# HECKE OPERATORS AND EQUIDISTRIBUTION OF HECKE POINTS

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## 1. Statements of Main results

Let $G$ be a connected almost simple simply connected linear algebraic group over $\mathbb{Q}$ with $G(\mathbb{R})$ non-compact and $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup. By the well known result of Borel and Harish-Chandra, $\Gamma$ is a lattice in $G(\mathbb{R})$. We remark that $G(\mathbb{R})$ is connected since $G$ is simply connected. We denote by $d\mu$ the normalized Haar measure on $\Gamma\backslash G(\mathbb{R})$.

We recall the definition of a Hecke operator:

**Hecke operator $T_a$.** Let $a \in G(\mathbb{Q})$. For $x \in \Gamma\backslash G(\mathbb{R})$, set $T_a x = \{[\Gamma a \Gamma x] \in \Gamma\backslash G(\mathbb{R})\}$. The Hecke operator $T_a$ on $L^2(\Gamma\backslash G(\mathbb{R}))$ is defined as follows: for any $f \in L^2(\Gamma\backslash G(\mathbb{R}))$,

$$T_a(f)(x) = \frac{1}{|T_a x|} \sum_{y \in T_a x} f(y)$$

where $|T_a x|$ denotes the cardinality of the set $T_a x$.

Since $\Gamma$ and $a^{-1}\Gamma a$ are commensurable with each other and $|\Gamma\backslash\Gamma a\Gamma| = [\Gamma : \Gamma \cap a^{-1}\Gamma a]$, the set $\Gamma\backslash\Gamma a \Gamma$ is finite for $a \in G(\mathbb{Q})$. The cardinality of this set will be denoted by $\deg(a)$. If $\Gamma a \Gamma = \amalg_{i=1}^{\deg(a)} \Gamma a_i$ for some $a_i \in G(\mathbb{Q})$, then

$$T_a(f)(x) = \frac{1}{\deg(a)} \sum_{i=1}^{\deg(a)} f(a_i x).$$

In particular, the above expression is independent of the choice of $a_i$'s. Note also that the operator norm of $T_a$ is precisely 1 due to the normalization.

Denote by $L_2^0(\Gamma\backslash G(\mathbb{R}))$ the orthogonal complement in $L^2(\Gamma\backslash G(\mathbb{R}))$ to the subspace of constant functions, that is,

$$L_0^2(\Gamma\backslash G(\mathbb{R})) = \{f \in L^2(\Gamma\backslash G(\mathbb{R})) \mid \int_{\Gamma\backslash G(\mathbb{R})} f \, d\mu = 0\}.$$

Then $T_a$ maps $L^2_0(\Gamma\backslash G(\mathbb{R}))$ into itself. We denote by $T_a^0 : L^2_0(\Gamma\backslash G(\mathbb{R})) \to L^2_0(\Gamma\backslash G(\mathbb{R}))$ the restriction of $T_a$ and set

$$\|T_a^0\| = \sup\{|\langle T_a f, h\rangle| \mid f, h \in L^2_0(\Gamma\backslash G(\mathbb{R})), \|f\| = 1, \|h\| = 1\}$$

where $\|f\|$ denotes the usual $L^2$-norm of $f$. In particular, for any $f \in L^2(\Gamma\backslash G(\mathbb{R}))$,

$$\|T_a f - \int_{\Gamma\backslash G(\mathbb{R})} f(x)d\mu(x)\| \le \|T_a^0\| \cdot \|f - \int_{\Gamma\backslash G(\mathbb{R})} f(x)d\mu(x)\| \le \|T_a^0\| \cdot \|f\|.$$

The main goal of this paper is to present an $L^2$-norm estimate of $T_a^0$ and, using that, to obtain an equidistribution of the sets $T_a x$ as $\deg(a)$ tends to $\infty$ with an estimate of equidistribution rate.

To formulate our main theorems, we now set up some notation. Let $R_f$ be the set of all finite primes. For each $p \in R_f$, let $A_p$ be a maximal $\mathbb{Q}_p$-split torus of $G(\mathbb{Q}_p)$ and $B_p$ a minimal $\mathbb{Q}_p$-parabolic subgroup of $G(\mathbb{Q}_p)$ containing $A_p$. Consider the set $\Phi_p$ of non-multipliable roots in the relative $\mathbb{Q}_p$-root system $\Phi(G, A_p)$ with the ordering given by $B_p$. If $G$ is quasi-split over $\mathbb{Q}_p$ and split over an unramified extension field over $\mathbb{Q}_p$, which is the case for almost all $p \in R_f$ [Ti1], let $K_p = G(\mathbb{Z}_p)$. Otherwise, we take $K_p$ to be a special subgroup of $G(\mathbb{Q}_p)$ associated to $A_p$ [Ti1]. Then $K_p$ is a good maximal compact subgroup of $G(\mathbb{Q}_p)$ for each $p \in R_f$. Hence we have the Cartan decomposition: $G(\mathbb{Q}_p) = K_p A_p^+ \Omega_p K_p$ where $A_p^+$ is the closed positive Weyl chamber. Here $\Omega_p$ is a finite subset of the centralizer of $A_p$, and $\Omega_p = \{e\}$ whenever $G$ is quasi-split and split over an unramified extension field over $\mathbb{Q}_p$. In particular, $\Omega_p = \{e\}$ for almost all $p \in R_f$.

A subset $\mathcal{S}_p \subset \Phi_p^+$ is called a *strongly orthogonal system* if for any two distinct $\alpha$ and $\beta$ in $\mathcal{S}_p$, neither of $\alpha \pm \beta$ belongs to $\Phi_p$.

We now recall the bi-$K_p$-invariant functions $\xi_{\mathcal{S}_p}$:

**Definition** [Oh2, Definition 4.10]**.** For a strongly orthogonal system $\mathcal{S}_p \subset \Phi_p^+$, the bi-$K_p$-invariant function $\xi_{\mathcal{S}_p}$ on $G(\mathbb{Q}_p)$ is defined by

$$\xi_{\mathcal{S}_p}(g) = \prod_{\alpha_p \in \mathcal{S}_p} \Xi_{PGL_2(\mathbb{Q}_p)} \begin{pmatrix} \alpha_p(g) & 0 \\ 0 & 1 \end{pmatrix}$$

where each root $\alpha_p \in \mathcal{S}_p$ is considered as a bi-$K_p$-invariant function on $G(\mathbb{Q}_p)$ by $\alpha_p(g) = \alpha_p(a)$ if $g = k_1 a d k_2$ for $k_1, k_2 \in K_p$, $a \in A_p^+$ and $d \in \Omega_p$, and $\Xi_{PGL_2(\mathbb{Q}_p)}$ denotes the Harish-Chandra function of $PGL_2(\mathbb{Q}_p)$ ([Ha], or see [Oh2, 2.2]).

