 1).
\]

This proves (3.9), and consequently $\xi$ is $u$-horospherical for any $u \notin R_+ v_0$. To show that $\xi$ is $v_0$-horospherical, fix any $u \notin R_+ v_0$ and $\tilde{\gamma}_j \in \Gamma$ be a sequence as in (3.9) associated to $u$. If we set $v_j = \beta_{\xi}(e, \tilde{\gamma}_j)$, then $\| v_j \|^{-1} v_j$ converges to a unit vector in $\text{int} \ L$ proportional to $u$. Therefore by repeating the same argument only now switching the roles of $v_0$ and $u$, we prove that $\xi$ is $v_0$-horospherical as well. This completes the proof. \qed
We may now prove theorem 3.1:

**Proof of theorem 3.1.** Let \( g \in G \) be such that \( \xi = g^+ \in \Lambda \) is a horospherical limit point. Set \( Y := [g]NM \). We claim that \( Y = \mathcal{E} \). By Benoist [1], the group generated by \( \lambda(\Gamma) \cap \text{int} \mathcal{L} \) is dense in \( \mathfrak{a} \). Hence for every \( \varepsilon > 0 \) there exist loxodromic elements \( \gamma_1, \ldots, \gamma_q \in \Gamma \) such that

\[
\lambda(\gamma_1), \ldots, \lambda(\gamma_q) \in \text{Int} \mathcal{L}
\]

and the group \( \mathbb{Z}\lambda(\gamma_1) + \cdots + \mathbb{Z}\lambda(\gamma_q) \) is an \( \varepsilon \)-net in \( \mathfrak{a} \), i.e., its \( \varepsilon \)-neighborhood covers all \( \mathfrak{a} \). Denote \( u_i = \lambda(\gamma_i) \) for \( i = 1, \ldots, q \). By Proposition 3.6, the point \( \xi \) is \( u_1 \)-horospherical. By Proposition 3.5, there exists a \( u_1 \)-periodic point \( x_1 \in \mathcal{E} \) contained in \( Y \), set

\[
Y_1 := x_1NM \subseteq Y.
\]

By Lemma 3.4, \( x_1^+ \) is \( u_1 \)-horospherical; in particular, it is a horospherical limit point. Therefore we can inductively find a \( u_i \)-periodic point \( x_i \) in \( Y_{i-1} = x_{i-1}NM \) for each \( 2 \leq i \leq q \). Since \( x_i(exp u_i)M = x_iM \), we have \( Y_i \exp Zu_i \subseteq Y_i \) for each \( 1 \leq i \leq q \). Therefore we obtain

\[
Y \supset Y_1 \exp Zu_1 \supset Y_2 \exp(Zu_1 + Zu_2) \supset \cdots \supset Y_q \exp(\sum_{i=1}^q Zu_i).
\]

Recalling the dependence of \( Y_q \) and \( \sum_{i=1}^q Zu_i \) on \( \varepsilon \), set

\[
Z_\varepsilon := Y_qMN \exp(\sum_{i=1}^q Zu_i) \subseteq Y.
\]

Since \( MN \exp(\sum_{i=1}^q Zu_i) \) is an \( \varepsilon \)-net of \( \mathcal{P} \) and \( \mathcal{E} \) is \( \mathcal{P} \)-minimal, \( Z_\varepsilon \) is a \( 2\varepsilon \)-net of \( \mathcal{E} \) for all \( \varepsilon > 0 \). Since \( Y \) contains a \( 2\varepsilon \)-net of \( \mathcal{E} \) for all \( \varepsilon > 0 \) and \( Y \) is closed, it follows that \( Y = \mathcal{E} \).

For the other direction, it suffices to consider the case when \( g = k \in K \) as the density of \([g]NM\) depends only on the coset \( gP \). Suppose that \( [k]NM \) is dense in \( \mathcal{E} \) for \( k \in K \). Since \( \Gamma kNM \supset kNAM \), it follows that for any \( T > 1 \) and for any cone \( \mathcal{C} \in \mathfrak{a}^+ \) with non-empty interior, \( \Gamma \cap kN \exp(\mathcal{C} - \mathcal{C}_T)K \neq \emptyset \) where \( \mathcal{C}_T = \{ v \in \mathcal{C} : \|v\| \leq T \} \). This implies that \( k^+ \) is a horospherical limit point by Definition 1.1. \( \square \)

4. **Topological mixing and directional limit points**

There is a close connection between denseness of \( N \)-orbits and the topological mixing of one-parameter diagonal flows with direction in \( \text{int} \mathcal{L} \). This connection allows us to make use of recent topological mixing results by Chow-Sarkar [2]:

**Theorem 4.1.** [2] *For any \( u \in \text{int} \mathcal{L} \), \( \{a_{tu} : t \in \mathbb{R}\} \) is topologically mixing on \( \Omega_0 \), i.e., for any open subsets \( \mathcal{O}_1, \mathcal{O}_2 \) of \( \Gamma \backslash G \) intersecting \( \Omega_0 \), \( \mathcal{O}_1 \exp tu \cap \mathcal{O}_2 \neq \emptyset \) for all large \( |t| \gg 1 \).*
The above theorem was predated by a result of Dang [5] in the case where \( M \) is abelian.

