ON DENSENESS OF HOROSPHERES IN HIGHER RANK HOMOGENEOUS SPACES

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Abstract. Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. Let N denote a maximal horospherical subgroup of G, and $P = MAN$ the minimal parabolic subgroup which is the normalizer of N. Let $\mathcal{E}$ denote the unique P-minimal subset of $\Gamma\backslash G$ and let $\mathcal{E}_0$ be a P°-minimal subset. We consider a notion of a horospherical limit point in the Furstenberg boundary G/P and show that the following are equivalent for any $[g] \in \mathcal{E}_0$,

(1) $gP \in G/P$ is a horospherical limit point;
(2) $[g]NM$ is dense in $\mathcal{E}$;
(3) $[g]N$ is dense in $\mathcal{E}_0$.

The equivalence of (1) and (2) is due to Dal'bo in the rank one case. We also show that unlike convex cocompact groups of rank one Lie groups, the NM-minimality of $\mathcal{E}$ does not hold in a general Anosov homogeneous space.

1. Introduction

Let G be a connected semisimple real algebraic group. Let (X, d) denote the associated Riemannian symmetric space. Let $P = MAN$ be a minimal parabolic subgroup of G with fixed Langlands decomposition, where A is a maximal real split torus of G, M the maximal compact subgroup of P commuting with A and N the unipotent radical of P. Note that N is a maximal horospherical subgroup of G, which is unique up to conjugations.

Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \log A$ so that $\log N$ consists of positive root subspaces, and we set $A^+ = \exp \mathfrak{a}^+$. This means that N is a contracting horospherical subgroup in the sense that for any a in the interior of A^+,

$$N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

Let Γ be a *Zariski dense* discrete subgroup of G. In this paper, we are interested in the topological behavior of the action of the horospherical subgroup N on $\Gamma\backslash G$ via the right translations. When $\Gamma < G$ is a cocompact lattice, every N-orbit is dense in $\Gamma\backslash G$, i.e., the N-action on $\Gamma\backslash G$ is minimal. This is due to Hedlund [11] for $G = \mathrm{PSL}_2(\mathbb{R})$ and to Veech [19] in general. Dani gave a full classification of possible orbit closures of N-action for any lattice $\Gamma < G$ [6].

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For a general discrete subgroup $\Gamma < G$, the quotient space $\Gamma\backslash G$ does not necessarily admit a dense N-orbit, even a dense NM-orbit, for instance in the case where Γ does not have a full limit set. Let $\mathcal{F}$ denote the Furstenberg boundary G/P. We denote by $\Lambda = \Lambda_\Gamma$ the limit set of Γ,

$$\Lambda = \{\lim_{i\to\infty} \gamma_i(o) \in \mathcal{F} : \gamma_i \in \Gamma\}$$

where $o \in X$ and the convergence is understood as in Definition 2.2. This definition is independent of the choice of $o \in X$. The limit set Λ is known to be the unique Γ-minimal subset of $\mathcal{F}$ (see [1, 9, 14]). Thus the set

$$\mathcal{E} = \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

is the unique P-minimal subset of $\Gamma\backslash G$. For a given point $[g] \in \mathcal{E}$, the topological behavior of the horospherical orbit $[g]N$ (or of $[g]NM$) is closely related to the ways in which the orbit $\Gamma(o)$ approaches gP along its limit cone. The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of Γ is defined as the smallest closed cone of $\mathfrak{a}^+$ containing the Jordan projection $\lambda(\Gamma)$. It is a convex cone with non-empty interior: $\operatorname{int}\mathcal{L} \neq \emptyset$ [1]. If $\operatorname{rank} G = 1$, then $\mathcal{L} = \mathfrak{a}^+$. In higher ranks, the limit cone of Γ depends more subtly on Γ.

Horospherical limit points. Recall that in the rank one case, a horoball in X based at $\xi \in \mathcal{F}$ is a subset of the form $gN(\exp \mathfrak{a}^+)(o)$ where $g \in G$ is such that $\xi = gP$ [5]. Our generalization to higher rank of the notion of a horospherical limit point involves the limit cone of Γ. By a Γ-tight horoball based at $\xi \in \mathcal{F}$, we mean a subset of the form $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$ where $g \in G$ is such that $\xi = gP$ and $\mathcal{C}$ is a closed cone contained in $\operatorname{int}\mathcal{L} \cup \{0\}$. For $T > 0$, we write

$$\mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o$$

where $\mathcal{C}_T = \{u \in \mathcal{C} : \|u\| < T\}$ for a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}$.

Definition 1.1. We call a limit point $\xi \in \Lambda$ a horospherical limit point of Γ if one of the following equivalent conditions holds:

- there exists a Γ-tight horoball $\mathcal{H}_\xi$ based at ξ such that for any $T > 1$, $\mathcal{H}_\xi(T)$ contains some point of $\Gamma(o)$;
- there exist a closed cone $\mathcal{C} \subset \operatorname{int}\mathcal{L} \cup \{0\}$ and a sequence $\gamma_j \in \Gamma$ satisfying that $\beta_\xi(o, \gamma_j o) \in \mathcal{C}$ for all $j \geq 1$ and $\beta_\xi(o, \gamma_j o) \to \infty$ as $j \to \infty$, where β denotes the $\mathfrak{a}$-valued Busemann map (Definition 2.3).

See Lemma 3.3 for the equivalence of the above two conditions. We denote by

$$\Lambda_h \subset \Lambda$$

the set of all horospherical limit points of Γ. The attracting fixed point y_γ of a loxodromic element $\gamma \in \Gamma$ whose Jordan projection $\lambda(\gamma)$ belongs to $\operatorname{int}\mathcal{L}$ is always a horospherical limit point (Lemma 3.5). Moreover, for any $u \in \operatorname{int}\mathcal{L}$, any u-directional radial limit point ξ (i.e, $\xi = gP$ for some $g \in G$ such that $\limsup_{t\to\infty} \Gamma g \exp(tu) \neq \emptyset$) is also a horospherical limit point (Lemma 5.3).

Remarks 1.2.

(1) There exists a notion of horospherical limit points in the geometric boundary associated to a symmetric space, see [10]. When $\operatorname{rank} G \geq 2$, this notion and the one considered here are different.
(2) Unlike the rank one case, a sequence $\gamma_i(o) \in \mathcal{H}_\xi(T_i)$, with $T_i \to \infty$, does not necessarily *converge* to ξ for a Γ-tight horoball $\mathcal{H}_\xi$ based at ξ. It is hence plausible that a general discrete group Γ would support a horospherical limit point outside of its limit set.

Denseness of horospheres. The following theorem generalizes Dal'bo's theorem [5] to discrete subgroups in higher rank semisimple Lie groups:

Theorem 1.3. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $[g] \in \mathcal{E}$, the following are equivalent:*

(1) $gP \in \Lambda_h$;
(2) $[g]NM$ is dense in $\mathcal{E}$.

Remarks 1.4. Conze and Guivarc'h considered the notion of a horospherical limit point for Zariski dense discrete subgroups Γ of $\mathrm{SL}_d(\mathbb{R})$ using the description of $\mathrm{SL}_d(\mathbb{R})/P$ as the full flag variety and the standard linear action of Γ on $\mathbb{R}^d$ [4]. By duality, this notion coincides with ours and hence the special case of Theorem 1.3 for $G = \mathrm{SL}_d(\mathbb{R})$ also follows from [4, Theorem 4.2].

In order to extend Theorem 1.3 to N-orbits, we fix a P°-minimal subset $\mathcal{E}_0$ of $\Gamma\backslash G$ where P° denotes the identity component of P. Clearly, $\mathcal{E}_0 \subset \mathcal{E}$. Since $P = P^\circ M$, any P°-minimal subset is a translate of $\mathcal{E}_0$ by an element of the finite group $M^\circ\backslash M$, where M° is the identity component of M. Denote by $\mathfrak{D}_\Gamma = \{\mathcal{E}_0, ..., \mathcal{E}_p\}$ the finite collection of all P°-minimal sets in $\mathcal{E}$. In order to understand N-orbit closures it is hence sufficient to restrict to $\mathcal{E}_0$.