Explicitly, $\Xi_{PGL_2(\mathbb{Q}_p)}$ is the bi-$PGL_2(\mathbb{Z}_p)$-function given by

$$\Xi_{PGL_2(\mathbb{Q}_p)} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{p}^n} \left( \frac{n(p-1) + (p+1))}{p+1} \right) \text{ for } |x|_p = p^{\pm n},\ n \in \mathbb{N}$$

where $|\cdot|_p$ denotes the $p$-adic valuation on $\mathbb{Q}_p$ (cf. [Oh2, 3.7]).

We now state the following norm estimate of Hecke operators.

**Theorem 1.1.** *Let $G$ be a connected almost simple simply connected linear algebraic group over $\mathbb{Q}$ with $G(\mathbb{R})$ non-compact. Let $\Gamma \subset G(\mathbb{Q})$ be a congruence subgroup. Set $R_1 = \{p \in R_f \mid rank_{\mathbb{Q}_p} G = 1\}$ and $R_2 = \{p \in R_f \mid rank_{\mathbb{Q}_p} G \geq 2\}$. Then there exists a constant $C$ such that for any $a \in G(\mathbb{Q})$*

$$\|T_a^0\| \leq \begin{cases} C\left(\prod_{p\in R_1} \xi_{\mathcal{S}_p}^{1/2}(a)\right)\left(\prod_{p\in R_2} \xi_{\mathcal{S}_p}(a)\right) & \text{if } rank_{\mathbb{Q}} G \geq 1; \\ C\left(\prod_{p\in R_2} \xi_{\mathcal{S}_p}(a)\right) & \text{if } rank_{\mathbb{Q}} G = 0 \end{cases}$$

*where $\mathcal{S}_p$ is a strongly orthogonal system of $\Phi_p$ for each $p \in R_1 \cup R_2$.*

*Remark.* If $G$ is $\mathbb{Q}$-split and $\Gamma = G(\mathbb{Z})$, then $C = 1$ and we can let $\mathcal{S}_p = \mathcal{S}$, $p \in R_f$ for any strongly orthogonal system $S$ of $\Phi$ where $\Phi$ is the set of non-multipliable roots in the relative root system with respect to a maximal $\mathbb{Q}$-split torus $A$. See the remark in 4.1 and 4.2 for a detailed account for $C$ in general cases. In particular, Theorem 4.2 presents a stronger version of Theorem 1.1 for a family of congruence subgroups.

In fact, for $p \in R_2$, the appearance of $\xi_{\mathcal{S}_p}$ in the above theorem is due to the result that $\xi_{\mathcal{S}_p}$ is a uniform pointwise bound for the matrix coefficients of all non-trivial unitary representations of $G(\mathbb{Q}_p)$ with respect to $K_p$-finite vectors [Oh2]. For $p \in R_1$, such a uniform bound does not exist since the group $G(\mathbb{Q}_p)$ does not have Kazhdan property (T). The presence of the function $\xi_{\mathcal{S}_p}^{1/2}$ in the above theorem for $p \in R_1$ is then due to a known bound towards the Ramanujan conjecture for $SL_2$ by Gelbart and Jacquet [GJ]. We note that the Gelbart-Jacquet estimate has been improved by Luo-Rudnick-Sarnak [LRS], by Shahidi [Sh] and most recently by Kim-Shahidi [KS] (also see the appendix by Kim and Sarnak in [KS] for a slightly better estimate). At the unramified primes, this gives a better estimate of the spherical function on $SL_2$ than the one we use. It is likely that the estimate could be extended to all matrix coefficients, thus reinforcing the second inequality in section 3.2 and hence Theorem 1.1 for $p \in R_1$.

If the Ramanujan conjecture for $SL_2$ is assumed, then we can replace $\xi_{\mathcal{S}_p}^{1/2}$ by $\xi_{\mathcal{S}_p}$.

**Theorem 1.2.** *Keeping the same notation as in Theorem 1.1, if we assume that $rank_{\mathbb{Q}} G \geq 1$ and that the Ramanujan conjecture holds at every finite prime $p$ for $SL_2$, then there exists a constant $C$ such that for any $a \in G(\mathbb{Q})$*

$$\|{T_a}^0\| \leq C \prod_{p\in R_f} \xi_{\mathcal{S}_p}(a).$$

We note that if $G$ is $\mathbb{Q}$-anisotropic and $p \in R_1$, the $\mathbb{Q}$-embedding of a form of $SL_2 \hookrightarrow G$ having the properties used in the proof of Theorem 3.4 does not necessarily exist. By using known cases of Langlands functoriality as well as restriction to other more complicated embeddings $H \hookrightarrow G$ of some $\mathbb{Q}$-subgroup $H$, we think that a non-trivial estimate on matrix coefficients can be obtained also at these primes. Thus if $\mathrm{rank}_{\mathbb{Q}} G = 0$, a suitable form of Theorem 1.1 could be proved controlling $\|T_a^0\|$ also at primes $p \in R_1$, but possibly with a weaker estimate. We thank the referee for this comment.

The only case where Theorem 1.1 has no content is when the absolute rank of $G$ is 1 while $\mathrm{rank}_{\mathbb{Q}} G = 0$ (if $\mathrm{rank}_{\mathbb{Q}} G = 0$ and $\mathrm{rank} G \geq 2$, then there exists infinitely many primes $p$ such that $\mathrm{rank}_{\mathbb{Q}_p} G \geq 2$). This happens only for the group $G = SL_1(D)$ where $D$ is a quaternion algebra over $\mathbb{Q}$ [Ti2]. The following is then a well known consequence of the Jacquet-Langlands correspondence [JL] based on the Gelbart-Jacquet estimate for a bound of the Ramanujan conjecture. Note that $R_1 = \{p \in R_f \mid D \text{ is split over } \mathbb{Q}_p\}$.