**N-orbits based at directional limit points along \( \text{int} \mathcal{L} \).**

**Definition 4.2.** For \( u \in \text{int} \mathfrak{a}^+ \), denote by \( \Lambda_u \) the set of all \( u \)-directional limit points, i.e., \( \xi \in \Lambda_u \) if and only if \( \limsup_{t \to +\infty} \Gamma g \exp(tu) \neq \emptyset \) for some (and hence any) \( g \in G \) with \( gP = \xi \).

It is easy to see that \( \Lambda_u \subset \Lambda \) for \( u \in \text{int} \mathfrak{a}^+ \).

**Proposition 4.3.** If \( [g] \in \mathcal{E}_0 \) satisfies \( g^+ \in \Lambda_u \) for some \( u \in \text{int} \mathcal{L} \), then \( [g]N = \mathcal{E}_0 \).

**Proof.** Since \( \Omega_0N = \mathcal{E}_0 \), we may assume without loss of generality that \( x = [g] \in \Omega_0 \). There exist \( \gamma_i \in \Gamma \) and \( t_i \to +\infty \) such that \( \gamma_ig_{a_{t_iu}} \) converges to some \( h \in G \). In particular, \( x \exp(t_iu) \to [h] \). Since \( x_{a_{t_iu}} \in \Omega_0 \) and \( \Omega_0 \) is \( \Lambda \)-invariant and closed, we have \( [h] \in \Omega_0 \). We write \( \gamma_ig_{a_{t_iu}} = hq_i \) where \( q_i \to e \) in \( G \). Therefore \( xN = [h]q_iNa_{-t_iu} \) for all \( i \geq 1 \). Let \( \Omega \subset \Gamma G \) be any open subset intersecting \( \Omega_0 \). It suffices to show that \( xN \cap \Omega \neq \emptyset \). Let \( \Omega_1 \) be an open subset intersecting \( \Omega_0 \) and \( V \subset P^+ \) be an open symmetric neighborhood of \( e \) such that \( \Omega_1 V \subset \Omega \).

Since \( q_i \to e \) and \( NV \) is an open neighborhood of \( e \) in \( G \), there exists an open neighborhood, say, \( U \) of \( e \) in \( G \) and \( i_0 \) such that \( U \subset q_iNV \) for all \( i \geq i_0 \). By Theorem 4.1, we can choose \( i > i_0 \) such that \( [h]U \cap \Omega_1 a_{t_iu} \neq \emptyset \). It follows that \( [h]q_iNV a_{-t_iu} \cap \Omega_1 \neq \emptyset \). Since \( V \subset a_{-t_iu}V a_{t_iu} \) as \( u \in \mathfrak{a}^+ \), we have
\[
[h]q_iNV a_{-t_iu} \cap \Omega_1 \subset [h]q_iNa_{-t_iu}V \cap \Omega_1.
\]
Since \( V = V^{-1} \), we get \( [h]q_iNa_{-t_iu} \cap \Omega_1 V \neq \emptyset \). Therefore \( xN \cap \Omega \neq \emptyset \), as desired. \( \Box \)

This immediately implies:

**Corollary 4.4.** If \( [g] \in \Omega_0 \) is \( u \)-periodic for some \( u \in \text{int} \mathcal{L} \), then \( [g]N = \mathcal{E}_0 \).

**Proof.** Since \( [g](\exp ku) = [g]m_k^0 \) for any integer \( k \) and \( M \) is compact, we have \( g^+ \in \Lambda_u \). Therefore the claim follows from Proposition 4.3. \( \Box \)

We may now conclude our main theorem in its fullest form:

**Theorem 4.5.** Let \( [g] \in \mathcal{E}_0 \). The following are equivalent:

1. \( g^+ \in \Lambda \) is a horospherical limit point;
2. \( [g]N \) is dense in \( \mathcal{E}_0 \);
3. \( [g]NM \) is dense in \( \mathcal{E} \).