The following is a refinement of Theorem 1.3:

Theorem 1.5. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $[g] \in \mathcal{E}_0$, the following are equivalent:*

(1) $gP \in \Lambda_h$;
(2) $[g]N$ is dense in $\mathcal{E}_0$.

Remark 1.6. We may consider horospherical limit points outside the context of Λ. In this case our proofs of Theorems 1.3 and 1.5 show that if $gP \in \mathcal{F}$ is a horospherical limit point, then the closures of $[g]MN$ and $[g]N$ contain $\mathcal{E}$ and $\mathcal{E}_i$, for some $\mathcal{E}_i \in \mathfrak{D}_\Gamma$, respectively.

For $G = \mathrm{SO}^\circ(n, 1)$, $n \geq 2$, Theorem 1.5 was proved in [16]. When G has rank one and $\Gamma < G$ is convex cocompact, every limit point is horospherical and Winter's mixing theorem [20] implies the N-minimality of $\mathcal{E}_0$.

Directional horospherical limit points. We also consider the following seemingly much stronger notion:

Definition 1.7. For $u \in \mathfrak{a}^+$, a point $\xi \in \mathcal{F}$ is called u-horospherical if there exists a sequence $\gamma_j \in \Gamma$ such that $\sup_j \|\beta_\xi(o, \gamma_j o) - \mathbb{R}_+ u\| < \infty$ and $\beta_\xi(o, \gamma_j o) \to \infty$ as $j \to \infty$.

Denote by $\Lambda_h(u)$ the set of u-horospherical limit points. Surprisingly, it turns out that every horospherical limit point is u-horospherical *for all* $u \in \operatorname{int} \mathcal{L}$:

Theorem 1.8. *For all* $u \in \operatorname{int} \mathcal{L}$*, we have*

$$\Lambda_h = \Lambda_h(u).$$

Existence of non-dense horospheres. A finitely generated subgroup $\Gamma < G$ is called an Anosov subgroup (with respect to P) if there exists $C > 0$ such that for all $\gamma \in \Gamma$, $\alpha(\mu(\gamma)) \geq C|\gamma| - C$ for all simple roots α of $(\mathfrak{g}, \mathfrak{a}^+)$, where $\mu(\gamma) \in \mathfrak{a}^+$ denotes the Cartan projection of γ and $|\gamma|$ is the word length of γ with respect to a fixed finite generating set of Γ.

For Zariski dense Anosov subgroups of G, almost all NM-orbits are dense in $\mathcal{E}$ and almost all N-orbits are dense in $\mathcal{E}_0$ with respect to any Patterson-Sullivan measure on Λ ([14], [15]). In particular, the set of all horospherical limit points has full Patterson-Sullivan measures.

On the other hand, as Anosov subgroups are regarded as higher rank generalizations of convex cocompact subgroups, it is a natural question whether the minimality of the NM-action persists in the higher rank setting. It turns out that it is not the case. Our example is based on Thurston's theorem [18, Theorem 10.7] together with the following observation on the implication of the existence of a Jordan projection of an element of Γ lying in the boundary $\partial\mathcal{L}$ of the limit cone.

Proposition 1.9. *Let* $\Gamma < G$ *be a Zariski dense discrete subgroup. For any loxodromic element* $\gamma \in \Gamma$*, we have*

$$\lambda(\gamma) \in \operatorname{int} \mathcal{L} \quad \text{if and only if} \quad \{y_\gamma, y_{\gamma^{-1}}\} \subset \Lambda_h$$

where y_γ *and* $y_{\gamma^{-1}}$ *denote the attracting fixed points of* γ *and* γ^{-1} *respectively.*

In particular, if $\lambda(\Gamma) \cap \partial\mathcal{L} \neq \emptyset$*, then* $\Lambda \neq \Lambda_h$ *and hence there exists a non-dense* NM*-orbit in* $\mathcal{E}$*.*

Thurston's work [18] provides many examples of Anosov subgroups satisfying that $\lambda(\Gamma) \cap \partial\mathcal{L} \neq \emptyset$. To describe them, let Σ be a a torsion-free cocompact lattice of $\mathrm{PSL}_2(\mathbb{R})$ and let $\pi : \Sigma \to \mathrm{PSL}_2(\mathbb{R})$ be a discrete faithful representation. Let $0 < d_-(\pi) \leq d_+(\pi) < \infty$ be the minimal and maximal geodesic stretching constants:

$$d_+(\pi) = \sup_{\sigma \in \Sigma - \{e\}} \frac{\ell(\pi(\sigma))}{\ell(\sigma)} \quad \text{and} \quad d_-(\pi) = \inf_{\sigma \in \Sigma - \{e\}} \frac{\ell(\pi(\sigma))}{\ell(\sigma)} \tag{1.1}$$

where $\ell(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\Sigma\backslash\mathbb{H}^2$ corresponding to σ and $\ell(\pi(\sigma))$ is defined similarly.

Consider the following self-joining subgroup

$$\Gamma_\pi := (\mathrm{id} \times \pi)(\Sigma) = \{(\sigma, \pi(\sigma)) : \sigma \in \Sigma\} < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}).$$

It is easy to see that Γ is an Anosov subgroup of $G = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Moreover when π is not a conjugate by a Möbius tranformation, Γ_π is Zariski dense in G (cf. [12, Lemma 4.1]). Identifying $\mathfrak{a} = \mathbb{R}^2$, the Jordan projection $\lambda(\gamma_\pi)$ of $\gamma_\pi = (\sigma, \pi(\sigma)) \in \Gamma_\pi$ is given by $(\ell(\sigma), \ell(\pi(\sigma))) \in \mathbb{R}^2$. Hence the limit cone $\mathcal{L}$ of Γ_π is given by

$$\mathcal{L} := \{(v_1, v_2) \in \mathbb{R}^2_{\geq 0} : d_-(\pi)v_1 \leq v_2 \leq d_+(\pi)v_1\}.$$

Thurston [18, Theorem 10.7] showed that $d_+(\pi)$ is realized by a simple closed geodesic of $\Sigma\backslash\mathbb{H}^2$ in *most of cases*, which hence provides infinitely many examples of Γ_π which satisfy $\lambda(\Gamma_\pi) \cap \partial\mathcal{L} \neq \emptyset$. Therefore Proposition 1.9 implies (in this case, we have $NM = N$):

Corollary 1.10. *There are infinitely many non-conjuagte Zariski dense Anosov subgroups $\Gamma_\pi < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ with non-dense NM-orbits in $\mathcal{E}$.*

We close the introduction by the following question (cf. [13],[17]):

Question 1.11. For a simple real algebraic group G with $\operatorname{rank} G \geq 2$, is every discrete subgroup $\Gamma < G$ with $\Lambda = \Lambda_h = \mathcal{F}$ necessarily a cocompact lattice in G?

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2. Preliminaries

Let G be a connected, semisimple real algebraic group. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$,

Choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set N to be the maximal contracting horospherical subgroup: for $a \in \operatorname{int} A^+$,

$$N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

We set $P = MAN$, which is the unique minimal parabolic subgroup of G, up to conjugation.

For $u \in \mathfrak{a}$, we write $a_u = \exp u \in A$. We denote by $\|\cdot\|$ the norm on $\mathfrak{g}$ induced by the Killing form. Consider the Riemannian symmetric space $X := G/K$ with the metric induced from the norm $\|\cdot\|$ on $\mathfrak{g}$ and $o = K \in X$.

Let $\mathcal{F} = G/P$ denote the Furstenberg boundary. Since K acts transitively on $\mathcal{F}$ and $K \cap P = M$, we may identify $\mathcal{F} = K/M$. We denote by $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$.