**Theorem 1.3.** *Let $G = SL_1(D)$ where $D$ is a quaternion algebra over $\mathbb{Q}$ such that $D$ is split over $\mathbb{R}$ and $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup. Then there exists a constant $C$ such that for any $a \in G(\mathbb{Q})$,*

$$\|T_a^0\| \leq C \prod_{p \in R_1} \xi_{\mathcal{S}_p}^{1/2}(a)$$

*where $\mathcal{S}_p$ is a strongly orthogonal system of $\Phi_p$ for each $p \in R_f$ where $D$ splits over $\mathbb{Q}_p$ (in fact, $\mathcal{S}_p = \{\alpha_p\}$ for the simple root $\alpha_p$ of $\Phi_p$).*

A strongly orthogonal system $\mathcal{S}$ in a reduced root system $\Phi$ is said to be *large* if every simple root of $\Phi$ has a non-zero coefficient in the formal sum $\sum_{\alpha \in \mathcal{S}} \alpha$; and *maximal* if the coefficient of each simple root in the formal sum $\sum_{\alpha \in \mathcal{S}} \alpha$ is not less than the one in $\sum_{\alpha \in \mathcal{O}} \alpha$ for any strongly orthogonal system $\mathcal{O}$ of $\Phi$ (cf. [Oh2]).

We obtain a direct corollary of Theorem 1.1 just by considering large strongly orthogonal systems $\mathcal{S}_p$, $p \in R_f$ (see 4.4).

**Corollary 1.4.** *Let $G$ and $\Gamma$ be as in Theorem 1.1. Let $\{a_n \in G(\mathbb{Q}) \mid n \in \mathbb{N}\}$ be any sequence with $\deg(a_n)$ tending to infinity. If $\mathrm{rank}_{\mathbb{Q}} G = 0$, assume that the diagonal embedding of the sequence $\{a_n\}$ into the direct product $\prod_{p \in R_2} G(\mathbb{Q}_p)$ is unbounded in $\prod_{p \in R_2} G(\mathbb{Q}_p)$. Then*

$$\lim_{n \to \infty} \|T_{a_n}^0\| = 0.$$

Clearly the convergence rate $\left(\prod_{p \in R_1} \xi_{\mathcal{S}_p}^{1/2}(a)\right)\left(\prod_{p \in R_2} \xi_{\mathcal{S}_p}(a)\right)$ in Theorem 1.1 is sharpest when maximal strongly orthogonal systems $\mathcal{S}_p$ are used for all $p \in R_1 \cup R_2$. A maximal strong orthogonal system for each irreducible reduced root system has been constructed in [Oh1]. In fact, it turns out that Theorem 1.1 yields the optimal convergence rate for the Hecke operator norms for the groups $SL_n$, $(n \geq 3)$ and $Sp_{2n}$, $(n \geq 2)$.

Write $G$ for $SL_n$ or $Sp_{2n}$. The group $Sp_{2n}$ will be defined by the bi-linear from $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. We write $s|t$ if $\frac{t}{s}$ is an integer. Let $A$ be the diagonal subgroup of $G$ and set

$$A^+ = \{\mathrm{diag}(a_1, \cdots, a_n) \mid a_i \in \mathbb{Q},\, a_i > 0,\, a_{i+1}|a_i \text{ for all } 1 \le i \le n-1\}$$

if $G = SL_n$ and

$$A^+ = \{\mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \mid a_i \in \mathbb{N},\, a_i \ge 1,\, a_{i+1}|a_i \text{ for all } 1 \le i \le n\}$$

if $G = Sp_{2n}$.

**Theorem 1.5.** *(cf. 5.1 and 5.3 below) Let $G = SL_n\,(n \ge 3)$ or $Sp_{2n}\,(n \ge 2)$, and $\Gamma = G(\mathbb{Z})$. Let $\mathcal{S}$ be a maximal strongly orthogonal system of $\Phi(G, A)$. Then there exists a function $f \in L^2_0(\Gamma\backslash G(\mathbb{R}))$ with $\|f\| = 1$ such that for any $\epsilon > 0$, one can find a constant $C$ (depending on $\epsilon$) such that*

$$C \prod_{p \in R_f} \xi_{\mathcal{S}}^{1+\epsilon}(a) \le \|T_a^0 f\| \le \prod_{p \in R_f} \xi_{\mathcal{S}}(a)$$

*for any $a \in A^+$.*

We remark that in [Oh2] the function $\xi_{\mathcal{S}}$ in Theorem 1.5 was shown to achieve the slowest decay in the spherical unitary dual of $G$. It is worthwhile to note that the same function $\xi_{\mathcal{S}}$ indeed comes from the spectrum of $L^2(\Gamma\backslash G(\mathbb{R}))$ (see 5.4 below).

We can easily deduce the following equidistribution statements from Theorem 1.1 (see 4.5):

**Theorem 1.6.** *Let $G$, $\Gamma$ and $\{a_n\}$ be as in Corollary 1.4. Then for any $x \in \Gamma\backslash G(\mathbb{R})$, the sets $T_{a_n}x$ are equidistributed with respect to $d\mu$, in the sense that*

$$\lim_{n\to\infty} T_{a_n} f(x) = \int_{\Gamma\backslash G(\mathbb{R})} f(g)\, d\mu(g)$$

*for any continuous function $f$ on $\Gamma\backslash G(\mathbb{R})$ with compact support.*

Under suitable differentiability assumption on $f$, the $L^2$-convergence rate obtained in Theorem 1.1 does survive in the equidistribution statement (see [CU, Corollary 8.3], or Theorem 4.6 below). For instance, we have the following:

**Theorem 1.7.** *Keeping the same notation from Theorem 1.1, let $f$ be a smooth function on $\Gamma\backslash G(\mathbb{R})$ with a compact support. Then for any $a \in G(\mathbb{Q})$ and for any $x \in \Gamma\backslash G(\mathbb{R})$,*

$$\left| T_a f(x) - \int_{\Gamma\backslash G(\mathbb{R})} f(g) d\mu(g) \right| \leq$$
$$\begin{cases} C \left( \prod_{p\in R_1} \xi_{\mathcal{S}_p}^{1/2}(a) \right) \left( \prod_{p\in R_2} \xi_{\mathcal{S}_p}(a) \right) & \text{if } rank_{\mathbb{Q}} G \geq 1; \\ C \left( \prod_{p\in R_2} \xi_{\mathcal{S}_p}(a) \right) & \text{if } rank_{\mathbb{Q}} G = 0 \end{cases}$$