**Proof.** The implication (2) \( \Rightarrow \) (3) is trivial and (3) \( \Rightarrow \) (1) was shown in Theorem 3.1. Hence let us prove (1) \( \Rightarrow \) (2).

Let \( x = [g] \in \mathcal{E}_0 \). Suppose that \( g^+ \in \Lambda_u \). Fix any \( u \in \lambda(\Gamma) \cap \text{int} \mathcal{L}_\Gamma \). By Propositions 3.6 and 3.5, \( xN \) contains a \( u \)-periodic point, say, \( x_0 \). Hence by Corollary 4.4, \( xN \supset \overline{x_0N} \supset \Omega_0N = \mathcal{E}_0 \). This proves (1) \( \Rightarrow \) (2). \( \Box \)
5. Conical limit points, Minimality and Jordan projection

A point $\xi \in F$ is called a conical limit point of $\Gamma$ if there exists a sequence $u_j \to \infty$ in $a^+$ such that for some (and hence every) $g \in G$ with $\xi = gP$

\[
\limsup_{j \to \infty} \Gamma_ga_{u_j} \neq \emptyset.
\]

A conical limit point of $\Gamma$ is indeed contained in $\Lambda$. We consider the following restricted notion:

**Definition 5.1.** We call $\xi \in F$ a strongly conical limit point of $\Gamma$ if there exists a closed cone $C \subset \text{int} L \cup \{0\}$ and a sequence $u_j \to \infty$ in $C$ such that for some (and hence every) $g \in G$ with $\xi = gP$,

\[
\limsup_{j \to \infty} \Gamma_ga_{u_j} \neq \emptyset.
\]

**Lemma 5.2.** Any strongly conical limit point of $\Gamma$ is horospherical.

**Proof.** Suppose that $\xi = gP$ is strongly conical, that is, there exist $\gamma_j \in \Gamma$ and $u_j \to \infty$ in some closed cone $C \subset \text{int} L \cup \{0\}$ such that $\gamma_jgau_j$ converges to some $h \in G$. Let $C'$ be a closed cone contained in $\text{int} L \cup \{0\}$ whose interior contains $C \setminus \{0\}$.

Then $\gamma_j^{-1} = gau_jq_j^{-1}h^{-1}$ and

\[
\beta_{gP}(e, \gamma_j^{-1}) = \beta_P(g^{-1}, au_jq_j^{-1}h^{-1}) = \beta_P(g^{-1}, q_j^{-1}h^{-1}) + \beta_P(e, au_j).
\]

Since $\beta_P(e, au_j) = u_j$ and $q_j^{-1}h^{-1}$ are uniformly bounded, the sequence

\[
\beta_{gP}(e, \gamma_j^{-1}) - u_j
\]

is uniformly bounded. Since $u_j \in C$ and $C \subset \text{int} C' \cup \{0\}$, it follows that

\[
\beta_{gP}(e, \gamma_j^{-1}) \in C'
\]

for all sufficiently large $j$. This proves that $\xi \in \Lambda_h$.

**Corollary 5.3.** For any $g \in G$ with strongly conical $g^+ \in F$, we have $|g|NM = \mathcal{E}$.

**Directionally conical limit points.** If $v \in \text{int} L$, then clearly $\Lambda_v$ is contained in the horospherical limit set of $\Gamma$, and hence any $NM$-orbit based at a point of $\Lambda_v$ is dense in $\mathcal{E}$. On the other hand, we would like to show in this section that the existence of a point in $\Lambda_v$ for $v \in \partial L_T$ implies the existence of a nondense $NM$-orbit in $\mathcal{E}$.

The flow $\exp(\mathbb{R}u)$ is said to be topologically transitive on $\Omega/M = \{\Gamma gM : g^\pm \in \Lambda\}$ if, for any open subsets $O_1, O_2$ intersecting $\Omega/M$, there exists a sequence $t_n \to +\infty$ such that $O_1 \cap O_2a_{t_n} \neq \emptyset$.

We make the following simple observation:

**Lemma 5.4.** For $g \in \Omega$, $gNM \supset \Omega$ if and only if $gw_0N^+M \supset \Omega$.
\textbf{Proof.} We have $N^+ = w_0NW_0^{-1}$. Note that $[g] \in \Omega$ if and only if $gw_0 \in \Omega$, since $(gw_0)^+ = g^+$. So $\Omega w_0 = \Omega$. Hence $gNM$ is dense in $\Omega$ if and only if $gw_0N^+Mw_0^{-1}$ is dense in $\Omega$ if and only if $[g]w_0N^+M$ is dense in $\Omega w_0 = \Omega$. \hfill \square

Since the opposition involution preserves $L$ and $\lambda(g^{-1}) = i\lambda(g)$ for any loxodromic element, it follows that $\lambda(\gamma) \in \partial L$ if and only if $\lambda(\gamma^{-1}) \in \partial L$.