Denote by $w_0 \in K$ the unique element in the Weyl group such that $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$; it is the longest Weyl element. We then have $\check{P} := w_0 P w_0^{-1}$ is an opposite parabolic subgroup of G, with $\check{N}$ its unipotent radical. The map $\mathrm{i} = -\operatorname{Ad}_{w_0} : \mathfrak{a}^+ \to \mathfrak{a}^+$ is called the opposition involution.

For $g \in G$, we consider the following visual maps

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := gw_0P \in \mathcal{F}.$$

Then $\mathcal{F}^{(2)} = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}$.

Any element $g \in G$ can be uniquely decomposed as the commuting product g_h, g_e, g_u, where g_h, g_e, and g_u are hyperbolic, elliptic and unipotent elements respectively. The Jordan projection of g is defined as the element $\lambda(g) \in \mathfrak{a}^+$ satisfying $g_h = \varphi \exp \lambda(g) \varphi^{-1}$ for some $\varphi \in G$.

An element $g \in G$ is called loxodromic if $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$; in this case, g_u is necessarily trivial. For a loxodromic element $g \in G$, the point $\varphi^+ \in \mathcal{F}$ is called the attracting fixed point of g, which we denote by y_g. For any loxodromic element $g \in G$ and $\xi \in \mathcal{F}$ with $(\xi, y_{g^{-1}}) \in \mathcal{F}^{(2)}$, we have $\lim_{k\to\infty} g^k \xi = y_g$ and the convergence is uniform on compact subsets.

Note that for any loxodromic element $g \in G$,

$$\lambda(g^{-1}) = \mathrm{i}\, \lambda(g).$$

Let $\Gamma < G$ be a Zariski dense discrete subgroup of G. The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of Γ is the smallest closed cone of $\mathfrak{a}^+$ containing $\lambda(\Gamma)$. It is a convex cone with non-empty interior [1].

We will use the following simple lemma.

Lemma 2.1. *For any $v \in \lambda(\Gamma)$ and $\zeta \in \mathcal{F}$, there exists a loxodromic element $\gamma \in \Gamma$ with $\lambda(\gamma) = v$ and a neighborhood U of ζ in $\mathcal{F}$ such that $\{y_\gamma\} \times U$ is a relatively compact subset of $\mathcal{F}^{(2)}$ and as $k \to \infty$,*

$$\gamma^{-k}\zeta \to y_{\gamma^{-1}} \quad \textit{uniformly on } U.$$

Proof. Let $\zeta \in \mathcal{F}$. Choose $\gamma_1 \in \Gamma$ such that $\lambda(\gamma_1) = v$. Since the set of all loxodromic elements of Γ is Zariski dense in G [2] and $\mathcal{F}^{(2)}$ is Zariski open in $\mathcal{F} \times \mathcal{F}$, there exists $\gamma_2 \in \Gamma$ such that $(\zeta, \gamma_2 y_{\gamma_1}) \in \mathcal{F}^{(2)}$. Let $\gamma = \gamma_2 \gamma_1 \gamma_2^{-1}$, so that $y_\gamma = \gamma_2 y_{\gamma_1}$. It now suffices to take any neighborhood U of ζ such that $U \times \{\gamma_2 y_{\gamma_1}\}$ is a relatively compact subset of $\mathcal{F}^{(2)}$. □

Convergence of a sequence in X to $\mathcal{F}$. By the Cartan decomposition $G = KA^+K$, for $g \in G$, we may write

$$g = \kappa_1(g) \exp(\mu(g)) \kappa_2(g) \in KA^+K$$

where $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, is uniquely determined, and $\kappa_1(g), \kappa_2(g) \in K$. If $\mu(g) \in \operatorname{int} \mathfrak{a}^+$, then $[\kappa_1(g)] \in K/M = \mathcal{F}$ is uniquely determined.

Let Π be the set of simple roots for $(\mathfrak{g}, \mathfrak{a})$. For a sequence $g_i \to G$, we say $g_i \to \infty$ regularly if $\alpha(\mu(g_i)) \to \infty$ for all $\alpha \in \Pi$. Note that if $g_i \to \infty$ regularly, then for all sufficiently large i, $\mu(g_i) \in \operatorname{int} \mathfrak{a}^+$ and hence $[\kappa_1(g_i)]$ is well-defined.

Definition 2.2. A sequence $p_i \in X$ is said to converge to $\xi \in \mathcal{F}$ if there exists $g_i \to \infty$ regularly in G with $p_i = g_i(o)$ and $\lim_{i\to\infty}[\kappa_1(g_i)] = \xi$.

P°-minimal subsets. We denote by $\Lambda \subset \mathcal{F}$ the limit set of Γ, which is defined as

$$\Lambda = \{\lim \gamma_i(o) : \gamma_i \in \Gamma\}. \tag{2.1}$$

For a non-Zariski dense subgroup, Λ may be an empty set. For $\Gamma < G$ Zariski dense, this is the unique Γ-minimal subset of $\mathcal{F}$ ([1], [14]).

It follows that the following set $\mathcal{E}$ is the unique P-minimal subset of $\Gamma\backslash G$:

$$\mathcal{E} = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}.$$

Let P° denote the identity component of P. Then $\mathcal{E}$ is a disjoint union of at most $[P : P^\circ]$-number of P°-minimal subsets. We fix one P°-minimal subset $\mathcal{E}_0$ once and for all. Note that any P°-minimal subset is then of the form $\mathcal{E}_0 m$ for some $m \in M$. We set

$$\Omega := \{[g] \in \Gamma\backslash G : g^+, g^- \in \Lambda\} \quad \text{and} \quad \Omega_0 := \Omega \cap \mathcal{E}_0. \tag{2.2}$$

Busemann map. The Iwasawa cocycle $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$ with $\xi = [k]$ for $k \in K$, $\exp \sigma(g, \xi)$ is the A-component of gk in the KAN decomposition, that is,

$$gk \in K \exp(\sigma(g, \xi))N.$$

The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times X \times X \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(ho, go) := \sigma(h^{-1}, \xi) - \sigma(g^{-1}, \xi).$$

We note that for any $g \in G$, $\xi \in \mathcal{F}$, and $x, y, z \in X$,

$$\beta_\xi(x, y) = \beta_{g\xi}(gx, gy), \quad \text{and} \quad \beta_\xi(x, y) = \beta_\xi(x, z) + \beta_\xi(z, y). \tag{2.3}$$

In particular, $\beta_\xi(o, go) \in \mathfrak{a}$ is defined by

$$g^{-1}k_\xi \in K \exp(-\beta_\xi(o, go))N, \tag{2.4}$$

and hence $\beta_P(o, a_u o) = u$ for any $u \in \mathfrak{a}$. For $h, g \in G$, we set $\beta_\xi(h, g) := \beta_\xi(ho, go)$.

Shadows. For $q \in X$ and $r > 0$, we set $B(q,r) = \{x \in X : d(x,q) \le r\}$. For $p = g(o) \in X$, the shadow of the ball $B(q,r)$ viewed from p is defined as

$$O_r(p,q) := \{(gk)^+ \in \mathcal{F} : k \in K,\ gk \operatorname{int} A^+ o \cap B(q,r) \neq \emptyset\}.$$

Similarly, for $\xi \in \mathcal{F}$, the shadow of the ball $B(q,r)$ as viewed from ξ is

$$O_r(\xi,q) := \{h^+ \in \mathcal{F} : h \in G \text{ satisfies } h^- = \xi,\ ho \in B(q,r)\}.$$

Lemma 2.3. [14, Lemma 5.6 and 5.7]

(1) There exists $\kappa > 0$ such that for any $g \in G$ and $r > 0$,

$$\sup_{\xi \in O_r(g(o),o)} \|\beta_\xi(g(o),o) - \mu(g^{-1})\| \le \kappa r.$$

(2) If a sequence $p_i \in X$ converges to $\xi \in \mathcal{F}$, then for any $0 < \varepsilon < r$, we have

$$O_{r-\varepsilon}(p_i,o) \subset O_r(\xi,o) \subset O_{r+\varepsilon}(p_i,o)$$

for all sufficiently large i.