*where the constant $C$ (depending on $f$) can be taken uniformly over compact subsets.*

*Remark.* The above results can be generalized in the following cases:

(1) Let $G$ be a connected almost simple simply connected $\mathbb{Q}$-group. Let $S$ be a finite set of primes including the archimedean one $\infty$. Assume that $G(\mathbb{Q}_p)$ is non-compact for some prime $p \in S$ (here we set $\mathbb{Q}_\infty = \mathbb{R}$). Let $\Gamma \subset G(\mathbb{Q})$ be an $S$-congruence subgroup, that is, $\Gamma = G(\mathbb{Q}) \cap U^S$ for some compact open subgroup $U^S$ of $G(\mathbb{A}_S)$ where $G(\mathbb{A}_S)$ denotes the subgroup of the adele group $G(\mathbb{A})$ consisting of elements with trivial components for all primes in $S$. Let $G_S = \prod_{p\in S} G(\mathbb{Q}_p)$. Then the diagonal embedding of $\Gamma$ into $G_S$, which will be identified with $\Gamma$, is a lattice in $G_S$. For each $a \in G(\mathbb{Q})$, we can consider the Hecke operator $T_a$ on $L^2(\Gamma\backslash G_S)$ acting in a diagonal way. It will be clear from our methods that all the results stated above remain valid for $T_a$ provided we replace $R_1$ and $R_2$ by $R_1 - S$ and $R_2 - S$ respectively.

(2) If $G$ is almost $\mathbb{Q}$-simple, then there exist a number field $K$ and a connected almost simple simply connected $K$-group $H$ such that $G = \mathrm{Rest}_{K/\mathbb{Q}} H$. The same statements as the above theorems are true if we replace $R_f$, $R_1$ and $R_2$ by the corresponding set of valuations of $K$ relative to $H$.

(3) Let $G$ be $GL_n$ or $GSp_{2n}$ (the group $GSp_{2n}$ is defined similarly as $Sp_{2n}$ except for allowing scalar multiplications). Let $\Gamma = G(\mathbb{Z})$. If we replace $\Gamma\backslash G(\mathbb{R})$ by $Z(\mathbb{R})\Gamma\backslash G(\mathbb{R})$ where $Z$ denotes the center of $G$, all the above theorems hold for those $G$ and $\Gamma$ where $R_1$ and $R_2$ are defined with respect to the semisimple ranks of $G(\mathbb{Q}_p)$'s (see 4.7).

We now discuss some applications of the above results on equidistribution of lattices in $\mathbb{R}^n$ with some properties. Via the natural isomorphism of the space $GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})$ with the space of lattices in $\mathbb{R}^n$ by $GL_n(\mathbb{Z})g \mapsto g^T\mathbb{Z}^n$, the space $Z(\mathbb{R})GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})$ is identified with the space $X$ of equivalence classes of lattices where $\Lambda \sim \Lambda'$ if and only if $\Lambda = c\Lambda'$ for some scalar $c$. We write $\bar{\Lambda}$ for the class in $X$ represented by a lattice $\Lambda$ in $\mathbb{R}^n$. For any $n$ positive integers $a_1, \cdots, a_n$ with $a_{i+1}|a_i$ for all $1 \leq i \leq n-1$ and $a_n = 1$, set

$$X_{\bar\Lambda}(a_1,\cdots,a_n)=\{\bar\Lambda'\in X\mid \Lambda'\subset\Lambda,\ \Lambda/\Lambda'\approx\sum_{i=1}^{n-1}\mathbb{Z}/a_i\mathbb{Z}\}.$$

Then

$$X_{\bar\Lambda}(a_1,\cdots,a_n)=T_{\mathrm{diag}(a_1,\cdots,a_n)}(\bar\Lambda).$$

Hence the following easily follows from Theorem 1.7 (see 3.1 below):

**Corollary 1.8.**

(1) *For any $\bar\Lambda\in X$, the sets $X_{\bar\Lambda}(a_1,\cdots,a_n)$ are equidistributed, that is, for any (nice) compact subset $\Omega\subset X$*

$$\frac{|\Omega\cap X_{\bar\Lambda}(a_1,\cdots,a_n)|}{|X_{\bar\Lambda}(a_1,\cdots,a_n)|}\sim\mu(\Omega)$$

*as diag$(a_1,\cdots,a_n)$ goes to infinity in $GL_n(\mathbb{Z})$ modulo its center.*

(2) *Let $n\geq 3$. For any smooth function $f$ on $X$ with compact support, we have that for any $\epsilon>0$, there exists a constant $C$ (depending on $\epsilon$ and $f$) such that for any such $(a_1,\cdots,a_n)$ as above,*

$$\left|\frac{\sum_{\bar\Lambda'\in X_{\bar\Lambda}(a_1,\cdots,a_n)}f(\bar\Lambda')}{|X_{\bar\Lambda}(a_1,\cdots,a_n)|}-\int_X f d\mu\right|\leq C\prod_{i=1}^{[n/2]}\left(\frac{a_i}{a_{n+1-i}}\right)^{-1/2+\epsilon}$$

*for any $\bar\Lambda\in X$.*

Here the notation $[\alpha]$ denotes the largest integer not bigger than $\alpha$.

On the other hand, if we set

$$X_{\bar\Lambda}(m)=\{\bar\Lambda'\in X\mid\Lambda'\subset\Lambda,\ \det(\Lambda')=m\},$$

$X_{\bar\Lambda}(m)$ is equal to the finite union of the sets $T_{\mathrm{diag}(a_1,\cdots,a_n)}\bar\Lambda$ where $(a_1,\cdots,a_n)$ reads through all $n$-tuples of positive integers such that $a_i\in\mathbb{N}$, $a_{i+1}|a_i$ for all $1\leq i\leq n-1$, $a_n=1$ and $\prod_{i=1}^n a_i=m$.