\textbf{Proposition 5.5.}

1. If $\Lambda = \Lambda_h$, then $\exp(\mathbb{R}v)$ is topologically transitive on $\Omega/M$ for any $v \in \text{int} \ a^+$ such that $\Lambda_v \neq \emptyset$.

2. For any loxodromic element $\gamma \in \Gamma$ with $y_\gamma, y_{\gamma^{-1}} \in \Lambda_h$, the flow $\exp(\mathbb{R}\lambda(\gamma))$ is topologically transitive on $\Omega/M$.

\textbf{Proof.} Assume that $\Lambda = \Lambda_h$; so the $NM$-action on $E$ is minimal. Suppose that $\Lambda_v \neq \emptyset$ for some $v \in \text{int} \ a^+$. We claim that for any $O_1, O_2$ be two right $M$-invariant open subsets intersecting $\Omega$, $O_1 \exp(t_i v) \cap O_2 \neq \emptyset$ for some sequence $t_i \to +\infty$. Choose $x = [g] \in \Omega$ so that $g^+ \in \Lambda_v$. Then there exists $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i} v$ converges to some $g_0$. Note that $x_0 := [g_0] \in \Omega$. So write $\gamma_i g a_{t_i} v = g_0 h_i$ with $h_i \to e$. By the $NM$-minimality assumption, $xNM$ intersects every open subset of $\Omega$. Since $v \in \text{int} \ a^+$ and hence $a_{-t_i} v a_{t_i} v \to e$ as $t \to +\infty$, we may assume without loss of generality that $x \in O_1$. Choose an open neighborhood $U$ of $e$ in $G$ so that $O_1 \supset xUM$. Note that there exists a sequence $T_i \to \infty$ as $i \to \infty$ such that for all $i$,

$$xUM a_{t_i} v \supset x a_{t_i} v a_{-t_i} v N_+^+ M a_{t_i} v \supset x_0 h_i N_T^+.$$ 

So $O_1 a_{t_i} v \supset x_0 h_i N_T^+ v$.

Choose an open neighborhood $V$ of $e$ in $G$ and some open subset $O_2$ intersecting $\Omega$ so that $O_2 \supset O_2' V$. Since $x_0 N^+ M$ is dense in $\Omega$, $x_0 n \in O_2'$ for some $n \in N^+$. Hence $x_0 h_i n = x_0 n (h_i^{-1} n) \in O_2' V \subset O_2$ for all $i$ large enough so that $n^{-1} h_i n \in V$. Therefore for all $i$ such that $n \in N_T^+$, we get $x_0 h_i n \in O_1 a_{t_i} v \cap O_2 \neq \emptyset$.

This proves the first claim.

Now suppose that $\gamma \in \Gamma$ is a loxodromic element with $y_\gamma, y_{\gamma^{-1}} \in \Lambda_h$. Write $\gamma = g m a_{-\sigma_\gamma} g^{-1}$ for some $g \in G$ and $m \in M$. Since $y_\gamma = g^+$ and $y_{\gamma^{-1}} = gw_0^+$, we have each $[g]NM$ and $[g]w_0 N M$ contains $\Omega$ in its closure. Now in the notation of the proof of the first claim, note that $x_0 = [g_0] \in [g] M$ since $[g] \exp(\mathbb{R}v) M$ is closed. Therefore each $x_0 N^+ M$ and $x_0 N^+ M$ contains $\Omega$. Based on this, the same argument as above shows the topological transitivity of $\exp \mathbb{R}v$, which finishes the proof since $v = \lambda(\gamma)$. \hfill \square

Together with the result of Dang and Glouriex [6, Proposition 4.7] which say that $\exp(\mathbb{R}u)$ is not topologically transitive on $\Omega/M$ for any $u \in \partial L \cap \text{int} \ a^+$, Proposition 5.5 implies the following:
Corollary 5.6.

(1) If $\Lambda_v \neq \emptyset$ for some $v \in \partial L \cap \text{int} a^+$, then $\Lambda \neq \Lambda_h$.

(2) If $\lambda(\gamma) \in \partial L$ for some loxodromic element $\gamma \in \Gamma$, then $\{y_\gamma, y_{\gamma^{-1}}\} \not\subset \Lambda_h$.

Hence, if $\Lambda = \Lambda_h$, then $\lambda(\Gamma) \subset \text{int} L$.

REFERENCES