3. Horospherical limit points

Let $\Gamma < G$ be a Zariski dense discrete subgroup. A Γ-tight horoball based at $\xi \in \mathcal{F}$ is a subset of the form $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$ where $g \in G$ is such that $\xi = gP$ and $\mathcal{C}$ is a closed cone contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. For $T > 0$, we write $\mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o$. We recall the definition from the introduction:

Definition 3.1. We say that $\xi \in \mathcal{F}$ is a horospherical limit point of Γ if there exists a Γ-tight horoball $\mathcal{H}_\xi$ based at ξ such that $\mathcal{H}_\xi(T) \cap \Gamma(o) \neq \emptyset$ for all $T > 1$.

In this section we provide a mostly self-contained proof of the following theorem:

Theorem 3.2. *Let $[g] \in \mathcal{E}$. The following are equivalent:*

(1) $g^+ = gP \in \Lambda$ is a horospherical limit point;

(2) $[g]NM$ is dense in $\mathcal{E}$.

The main external ingredient in our proof is the density of the group generated by the Jordan projection $\lambda(\Gamma)$, due to Benoist [2], that is,

$$\mathfrak{a} = \overline{\langle \lambda(\Gamma) \rangle}$$

for every Zariski dense discrete subgroup $\Gamma < G$. In fact, for every cone $\mathcal{C} \subset \mathcal{L}$ with non-empty interior, there exists a Zariski dense subgroup $\Gamma' < \Gamma$ with $\mathcal{L}_{\Gamma'} \subset \mathcal{C}$ (see [1]); therefore we have

$$\mathfrak{a} = \overline{\langle \lambda(\Gamma) \cap \operatorname{int} \mathcal{L} \rangle}.$$

It is convenient to use a characterization of horospherical limit points in terms of the Busemann function.

Lemma 3.3. *For $\xi \in \Lambda$, we have $\xi \in \Lambda_h$ if and only if there exists a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ and a sequence $\gamma_j \in \Gamma$ satisfying*

$$\beta_\xi(o, \gamma_j o) \to \infty \quad \textit{and} \quad \beta_\xi(o, \gamma_j o) \in \mathcal{C} \textit{ for all large } j \geq 1. \tag{3.1}$$

Proof. Let $\xi = gP \in \Lambda_h$ be as defined in Definition 3.1. Then there exists $\gamma_j = gpn_j a_{u_j} k_j \in \Gamma$ for some $p \in P$, $n_j \in N$, $k_j \in K$ and $u_j \to \infty$ in some closed cone $\mathcal{C}$ contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. Fix some closed cone $\mathcal{C}' \subset \operatorname{int} \mathcal{L} \cup \{0\}$ whose interior contains $\mathcal{C}$. Note that

$$\begin{aligned}
\beta_\xi(o, \gamma_j o) &= \beta_{gP}(e, g) + \beta_{gP}(g, gpn_j a_{u_j}) \\
&= \beta_P(g^{-1}, e) + \beta_P(e, p) + \beta_P(e, n_j) + \beta_P(e, a_{u_j}) \\
&= \beta_P(g^{-1}, p) + u_j.
\end{aligned}$$

Therefore the sequence $\beta_\xi(o, \gamma_j) - u_j$ is uniformly bounded. Since $u_j \in \mathcal{C}$, $\beta_\xi(o, \gamma_j o) \in \mathcal{C}'$ for all large j. Therefore (3.1) holds. For the other direction, let γ_j and $\mathcal{C}$ satisfy (3.1) for $\xi = gP$ for $g \in G$. Since $G = gNAK$, we may write $\gamma_j = gn_j a_{u_j} k_j$ for some $n_j \in N, u_j \in \mathfrak{a}$ and $k_j \in K$. By a similar computation as above, the sequence $\beta_\xi(o, \gamma_j o) - u_j$ is uniformly bounded. It follows that $u_j \in \mathcal{C}'$ for all large j and $u_j \to \infty$. Therefore for any $T > 1$, there exists $j > 1$ such that $\gamma_j(o) \in gN \exp(\mathcal{C}' - \mathcal{C}'_T)(o)$. This proves $\xi \in \Lambda_h$. ☐

We note that condition (3.1) is independent of the choice of basepoint o. Indeed, for any $g \in G$ and $\xi \in \mathcal{F}$ and for all $\gamma \in \Gamma$ we have

$$\beta_\xi(o, \gamma o) = \beta_\xi(o, go) + \beta_\xi(go, \gamma go) + \beta_\xi(\gamma go, \gamma o),$$

and hence

$$\begin{aligned}
\|\beta_\xi(o, \gamma o) - \beta_\xi(go, \gamma go)\| &= \|\beta_\xi(o, go) + \beta_\xi(\gamma go, \gamma o)\| \\
&= \|\beta_\xi(o, go) - \beta_{\gamma^{-1}\xi}(o, go)\| \\
&\leq 2 \cdot \max_{\eta \in \mathcal{F}} \|\beta_\eta(o, go)\|.
\end{aligned}$$

Since this bound is independent of $\gamma \in \Gamma$, condition (3.1) implies that for any $p = go \in X$,

$$\beta_\xi(p, \gamma_j p) \to \infty \quad \text{and} \quad \beta_\xi(p, \gamma_j p) \in \mathcal{C} \text{ for all large } j. \tag{3.2}$$

Let us now consider the following seemingly stronger condition for a limit point being horospherical:

Definition 3.4. For $u \in \mathfrak{a}^+$, a point $\xi \in \mathcal{F}$ is called a u-horospherical limit point if for some $p \in X$ (and hence for any $p \in X$), there exists a constant $R > 0$ and a sequence $\gamma_j \in \Gamma$ satisfying

$$\beta_\xi(p, \gamma_j p) \to \infty \quad \text{and} \quad \|\beta_\xi(p, \gamma_j p) - \mathbb{R}_+ u\| < R \quad \text{for all } j.$$

We denote the set of u-horospherical limit points by $\Lambda_h(u)$.

By G-invariance of the Busemann map, the set of horospherical (resp. u-horospherical) limit points is Γ-invariant. Therefore for $x = [g] \in \Gamma\backslash G$, we may say $x^+ := \Gamma g P$ horospherical (resp. u-horospherical) if g^+ is.

For $u \in \mathfrak{a}$, we call $x \in \Gamma\backslash G$ a u-periodic point if $xa_u = xm_0$ for some $m_0 \in M$; note that $xa_{\mathbb{R}u}M_0$ is then compact. Note that for $u \in \operatorname{int}\mathfrak{a}^+$, the existence of a u-periodic point is equivalent to the condition that $u \in \lambda(\Gamma)$.

Lemma 3.5. *Let $u \in \mathfrak{a}^+$. If $x \in \Gamma\backslash G$ is u-periodic, then $x^+ \in \mathcal{F}$ is a u-horospherical limit point.*

Proof. Since x is u-periodic, there exist $g \in G$ with $x = [g]$ and $\gamma \in \Gamma$ such that $\gamma = ga_u m g^{-1}$ for some $m \in M$, and $y_\gamma = g^+ \in \Lambda$. Moreover, for any $k \geq 1$

$$\beta_{gP}(go, \gamma^k go) = \beta_P(o, a_u^k o) = ku.$$

This implies gP is u-horospherical. □

Proposition 3.6. *Let $x \in \Gamma\backslash G$. If x^+ is u-horospherical for some $u \in \lambda(\Gamma)$ then the closure $\overline{xN}$ contains a u-periodic point.*

Proof. Choose $g \in G$ so that $x = [g]$. We may assume without loss of generality that $g = k \in K$, since $kanN = kNa$, and a translate of a u-periodic point by an element of A is again a u-periodic point. Since $u \in \lambda(\Gamma)$, there exists a u-periodic point, say, $x_0 \in \Gamma\backslash G$. It suffices to show that

$$\overline{[k]N} \cap x_0 AM \neq \emptyset \tag{3.3}$$

as every point in x_0AM is u-periodic.