The following will be deduced from Theorem 1.7 in section 5.2:

**Corollary 1.9.**

(1) *For any $\bar\Lambda\in X$ and for any (nice) compact subset $\Omega\subset X$,*

$$\frac{|\Omega\cap X_{\bar\Lambda}(m)|}{|X_{\bar\Lambda}(m)|}\sim\mu(\Omega)\qquad as\ m\to\infty\ .$$

(2) *Let $n\geq 3$. Let $f$ be a smooth function on $X$ with compact support and $\bar\Lambda$ any element in $X$. Then for any $\epsilon>0$, we have*

$$\frac{\sum_{\bar\Lambda'\in X_{\bar\Lambda}(m)}f(\bar\Lambda')}{|X_{\bar\Lambda}(m)|}=\int_X f d\mu+O(m^{-1/2+\epsilon})\qquad as\ m\to\infty\ .$$

The condition of compact support for $f$ is unnecessary in the above corollaries, for example, the Gaussian function for the Riemannian structure on $X$ will do (see 4.6). Again, if the Ramanujan conjecture for $SL_2$ is assumed, the rate in the second part in the above Corollaries is valid for $n = 2$ as well. In view of Theorem 1.5, the rate $m^{-1/2+\epsilon}$ is an optimal one.

The first claim in Corollary 1.9 was in fact considered by Linnik and Skubenko [LS] in a slightly different form. With the use of ergodic method, they showed the equidistribution of the sets $R(m) = \{x \in M_{n\times n}(\mathbb{Z}) \mid \det x = m\}$, when projected to $SL_n(\mathbb{R})$, as $m \to \infty$ (without obtaining any rate of convergence). This problem was interpreted in terms of Hecke operators first by Sarnak who also deduced the equidistribution of $X_{\bar{\Lambda}}(m)$, or equivalently $R(m)$, from convergence of the norms of Hecke operators (see [Sa] for details).

Let $K = Sp_{2n}(\mathbb{R}) \cap SU_{2n}$ be the maximal compact subgroup of $Sp_{2n}(\mathbb{R})$. Then the space $Sp_{2n}(\mathbb{Z})\backslash Sp_{2n}(\mathbb{R})/K$ is the moduli space of principally polarized abelian varieties. We think that our results on Hecke operators may yield to an equidistribution theorem for the set of points with complex multiplication by a given number field $F$. For $n = 1$, this has been explained in [CU]. Using the results of Duke [Du], this gives the equidistribution of the set of elliptic curves with complex multiplication by an order $\mathcal{O}_{F,f}$ ($f$ a conductor) in the ring of integers $\mathcal{O}_F$ of an imaginary quadratic extension field $F$ of $\mathbb{Q}$ when the discriminant $d_{F,f}$ of $\mathcal{O}_{F,f}$ tends to infinity. Some motivations for such results related to the André-Oort conjecture are given in the introduction of [CU].

Our results may also be used to generalize the work of Lubotzky, Phillips and Sarnak [LPS, I, II] on distributing (Hecke) points on $S^2$ to a connected compact simple Lie group with discrepancy estimates.

In section 2, we discuss how the Hecke operators are related with unitary representations of $G(\mathbb{Q}_p)$'s. Proposition 2.6 reduces the norm estimate of Hecke operators to the pointwise bound for matrix coefficients of unitary representations of the groups $G(\mathbb{Q}_p)$, $p \in R_f$. When the $\mathbb{Q}_p$-rank of $G(\mathbb{Q}_p)$ is at least 2, we use the bound constructed in [Oh2], which is presented in Theorem 3.1. For the case when the $\mathbb{Q}_p$-rank of $G(\mathbb{Q}_p)$ is 1, we lift the Ramanujan bound for $SL_2$ to $G(\mathbb{Q}_p)$ using the method developed by Burger and Sarnak in [BS] (see [CU] for its $p$-adic version). This process is explained in 3.3 and 3.4. In section 4, we give proofs of all theorems listed in the introduction except Theorem 1.5, and in section 5, we write down the bounds in Theorem 1.1 explicitly for the groups $SL_n$ (or $GL_n$) and $Sp_{2n}$ (or $GSp_{2n}$) using the maximal strongly orthogonal system constructed in [Oh1] and obtain Theorem 1.5. We also prove Corollary 1.9 in 5.2.

A special case of Theorem 1.1 for $G = SL_n$ was first announced by Sarnak in his address in the 1991 international congress of mathematics in Kyoto [Sa]. Chiu afterwards obtained some analogues of Theorem 1.1 for $SL_2$ and $SL_3$ in [Ch] for special types of Hecke operators. The first and the third named authors, in their joint work [CU], have also obtained some special cases of the above results for $GL_n$ and $GSp_{2n}$. We also mention that Theorem 1.6 has been stated in [BS] without proofs when the sequence $a_n$ converges to an element of $G(\mathbb{R})$ not in the commensurator of $\Gamma$. We emphasize that the novelty of our results lies not just in obtaining the equidistribution statement but also in getting (sharp) rates of equidistribution. While all previous results were limited to some directions of $a_n$ going to $\infty$ we have no such restrictions in our theorems.

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## 2. Adelic interpretation of $T_a$

**2.1.** Unless mentioned otherwise, in the whole section 2, let $G$ be a connected almost simple simply connected $\mathbb{Q}$-linear group such that $G(\mathbb{R})$ is non-compact. Let $R$ be the set of all primes including $\infty$ and $R_f$ the finite primes. We set $\mathbb{Q}_\infty = \mathbb{R}$. We denote by $G(\mathbb{A})$ the adele group associated to $G$, that is, $G(\mathbb{A})$ is the restricted topological product of the groups $G(\mathbb{Q}_p)$, $p \in R$, with respect to a family of compact open subgroups $K_p \subset G(\mathbb{Q}_p)$, $p \in R_f$. Recall that $G$ is called *unramified* over $\mathbb{Q}_p$ if $G$ is quasi-split over $\mathbb{Q}_p$ and splits over an unramified extension field over $\mathbb{Q}_p$. Set $R_h = \{p \in R_f \mid G \text{ is unramified over } \mathbb{Q}_p\}$. Thus $R_f - R_h$ is a finite set [Ti1]. If $p \in R_h$, we can take $K_p$ to be a hyper-special maximal compact open subgroup of $G(\mathbb{Q}_p)$. For other primes $p \in R_f$, as in the introduction, $K_p$ is simply a special maximal compact open subgroup of $G(\mathbb{Q}_p)$. We may assume that for almost all $p \in R_f$, $K_p = \mathcal{G}(\mathbb{Z}_p)$ for a smooth model of $G$ over $\mathbb{Z}[1/N]$ ($N$ some appropriate integer). Denote by $G(\mathbb{A}_f)$ the subgroup of $G(\mathbb{A})$ consisting of adeles with the trivial element at $\infty$. Let $\Gamma$ be a congruence subgroup of $G$. Then $\Gamma$ is of the form $G(\mathbb{Q}) \cap (G(\mathbb{R}) \times \prod_{p \in R_f} U_p)$ where $U_p \subset G(\mathbb{Q}_p)$ are compact open subgroups and $U_p = K_p$ for almost all $p \in R_f$. Set $U_f = \prod_{p \in R_f} U_p$. Since $U_f$ is an open subgroup of $G(\mathbb{A}_f)$, by the strong approximation theorem (cf. [PR, Theorem 7.12]), we have