Since k^+ is u-horospherical and using (2.4), there exists $R > 0$ and sequences $\gamma_j \in \Gamma$, $u_j \to \infty$ in $\mathfrak{a}^+$ and $k_j \in K$ and $n_j \in N$ satisfying $\gamma_j^{-1}k = k_j a_{-u_j} n_j$ or

$$k_j = \gamma_j^{-1} k n_j^{-1} a_{u_j}, \tag{3.4}$$

with $\|\mathbb{R}_+ u - u_j\| < R$ for all j. Let $\ell_j \to \infty$ be a sequence of integers satisfying

$$\|\ell_j u - u_j\| < R + \|u\| \quad \text{ for all } j \geq 1. \tag{3.5}$$

By passing to a subsequence, we may assume without loss of generality that $\gamma_j^{-1}kP$ converges to some $\xi_0 \in \mathcal{F}$. Since $\check{N}P$ is Zariski open and Γ is Zariski dense, we may choose $g_0 \in G$ such that $x_0 = [g_0]$ and $g_0^{-1}\xi_0 \in \check{N}P$. Let $h_0 \in \check{N}$ be such that $\xi_0 = g_0 h_0 P$. Since $g_0\check{N}P$ is open and $\gamma_j^{-1}kP \to g_0h_0P$, we may assume that for all j, there exists $h_j \in \check{N}$ satisfying $g_0 h_j P = \gamma_j^{-1}kP = k_jP$ with $h_j \to h_0$. Let $p_j = a_{v_j} m_j \tilde{n}_j \in P = AMN$ be such that $g_0 h_j p_j = k_j$; since $h_j \to h_0$ and the product map $\check{N} \times P \to \check{N}P$ is a diffeomorphism, the sequence p_j, as well as $v_j \in \mathfrak{a}$, are bounded.

Therefore by (3.4), we get for all j,

$$\begin{aligned} g_0 &= k_j p_j^{-1} h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} a_{u_j} (\tilde{n}_j^{-1} m_j^{-1} a_{-v_j}) h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} (a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) a_{u_j} m_j^{-1} a_{-v_j} h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} (a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) m_j^{-1} (a_{u_j - v_j} h_j^{-1} a_{-u_j + v_j}) a_{u_j - v_j}. \end{aligned}$$

Since $h_j^{-1} \in \check{N}$ and $v_j \in \mathfrak{a}$ are uniformly bounded and since $u_j \to \infty$ within a bounded neighborhood of the ray $\mathbb{R}_+ u \in \operatorname{int} \mathfrak{a}^+$, we have

$$\tilde{h}_j = a_{u_j - v_j} h_j^{-1} a_{-u_j + v_j} \to e \quad \text{in } \check{N}.$$

By setting $n'_j = n_j^{-1}(a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) \in N$, we may now write

$$g_0 = \gamma_j^{-1} k n'_j m_j^{-1} \tilde{h}_j a_{u_j - v_j}.$$

Since x_0 is u-periodic, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0 = g_0 a_u m_0 g_0^{-1}$ for some $m_0 \in M$. Hence for all $j \geq 1$,

$$\gamma_0^{-\ell_j} = g_0 a_{-\ell_j u} m_0^{-\ell_j} g_0^{-1} = (\gamma_j^{-1} k n'_j m_j^{-1} \tilde{h}_j a_{u_j - v_j})(a_{-\ell_j u} m_0^{-\ell_j}) g_0^{-1}.$$

In other words,

$$\gamma_j^{-1} k n'_j = \gamma_0^{-\ell_j} g_0 m_0^{\ell_j} a_{-u_j + \ell_j u + v_j} \tilde{h}_j^{-1} m_j.$$

Since the sequence $-u_j + \ell_j u + v_j \in \mathfrak{a}$ is uniformly bounded by (3.5) and $\tilde{h}_j \to e$ in $\check{N}$, we conclude that the sequence $\Gamma k n'_j$ has an accumulation point in $\Gamma g_0 AM$. This proves (3.3). □

It turns out that a horospherical limit point is also u-horospherical for any $u \in \operatorname{int} \mathcal{L}$:

Proposition 3.7. *For each $u \in \operatorname{int} \mathcal{L}$, we have $\Lambda_h = \Lambda_h(u)$.*

Proof. Let $\xi \in \Lambda_h$. By definition, there is a sequence $\gamma_j \in \Gamma$ satisfying $v_j := \beta_\xi(e, \gamma_j) \to \infty$ with the sequence $\|v_j\|^{-1} v_j$ converging to some point $v_0 \in \operatorname{int} \mathcal{L}$. By passing to a subsequence, we may assume that $\gamma_j^{-1} \xi$ converges to some $\xi_0 \in \mathcal{F}$.

Let $u \in \operatorname{int} \mathcal{L}$. We claim that $\xi \in \Lambda_h(u)$. We first consider the case $u \notin \mathbb{R}_+ v_0$. Let $r := \operatorname{rank} G - 1 \geq 0$. Since $\cup_{\gamma \in \Gamma} \mathbb{R}_+ \lambda(\gamma)$ is dense in $\mathcal{L}$, there exist $w_1, \cdots, w_r \in \lambda(\Gamma)$ such that v_0 belongs to the interior of the convex cone spanned by $u, w_1, \cdots, w_r$, so that

$$v_0 = c_0 u + \sum_{\ell=1}^{r} c_\ell w_\ell$$

for some positive constants $c_0, \cdots, c_\ell$.

Since $\|v_j\|^{-1}v_j \to v_0$, we may assume, by passing to a subsequence, that for each $j \geq 1$, we have

$$\|v_j\|^{-1}v_j = c_{0,j}u + \sum_{\ell=1}^{r} c_{\ell,j}w_\ell \tag{3.6}$$

for some positive $c_{\ell,j}$, $\ell = 0, \cdots, r$. Note that for each $0 \leq \ell \leq r$, $c_{\ell,j} \to c_\ell$ as $j \to \infty$.

By Lemma 2.1, we can find a loxodromic element $g_1 \in \Gamma$ and a neighborhood U_1 of ξ_0 such that $\lambda(g_1^{-1}) = w_1$, $\{y_{g_1}\} \times U_1 \subset \mathcal{F}^{(2)}$ and $g_1^{-k}U_1 \to y_{g_1^{-1}}$ uniformly. Applying Lemma 2.1 once more, we can find $g_2 \in \Gamma$ satisfying $\lambda(g_2^{-1}) = w_2$ and a neighborhood $U_2 \subset \mathcal{F}$ of $y_{g_1^{-1}}$ satisfying $\{y_{g_2}\} \times U_2 \subset \mathcal{F}^{(2)}$ and that $g_2^{-k}U_2 \to y_{g_2^{-1}}$ uniformly.

Continuing inductively, we get elements $g_1, ..., g_r \in \Gamma$ and open sets $U_1, ..., U_r \subset \mathcal{F}$ satisfying that for all $\ell = 1, ..., r$,

(1) $w_\ell = \lambda(g_\ell^{-1})$;
(2) $y_{g_{\ell-1}^{-1}} \in U_\ell$;
(3) $g_\ell^{-k}U_\ell \to y_{g_\ell^{-1}}$ uniformly; and
(4) $\{y_{g_\ell}\} \times U_\ell$ is a relatively compact subset of $\mathcal{F}^{(2)}$.

We set $\xi_\ell := y_{g_\ell^{-1}}$ for each $1 \leq \ell \leq r$; so U_ℓ is a neighborhood of $\xi_{\ell-1}$ for each $1 \leq \ell \leq r$.

Since $\mathcal{Q}_{\eta_0} := \{\eta \in \mathcal{F} : (\eta_0, \eta) \in \mathcal{F}^{(2)}\} = \bigcup_{R>0} O_R(\eta_0, o)$ for any $\eta_0 \in \mathcal{F}$ and $U_\ell \subset \mathcal{Q}_{y_{g_\ell}}$ is a relatively compact subset of $\mathcal{F}^{(2)}$, there exists $R_\ell > 0$ such that $U_\ell \subset O_{R_\ell}(y_{g_\ell}, o)$. Since $g_\ell^k o$ converges to y_{g_ℓ} as $k \to +\infty$, by Lemma 2.3(2),

$$O_{R_\ell}(y_{g_\ell}o, o) \subset O_{R_\ell+1}(g_\ell^k o, o) \tag{3.7}$$

for all sufficiently large $k > 1$.

For each $1 \leq \ell \leq r$ and $j \geq 1$, let $k_{\ell,j}$ be the largest integer smaller than $c_{\ell,j}\|v_j\|$. As $\|v_j\| \to \infty$, and $c_{\ell,j} \to c_\ell$, we have $k_{\ell,j} \to \infty$ as $j \to \infty$. By the uniform contraction $g_\ell^{-k}U_i \to \xi_\ell$, there exists $j_0 > 1$ such that for all $j \geq j_0$,

$$\gamma_j^{-1}\xi \in U_1, \quad g_\ell^{-k_{\ell,j}}U_\ell \subseteq U_{\ell+1}, \quad \text{and} \quad U_\ell \subset O_{R_\ell+1}(g_\ell^{k_{\ell,j}}o, o) \tag{3.8}$$

for all $\ell = 1, ..., r$.

For each $j \geq j_0$, we now set

$$\tilde{\gamma}_j := \gamma_j g_1^{k_{1,j}} g_2^{k_{2,j}} \cdots g_r^{k_{r,j}} \in \Gamma.$$

We claim that $\beta_\xi(e, \tilde{\gamma}_j) \to \infty$ as $j \to \infty$ and that

$$\sup_{j \geq j_0} \|\beta_\xi(e, \tilde{\gamma}_j) - \mathbb{R}_+u\| < \infty; \tag{3.9}$$

this proves that ξ is u-horospherical.

Fix $j \geq j_0$ and for each $1 \leq \ell \leq r$, let $k_\ell := k_{\ell,j}$, $b_\ell := c_{\ell,j}\|v_j\|$, and set

$$h_\ell = g_1^{k_1} g_2^{k_2} \cdots g_\ell^{k_\ell},$$

and $g_0 = e$. The cocycle property of the Busemann function gives that

$$\beta_\xi(e, \tilde{\gamma}_j) = \beta_\xi(e, \gamma_j) - \sum_{\ell=1}^{r} \beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e). \tag{3.10}$$

By (3.8), $\gamma_j^{-1}\xi \in U_1$ and for each $1 \leq \ell \leq r$,

$$h_{\ell-1}^{-1}\gamma_j^{-1}\xi \in g_\ell^{-k_\ell} \cdots g_1^{-k_1} U_1 \subset U_{\ell+1} \subset O_{R_\ell+1}(g_\ell^{k_\ell} o, o).$$

Hence by Lemma 2.3(1), there exists $\kappa \geq 1$ such that for each $1 \leq \ell \leq r$

$$\|\beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e) - \mu(g_\ell^{-k_\ell})\| \leq \kappa(R_\ell + 1).$$

Note that for some $C_\ell > 0$, $\|\mu(g_\ell^{-k}) - k\lambda(g_\ell^{-1})\| \leq C_\ell$ for all $k \geq 1$. Since $\lambda(g_\ell^{-1}) = w_\ell$, we get

$$\|\beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e) - k_\ell w_\ell\| \leq \kappa(R_\ell + 1) + C_\ell.$$

Therefore by (3.10), we obtain

$$\|\beta_\xi(e, \tilde{\gamma}_j) - (v_j - \sum_{\ell=1}^{r} k_\ell w_\ell)\| \leq \kappa \sum_{\ell=1}^{r} (R_\ell + C_\ell + 1).$$

By (3.6), we have

$$c_{0,j}\|v_j\|u = v_j - \sum_{\ell=1}^{r} b_\ell w_\ell.$$

Since $|b_\ell - k_\ell| \leq 1$ and $c_{0,j} > 0$, we deduce that for all $j \geq j_0$,

$$\begin{aligned}
\|\beta_\xi(e, \tilde{\gamma}_j) - \mathbb{R}_+ u\| &\leq \|\beta_\xi(e, \tilde{\gamma}_j) - c_{0,j}\|v_j\| \cdot u\| \\
&\leq \left\| \beta_\xi(e, \tilde{\gamma}_j) - (v_j - \sum_{\ell=1}^{r} k_\ell w_\ell) \right\| + \sum_{\ell=1}^{r} \|k_\ell w_\ell - b_\ell w_\ell\| \\
&\leq \kappa \sum_{\ell=1}^{r} (R_\ell + C_\ell + \|w_\ell\| + 1).
\end{aligned}$$

This proves (3.9), and consequently ξ is u-horospherical for any $u \notin \mathbb{R}_+ v_0$. To show that ξ is v_0-horospherical, fix any $u \notin \mathbb{R}_+ v_0$ and $\tilde{\gamma}_j \in \Gamma$ be a sequence as in (3.9) associated to u. If we set $\tilde{v}_j = \beta_\xi(e, \tilde{\gamma}_j)$, then $\|\tilde{v}_j\|^{-1}\tilde{v}_j$ converges to a unit vector in $\operatorname{int}\mathcal{L}$ proportional to u. Therefore by repeating the same argument only now switching the roles of v_0 and u, we prove that ξ is v_0-horospherical as well. This completes the proof. □

We may now prove theorem 3.2:

Proof of theorem 3.2. Let $g \in G$ be such that $\xi = g^+ \in \Lambda$ is a horospherical limit point. Set $Y := \overline{[g]NM}$. We claim that $Y = \mathcal{E}$. By Benoist [1], the group generated by $\lambda(\Gamma) \cap \operatorname{int} \mathcal{L}$ is dense in $\mathfrak{a}$. Hence for every $\varepsilon > 0$ there exist loxodromic elements $\gamma_1, ..., \gamma_q \in \Gamma$ such that

$$\lambda(\gamma_1), ..., \lambda(\gamma_q) \in \operatorname{Int}\mathcal{L}$$

and the group $\mathbb{Z}\lambda(\gamma_1) + \cdots + \mathbb{Z}\lambda(\gamma_q)$ is an ε-net in $\mathfrak{a}$, i.e., its ε-neighborhood covers all $\mathfrak{a}$. Denote $u_i = \lambda(\gamma_i)$ for $i = 1, ..., q$. By Proposition 3.7, the point ξ is u_1-horospherical. By Proposition 3.6, there exists a u_1-periodic point $x_1 \in \mathcal{E}$ contained in Y, set

$$Y_1 := \overline{x_1 NM} \subset Y.$$

By Lemma 3.5, x_1^+ is u_1-horospherical; in particular, it is a horospherical limit point. Therefore we can inductively find a u_i-periodic point x_i in $Y_{i-1} = \overline{x_{i-1}NM}$ for each $2 \leq i \leq q$. By periodicity $x_i(\exp u_i)M = x_i M$, and hence $Y_i \exp \mathbb{Z}u_i = Y_i$ for each $1 \leq i \leq q$. Therefore we obtain

$$Y \supset Y_1 \exp \mathbb{Z}u_1 \supset Y_2 \exp(\mathbb{Z}u_1 + \mathbb{Z}u_2) \supset \cdots \supset Y_q \exp\big(\sum_{i=1}^{q} \mathbb{Z}u_i\big).$$

Recalling the dependence of Y_q and $\sum_{i=1}^q \mathbb{Z}u_i$ on ε, set

$$Z_\varepsilon := Y_q MN \exp\big(\sum_{i=1}^{q} \mathbb{Z}u_i\big) \subset Y.$$

Since $MN \exp(\sum_{i=1}^q \mathbb{Z}u_i)$ is an ε-net of P and $\mathcal{E}$ is P-minimal, Z_ε is a 2ε-net of $\mathcal{E}$ for all $\varepsilon > 0$. Since Y contains a 2ε-net of $\mathcal{E}$ for all $\varepsilon > 0$ and Y is closed, it follows that $Y = \mathcal{E}$.