$$G(\mathbb{A}) = G(\mathbb{Q})\, G(\mathbb{R}) U_f.$$

Since

$$\Gamma = G(\mathbb{Q}) \cap (G(\mathbb{R}) \times U_f),$$

the spaces $G(\mathbb{Q})\backslash G(\mathbb{A})/U_f$ and $\Gamma\backslash G(\mathbb{R})$ are naturally identified with each other and furthermore this identification is a homeomorphism. Let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/U_f)$ denote the space of $L^2$-integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/U_f$ with respect to the pull back of the measure $\mu$ under the above identification. Let $\mu_p$ denote the Haar measure on $G(\mathbb{Q}_p)$ with $\mu_p(U_p) = 1$ for each $p \in R_f$. It is well known that the collection $\mu$, $\mu_p$, $p \in R_f$ defines a Haar measure on $G(\mathbb{A})$ and hence on $G(\mathbb{Q})\backslash G(\mathbb{A})$, which will be denoted by $\mu_{\mathbb{A}}$. Consider the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{U_f}$ of $L^2$-integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ (with respect to $\mu_{\mathbb{A}}$) which are right $U_f$-invariant. Then the canonical embedding of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{U_f}$ into $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/U_f)$ is an isometry under which the constant functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ correspond to the constant functions $G(\mathbb{Q})\backslash G(\mathbb{A})/U_f$ [GGP, Appendix II to Ch 4]. We will not distinguish these two spaces as well as their orthogonal complements to the space of constant functions hereafter.

The map

$$\phi : L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{U_f} \to L^2(\Gamma\backslash G(\mathbb{R}))$$

given by $\phi(f)(x) = f(x, (e)_{p\in R_f})$, $x \in G(\mathbb{R})$, defines a one-to-one correspondence, which is an isometry. Here $e$ denotes the identity element in $G$.

**2.2.** For $a \in G(\mathbb{Q})$, set $\deg_p(a)$ to be the cardinality of the set $U_p\backslash U_p a U_p$ for each $p \in R_f$. Note that $\deg_p(a) = 1$ for almost all $p \in R_f$.

**Lemma.** *For $a \in G(\mathbb{Q})$,*

1. $\deg(a) = [\Gamma : \Gamma \cap a^{-1}\Gamma a]$*;*
2. $\deg_p(a) = [U_p : U_p \cap a^{-1}U_p a]$*;*
3. $\deg(a) = \prod_{p\in R_f} \deg_p(a)$*.*

*Proof.* Consider the $\Gamma$-orbit of $a$ in $\Gamma\backslash G(\mathbb{R})$ under the translation from the right, that is, $\gamma \mapsto \Gamma a\gamma^{-1}$. Then the stabilizer $\{\gamma \in \Gamma \mid \Gamma a\gamma^{-1} = \Gamma a\}$ is precisely $\Gamma\cap a^{-1}\Gamma a$. Hence we obtain a bijection between $\Gamma\backslash\Gamma a\Gamma$ and $(\Gamma \cap a^{-1}\Gamma a)\backslash\Gamma$; hence $\deg(a) = [\Gamma : \Gamma \cap a^{-1}\Gamma a]$. Similarly, we have $\deg_p(a) = [U_p : U_p\cap a^{-1}U_p a]$. Denote by $i$ the diagonal embedding of $\Gamma$ into $U_f$ and by $\pi$ the natural projection of $U_f$ onto the coset space $(U_f\cap a^{-1}U_f a)\backslash U_f$. Then the map $\pi \circ i$ factors through $(\Gamma \cap a^{-1}\Gamma a)\backslash\Gamma$. On the other hand, by the strong approximation, we have $(U_f \cap a^{-1}U_f a)\Gamma = U_f$. Hence this yields a bijection between $(\Gamma \cap a^{-1}\Gamma a)\backslash\Gamma$ and $(U_f \cap a^{-1}U_f a)\backslash U_f$. Since $[U_f : U_f \cap a^{-1}U_f a] = \prod_{p\in R_f}[U_p : U_p \cap a^{-1}U_p a]$, the lemma is proved. □

**2.3.** For each $a \in G(\mathbb{Q})$ and $p \in R_f$, we now define the local Hecke operator $T_{a(p)}$ acting on the space of right $U_f$-invariant functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. If $U_p a U_p = \amalg_{i=1}^{\deg_p(a)} U_p a_{ip}$ for $a_{ip} \in G(\mathbb{Q}_p)$, then for $(x_q)_{q\in R} \in G(\mathbb{Q})\backslash G(\mathbb{A})$,

$$T_{a(p)}(f)((x_q)_{q\in R}) = \frac{1}{\deg_p(a)} \sum_{i=1}^{\deg_p(a)} f((x_q)_{q\neq p}, (x_p a_{ip}^{-1})).$$

One can easily check that this is a well defined operator and that for primes $p \neq q$, $T_{a(p)}$ and $T_{a(q)}$ commute with each other. Hence the product operator $\hat{T}_a := \prod_{p\in R_f} T_{a(p)}$ acts on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{U_f}$. The following is well known:

**Theorem.** *For any $a \in G(\mathbb{Q})$ and $f \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{U_f}$,*

$$\phi(\hat{T}_a(f)) = T_a(\phi(f)).$$

*Proof.* Let $S \subset R_f$ be a finite subset such that $a \in U_p$ for each $p \in R_f - S$. Let $\Gamma a \Gamma = \amalg_{i=1}^{\deg(a)} \Gamma a_i$ and $U_p a U_p = \amalg_{j_p=1}^{\deg_p(a)} U_p a_{j_p}$ for each $p \in S$. We will understand $f$ as a function on $G(\mathbb{A})$ which is $G(\mathbb{Q})$-invariant from the left and $U_f$-invariant from the right. Note that for $g \in G(\mathbb{R})$,

$$T_a(\phi(f))(g) = \frac{1}{\deg(a)} \sum_{i=1}^{\deg(a)} f\left(a_i g, (e)_{p\in R_f}\right)$$

and

$$\phi(\hat{T}_a(f))(g) = \frac{1}{\prod_{p\in S}\deg_p(a)} \sum_{\{(j_p)_{p\in S} | 1\le j_p \le \deg_p(a)\}} f(g, (a_{j_p}^{-1})_{p\in S}, (e)_{p\notin S}).$$