For the other direction, it suffices to consider the case when $g = k \in K$ as the density of $[g]NM$ depends only on the coset gP. Suppose that $[k]NM$ is dense in $\mathcal{E}$ for $k \in K$. Since $\overline{\Gamma kNM} \supset kNAM$, it follows that for any $T > 1$ and for any cone $\mathcal{C} \in \mathfrak{a}^+$ with non-empty interior, $\Gamma \cap kN \exp(\mathcal{C} - \mathcal{C}_T)K \neq \emptyset$ where $\mathcal{C}_T = \{v \in \mathcal{C} : \|v\| \leq T\}$. This implies that k^+ is a horospherical limit point by Definition 3.1. ☐

4. Topological mixing and directional limit points

There is a close connection between denseness of N-orbits and the topological mixing of one-parameter diagonal flows with direction in $\operatorname{int} \mathcal{L}$. This connection allows us to make use of recent topological mixing results by Chow-Sarkar [3]: recall the notation Ω_0 from (2.2).

Theorem 4.1. [3] *For any $u \in \operatorname{int} \mathcal{L}$, $\{a_{tu} : t \in \mathbb{R}\}$ is topologically mixing on Ω_0, i.e., for any open subsets $\mathcal{O}_1, \mathcal{O}_2$ of $\Gamma\backslash G$ intersecting Ω_0,*

$$\mathcal{O}_1 \exp tu \cap \mathcal{O}_2 \neq \emptyset \quad \text{for all large } |t| \gg 1.$$

The above theorem was predated by a result of Dang [7] in the case where M is abelian.

N-orbits based at directional limit points along $\operatorname{int}\mathcal{L}$.

Definition 4.2. For $u \in \operatorname{int}\mathfrak{a}^+$, denote by Λ_u the set of all u-directional limit points, i.e., $\xi \in \Lambda_u$ if and only if $\limsup_{t\to+\infty} \Gamma g \exp(tu) \neq \emptyset$ for some (and hence any) $g \in G$ with $gP = \xi$.

It is easy to see that $\Lambda_u \subset \Lambda$ for $u \in \operatorname{int}\mathfrak{a}^+$.

Proposition 4.3. *If $[g] \in \mathcal{E}_0$ satisfies $g^+ \in \Lambda_u$ for some $u \in \operatorname{int}\mathcal{L}$, then*

$$\overline{[g]N} = \mathcal{E}_0.$$

Proof. Since $\Omega_0 N = \mathcal{E}_0$, we may assume without loss of generality that $x = [g] \in \Omega_0$. There exist $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i u}$ converges to some $h \in G$. In particular, $x \exp(t_i u) \to [h]$. Since $x a_{t_i u} \in \Omega_0$ and Ω_0 is A-invariant and closed, we have $[h] \in \Omega_0$. We write $\gamma_i g a_{t_i u} = h q_i$ where $q_i \to e$ in G. Therefore $xN = [h] q_i N a_{-t_i u}$ for all $i \geq 1$. Let $\mathcal{O} \subset \Gamma\backslash G$ be any open subset intersecting Ω_0. It suffices to show that $xN \cap \mathcal{O} \neq \emptyset$. Let $\mathcal{O}_1$ be an open subset intersecting Ω_0 and $V \subset \check{P}$ be an open symmetric neighborhood of e such that $\mathcal{O}_1 V \subset \mathcal{O}$.

Since $q_i \to e$ and NV is an open neighborhood of e in G, there exists an open neighborhood, say, U of e in G and i_0 such that $U \subset q_i NV$ for all $i \geq i_0$. By Theorem 4.1, we can choose $i > i_0$ such that $[h]U \cap \mathcal{O}_1 a_{t_i u} \neq \emptyset$. It follows that $[h] q_i N V a_{-t_i u} \cap \mathcal{O}_1 \neq \emptyset$. Since $V \subset a_{-t_i u} V a_{t_i u}$ as $u \in \mathfrak{a}^+$, we have

$$[h] q_i N V a_{-t_i u} \cap \mathcal{O}_1 \subset [h] q_i N a_{-t_i u} V \cap \mathcal{O}_1.$$

Since $V = V^{-1}$, we get $[h] q_i N a_{-t_i u} \cap \mathcal{O}_1 V \neq \emptyset$. Therefore $xN \cap \mathcal{O} \neq \emptyset$, as desired. $\square$

This immediately implies:

Corollary 4.4. *If $[g] \in \Omega_0$ is u-periodic for some $u \in \operatorname{int}\mathcal{L}$, then*

$$\overline{[g]N} = \mathcal{E}_0.$$

Proof. Since $[g](\exp ku) = [g] m_0^k$ for any integer k and M is compact, we have $g^+ \in \Lambda_u$. Therefore the claim follows from Proposition 4.3. $\square$

We may now conclude our main theorem in its fullest form:

Theorem 4.5. *Let $[g] \in \mathcal{E}_0$. The following are equivalent:*

(1) $g^+ \in \Lambda$ is a horospherical limit point;
(2) $[g]N$ is dense in $\mathcal{E}_0$;
(3) $[g]NM$ is dense in $\mathcal{E}$.

Proof. The implication (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) was shown in Theorem 3.2. Hence let us prove (1) $\Rightarrow$ (2).

Let $x = [g] \in \mathcal{E}_0$. Suppose that $g^+ \in \Lambda_h$. Fix any $u \in \lambda(\Gamma) \cap \operatorname{int}\mathcal{L}_\Gamma$. By Propositions 3.7 and 3.6, xN contains a u-periodic point, say, x_0. Hence by Corollary 4.4, $\overline{xN} \supset \overline{x_0 N} \supset \Omega_0 N = \mathcal{E}_0$. This proves (1) $\Rightarrow$ (2). $\square$

5. Conical limit points, Minimality and Jordan projection

A point $\xi \in \mathcal{F}$ is called a *conical* limit point of Γ if there exists a sequence $u_j \to \infty$ in $\mathfrak{a}^+$ such that for some (and hence every) $g \in G$ with $\xi = gP$

$$\limsup_{j\to\infty} \Gamma g a_{u_j} \neq \emptyset.$$

A conical limit point of Γ is indeed contained in Λ. We consider the following restricted notion:

Definition 5.1. We call $\xi \in \mathcal{F}$ a *strongly conical* limit point of Γ if there exists a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ and a sequence $u_j \to \infty$ in $\mathcal{C}$ such that for some (and hence every) $g \in G$ with $\xi = gP$,

$$\limsup_{j\to\infty} \Gamma g a_{u_j} \neq \emptyset.$$

Remarks 5.2. We mention that a conical limit point defined in [4] for $\Gamma < \mathrm{SL}_d(\mathbb{R})$ coincides with our strongly conical limit point.

Lemma 5.3. *Any strongly conical limit point of* Γ *is horospherical.*

Proof. Suppose that $\xi = gP$ is strongly conical, that is, there exist $\gamma_j \in \Gamma$ and $u_j \to \infty$ in some closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ such that $\gamma_j g a_{u_j}$ converges to some $h \in G$. Write $\gamma_j g a_{u_j} = h q_j$ where $q_j \to e$ in G. Let $\mathcal{C}'$ be a closed cone contained in $\operatorname{int} \mathcal{L} \cup \{0\}$ whose interior contains $\mathcal{C} \setminus \{0\}$.