From the way we obtained the bijection between $\Gamma\backslash\Gamma a\Gamma$ and $U_f\backslash U_f a U_f$ in the proof of Lemma 2.2, it is clear that for any $(j_p)_{p\in S}$, there exists one and only one $a_i$ for some $1 \le i \le \deg(a)$ such that $a_i \in U_p a_{j_p}$, or equivalently $a_i^{-1} \in a_{j_p}^{-1} U_p$, for each $p \in S$. Since $f$ is $G(\mathbb{Q})$-invariant from the left and $U_f$-invariant from the right,

$$\sum_{\{(j_p)_{p\in S} | 1\le j_p \le \deg_p(a)\}} f(g, (a_{j_p}^{-1})_{p\in S}, (e)_{p\notin S}) = \sum_{i=1}^{\deg(a)} f(g, (a_i)^{-1}{}_{p\in S}, (e)_{p\notin S}),$$

which is again equal to $\sum_{i=1}^{\deg(a)} f(a_i g, (e)_{p\in R_f})$. Since $\deg(a) = \prod_{p\in S}\deg_p(a)$ by Lemma 2.2, we have

$$\phi(\hat{T}_a(f))(g) = T_a(\phi(f))(g)$$

for any $g \in G(\mathbb{R})$. □

**2.4.** Let $\mathbb{C}$ denote the space of constant functions on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We will write $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \mathbb{C} \oplus L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$. If $S \subset R$ is a finite subset such that $R_f - S \subset R_h$ and $U_p = K_p = G(\mathbb{Z}_p)$ for all $p \in R_f - S$, then the subgroup $\prod_{p\in R_f - S} U_p$ acts trivially on this space, which we will view then as a module under $\prod_{p\in S} G(\mathbb{Q}_p)$. Set

$G_S = \prod_{p\in S} G(\mathbb{Q}_p)$. As such (all groups having type one), the space $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ has a decomposition as a Hilbert integral. Write

$$L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}))^{U_f} = \int_X m_x \rho_x^{U_f} \, d\nu(x)$$

where $X = \widehat{G}_S$, $\rho_x = \otimes_{p\in S}\rho_{x(p)}$ is irreducible with a non-trivial $U_f$-invariant vector, $m_x$ is a multiplicity for each $x \in X$ and $\nu$ is a measure on $X$. Here each $\rho_{x(p)}$ is an irreducible unitary representation of $G(\mathbb{Q}_p)$ with a non-trivial $U_p$-invariant vector [Fl].

From the strong approximation and the spectral decomposition (cf. [Ar]), we have:

**Proposition.** *For each $p \in S$, $\rho_{x(p)}$ is non-trivial for almost all $x \in X$.*

We may therefore assume that the above proposition holds for all $x \in X$ without loss of generality.

**2.5.** Since both $K_p$ and $U_p$ are compact open subgroups of $G(\mathbb{Q}_p)$, they are commensurable with each other. It is easy to see:

**Lemma.** *If $\rho$ is a unitary representation of $G(\mathbb{Q}_p)$ and $v$ is a $U_p$-invariant vector, then*

$$dim\langle K_p v\rangle \le [K_p : K_p \cap U_p]$$

*where $\langle K_p v\rangle$ denotes the subspace spanned by $K_p v$.*

**2.6.** Recall that for two unitary representations $\rho_1$ and $\rho_2$ of $G(\mathbb{Q}_p)$, $\rho_1$ is said to be weakly contained in $\rho_2$ if every diagonal matrix coefficients of $\rho_1$ can be approximated uniformly on compact sets by convex combinations of diagonal matrix coefficients of $\rho_2$. For each $p \in R_f$, denote by $\hat{G}_p$ the unitary dual of $G(\mathbb{Q}_p)$ and by $\hat{G}_p^{Aut} \subset \hat{G}_p$ the set of irreducible unitary representations of $G(\mathbb{Q}_p)$ which are weakly contained in the representations appearing as $\mathbb{Q}_p$-components of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/O_f)$ for some compact open subgroup $O_f \subset G(\mathbb{A}_f)$.

We are now ready to prove a main proposition which provides relation between Hecke operators and unitary representations.

**Proposition.** *Let $G$ be a connected almost simple simply connected $\mathbb{Q}$-group with $G(\mathbb{R})$ non-compact and $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup of the form $\Gamma = G(\mathbb{Q}) \cap (G(\mathbb{R}) \times \prod_{p\in R_f} U_p)$. Suppose that for each $p \in R_f$, there exists a bi-$K_p$-invariant positive function $F_p$ on the group $G(\mathbb{Q}_p)$ such that for any non-trivial $\rho_p \in \hat{G}_p^{Aut}$ with $K_p$-finite unit vectors $v$ and $w$,*

$$|\langle \rho_p(g)v, w\rangle| \le (dim\langle K_p v\rangle dim\langle K_p w\rangle)^{1/2} \cdot F_p(g) \ \text{ for any } g \in G(\mathbb{Q}_p).$$

*Assume moreover that $F_p(e) = 1$ for almost all $p \in R_f$. Then for any $a \in G(\mathbb{Q})$,*

$$\|T_a^0\| \leq C \prod_{p \in R_f} F_p(a)$$

*where $C = \prod_{p \in R_f} [K_p : K_p \cap U_p]$.*

*Proof.* Let $S \subset R$ be a finite subset such that $R_f - S \subset R_h$ and $U_p = K_p$ for each $p \in R_f - S$. We assume moreover that $a \in K_p$ for all $p \in R_f - S$. Write $\mathcal{L}_0 = L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. As a representation of $G_S$, we can write

$$\mathcal{L}_0 = \int_{x \in X} m_x \rho_x d\nu(x)$$

where $X$, $\rho_x$, $m_x$ and $\nu$ are as in 2.4. If $\mathcal{L}_x \cong \oplus^{m_x} \rho_x$ is the corresponding summand,

$$\langle v, w \rangle = \int_{x \in X} \langle v_x, w_x \rangle_{\mathcal{L}_x} d\nu(x).$$