Then $\gamma_j^{-1} = g a_{u_j} q_j^{-1} h^{-1}$ and

$$\beta_{gP}(e, \gamma_j^{-1}) = \beta_P(g^{-1}, a_{u_j} q_j^{-1} h^{-1}) = \beta_P(g^{-1}, q_j^{-1} h^{-1}) + \beta_P(e, a_{u_j}).$$

Since $\beta_P(e, a_{u_j}) = u_j$ and $q_j^{-1} h^{-1}$ are uniformly bounded, the sequence

$$\beta_{gP}(e, \gamma_j^{-1}) - u_j$$

is uniformly bounded. Since $u_j \in \mathcal{C}$ and $\mathcal{C} \subset \operatorname{int} \mathcal{C}' \cup \{0\}$, it follows that

$$\beta_{gP}(e, \gamma_j^{-1}) \in \mathcal{C}'$$

for all sufficiently large j. This proves that $\xi \in \Lambda_h$.

$\square$

Corollary 5.4. *For any* $g \in G$ *with strongly conical* $g^+ \in \mathcal{F}$*, we have*

$$\overline{[g]NM} = \mathcal{E}.$$

Directionally conical limit points. If $v \in \operatorname{int} \mathcal{L}$, then clearly Λ_v is contained in the horospherical limit set of Γ, and hence any NM-orbit based at a point of Λ_v is dense in $\mathcal{E}$. On the other hand, we would like to show in this section that the existence of a point in Λ_v for $v \in \partial \mathcal{L}_\Gamma$ implies the existence of a nondense NM-orbit in $\mathcal{E}$.

The flow $\exp(\mathbb{R}u)$ is said to be topologically transitive on $\Omega/M = \{\Gamma g M : g^\pm \in \Lambda\}$ if, for any open subsets $\mathcal{O}_1, \mathcal{O}_2$ intersecting Ω/M, there exists a sequence $t_n \to +\infty$ such that $\mathcal{O}_1 \cap \mathcal{O}_2 a_{t_n u} \neq \emptyset$.

We make the following simple observation:

Lemma 5.5. *For $g \in \Omega$, we have*

$$\overline{gNM} \supset \Omega \quad \text{if and only if} \quad \overline{gw_0\check{N}M} \supset \Omega.$$

Proof. We have $\check{N} = w_0 N w_0^{-1}$. Note that $[g] \in \Omega$ if and only if $[gw_0] \in \Omega$, since $(gw_0)^{\pm} = g^{\mp}$. So $\Omega w_0 = \Omega$. Hence gNM is dense in Ω if and only if $gw_0\check{N}Mw_0^{-1}$ is dense in Ω if and only if $[g]w_0\check{N}M$ is dense in $\Omega w_0 = \Omega$. □

Since the opposition involution preserves $\mathcal{L}$ and $\lambda(g^{-1}) = \mathrm{i}\,\lambda(g)$ for any loxodromic element, it follows that $\lambda(\gamma) \in \partial\mathcal{L}$ if and only if $\lambda(\gamma^{-1}) \in \partial\mathcal{L}$.

Proposition 5.6.

(1) If $\Lambda = \Lambda_h$, then $\exp(\mathbb{R}v)$ is topologically transitive on Ω/M for any $v \in \operatorname{int}\mathfrak{a}^+$ such that $\Lambda_v \neq \emptyset$.

(2) For any loxodromic element $\gamma \in \Gamma$ with $\{y_\gamma, y_{\gamma^{-1}}\} \subset \Lambda_h$, the flow $\exp(\mathbb{R}\lambda(\gamma))$ is topologically transitive on Ω/M.

Proof. Assume that $\Lambda = \Lambda_h$; so the NM-action on $\mathcal{E}$ is minimal. Suppose that $\Lambda_v \neq \emptyset$ for some $v \in \operatorname{int}\mathfrak{a}^+$. We claim that for any $\mathcal{O}_1, \mathcal{O}_2$ be two right M-invariant open subsets intersecting Ω, $\mathcal{O}_1 \exp(t_i v) \cap \mathcal{O}_2 \neq \emptyset$ for some sequence $t_i \to +\infty$. Choose $x = [g] \in \Omega$ so that $g^+ \in \Lambda_v$. Then there exists $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i v}$ converges to some g_0. Note that $x_0 := [g_0] \in \Omega$. So write $\gamma_i g a_{t_i v} = g_0 h_i$ with $h_i \to e$. By the NM-minimality assumption, xNM intersects every open subset of Ω. Since $v \in \operatorname{int}\mathfrak{a}^+$ and hence $a_{-tv} n a_{tv} \to e$ as $t \to +\infty$, we may assume without loss of generality that $x \in \mathcal{O}_1$. Choose an open neighborhood U of e in G so that $\mathcal{O}_1 \supset xUM$. Note that there exists a sequence $T_i \to \infty$ as $i \to \infty$ such that for all i,

$$xUMa_{t_i v} \supset x a_{t_i v} a_{-t_i v} \check{N}_\varepsilon M a_{t_i v} \supset x_0 h_i \check{N}_{T_i},$$

where $\check{N}_R = \check{N} \cap B_R^G$ is the the set of elements of $\check{N}$ of norm$\leq R$. So $\mathcal{O}_1 a_{t_i v} \supset x_0 h_i \check{N}_{T_i}$.

Choose an open neighborhood V of e in G and some open subset $\mathcal{O}_2'$ intersecting Ω so that $\mathcal{O}_2 \supset \mathcal{O}_2' V$. Since $x_0\check{N}M$ is dense in Ω, $x_0 n \in \mathcal{O}_2'$ for some $n \in \check{N}$. Hence $x_0 h_i n = x_0 n(n^{-1} h_i n) \in \mathcal{O}_2' V \subset \mathcal{O}_2$ for all i large enough so that $n^{-1} h_i n \in V$. Therefore for all i such that $n \in \check{N}_{T_i}$, we get

$$x_0 h_i n \in \mathcal{O}_1 a_{t_i v} \cap \mathcal{O}_2 \neq \emptyset.$$

This proves the first claim.

Now suppose that $\gamma \in \Gamma$ is a loxodromic element with $y_\gamma, y_{\gamma^{-1}} \in \Lambda_h$. Write $\gamma = g m a_v g^{-1}$ for some $g \in G$ and $m \in M$. Since $y_\gamma = g^+$ and $y_{\gamma^{-1}} = g w_0^+$, we have each $[g]NM$ and $[g]w_0NM$ contains Ω in its closure. Now in the notation of the proof of the first claim, note that $x_0 = [g_0] \in [g]M$ since $[g]\exp(\mathbb{R}v)M$ is closed. Therefore each $\overline{x_0NM}$ and $\overline{x_0\check{N}M}$ contains Ω. Based on this, the same argument as above shows the topological transitivity of $\exp\mathbb{R}v$, which finishes the proof since $v = \lambda(\gamma)$. □

Since $\mathcal{L}$ is invariant under the opposition involution i and $\lambda(\gamma) = \mathrm{i}\,\lambda(\gamma^{-1})$ for any loxodromic element $\gamma \in \Gamma$, the Jordan projection $\lambda(\gamma)$ belongs to $\partial\mathcal{L}$ if and only if the Jordan projection $\lambda(\gamma^{-1})$ belongs to $\partial\mathcal{L}$. Together with the result of Dang and Gloriuex [8, Proposition 4.7] which say that $\exp(\mathbb{R}u)$ is not topologically transitive on Ω/M for any $u \in \partial\mathcal{L} \cap \operatorname{int}\mathfrak{a}^+$, Proposition 5.6 implies the following:

Corollary 5.7.

(1) If $\Lambda_v \neq \emptyset$ for some $v \in \partial\mathcal{L} \cap \operatorname{int}\mathfrak{a}^+$, then

$$\Lambda \neq \Lambda_h.$$

(2) For any loxodromic element $\gamma \in \Gamma$, we have $\lambda(\gamma) \in \partial\mathcal{L}$ if and only if

$$\{y_\gamma, y_{\gamma^{-1}}\} \not\subset \Lambda_h.$$

Hence, if $\Lambda = \Lambda_h$, then $\lambda(\Gamma) \subset \operatorname{int}\mathcal{L}$.

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