As in 2.3, $T_a^0$ acts on $\mathcal{L}_0^{U_f}$ by the product $\prod_{p \in S} T_{a(p)}$ where each $T_{a(p)}$ acts as a local Hecke operator on the $p$-factor $\rho_{x(p)}^{U_p}$ of $\rho_x$ as follows: if $v$ is a $U_p$-invariant vector, then

$$T_{a(p)}(v) = \frac{1}{\deg_p(a)} \chi_{U_p a U_p} * v = \frac{1}{\deg_p(a)} \int_{G(\mathbb{Q}_p)} \chi_{U_p a U_p}(g)\, \rho_{x(p)}(g)(v)\, d\mu_p(g)$$

where $\mu_p$ is the Haar measure on $G(\mathbb{Q}_p)$ with $\mu_p(U_p) = 1$. Hence if $U_p a U_p = \amalg_{j_p=1}^{\deg_p(a)} U_p a_{j_p}$,

$$T_{a(p)}(v) = \frac{1}{\deg_p(a)} \sum_{j_p=1}^{\deg_p(a)} \rho_{x(p)}(a_{j_p}) v.$$

By the integral representation, it now suffices to show that for each $p \in S \cap R_f$ and for any $U_p$-invariant vectors $v$, $w$ in the space of $\rho_{x(p)}$,

$$|\langle T_{a(p)}(v), w \rangle| \leq [K_p : K_p \cap U_p]\, F_p(a)\, \|v\| \cdot \|w\|.$$

Let $p \in S \cap R_f$. Since $a_{j_p} \in U_p a U_p$ for each $1 \leq j_p \leq \deg_p(a)$, and $v$ and $w$ are $U_p$-invariant vectors, we have

$$\langle T_{a(p)} v, w \rangle = \langle \rho_{x(p)}(a) v, w \rangle.$$

Since $\rho_{x(p)}$ is non-trivial (Proposition 2.4) and the dimensions of $\langle K_p v \rangle$ and $\langle K_p w \rangle$ are at most $[K_p : K_p \cap U_p]$ (Lemma 2.5), we have

$$|\langle T_{a(p)} v, w \rangle| \leq [K_p : K_p \cap U_p]\, F_p(a)\, \|v\| \cdot \|w\|$$

by the assumption. Integrating over $X$, we conclude that for any $v$ and $w \in \mathcal{L}_0$,

$$|\langle \hat{T}_a v, w\rangle| \leq C \prod_{p\in R_f} F_p(a)\, \|v\| \cdot \|w\|$$

by the Cauchy-Schwartz inequality, where $C$ is the product of the finite primes $p$ such that $U_p \neq K_p$ of $[K_p, K_p \cap U_p]$. Since the map $\phi$ in 2.1 is an isometry, this finishes the proof. □

Since $K_p$ and $U_p$ are commensurable for each $p \in R_f$ and $K_p = U_p$ for almost all $p \in R_f$, the product $\prod_{p\in R_f} [K_p : K_p \cap U_p]$ is well defined.

*2.7. Remark.* In fact, in the above proposition, we obtain that

$$\|T_a^0\| \leq \prod_{p\in R_0} ([K_p : K_p \cap U_p]\, F_p(a))$$

for any finite subset $R_0 \subset R_f$. This follows from the observation that for any $p \in R_f - R_0$, we can use the trivial upper bound $\|v\| \cdot \|w\|$ for $\langle T_{a(p)} v, w\rangle$ instead of $[K_p : K_p \cap U_p]\, F_p(a)$. This fact will be used in 4.2.

## 3. Bounds for matrix coefficients

**3.1.** In this subsection, let $G_p$ be any connected reductive $\mathbb{Q}_p$-group with $G_p/Z(G_p)$ almost $\mathbb{Q}_p$-simple. Let $A_p$ be a maximal $\mathbb{Q}_p$-split torus, $B_p$ a minimal parabolic subgroup of $G_p$ containing $A_p$ and $K_p$ a good maximal compact subgroup of $G_p(\mathbb{Q}_p)$ with Cartan decomposition $G_p(\mathbb{Q}_p) = K_p A_p^+ \Omega_p K_p$ where $A_p^+$ denotes the closed positive Weyl chamber. In particular, for any $g \in G_p(\mathbb{Q}_p)$, there exist unique elements $a \in A_p^+$ and $d \in \Omega_p$ such that $g \in K_p a d K_p$. We remark that $\Omega_p = \{e\}$ for any $p \in R_h$ (see 2.1). Let $\mathcal{S}_p$ be a strongly orthogonal system of $\Phi_p$ where $\Phi_p$ denotes the set of non-multipliable roots in the relative root system $\Phi(G_p, A_p)$. Recall the bi-$K_p$-invariant function $\xi_{\mathcal{S}_p}$ from the introduction. In particular, if we set

$$n_{\mathcal{S}_p}(g) = \frac{1}{2} \sum_{\alpha\in\mathcal{S}_p} \log_p |\alpha(g)|_p,$$

then

$$\xi_{\mathcal{S}_p}(g) = p^{-n_{\mathcal{S}_p}(g)} \prod_{\alpha\in\mathcal{S}_p} \left( \frac{(\log_p |\alpha(g)|_p)(p-1) + (p+1)}{p+1} \right).$$

It follows that for any $\epsilon > 0$, there exists constants $C_1$ and $C_2(\epsilon)$ such that for any $g \in G(\mathbb{Q}_p)$,

$$C_1\, p^{-n_{\mathcal{S}_p}(g)} \leq \xi_{\mathcal{S}_p}(g) \leq C_2(\epsilon)\, p^{-n_{\mathcal{S}_p}(g)(1-\epsilon)}$$

(see [Oh2, 5.9]).

**Theorem** [Oh2, Theorem 1.1]**.** *Assume that the semisimple $\mathbb{Q}_p$-rank of $G_p$ is at least 2. Let $\mathcal{S}_p$ be any strongly orthogonal system of $\Phi_p$. Then for any irreducible infinite dimensional unitary representation $\rho$ of $G_p(\mathbb{Q}_p)$ with $K_p$-finite unit vectors $v$ and $w$,*

$$|\langle\rho(g)v,w\rangle| \le \big([K_p : K_p \cap dK_pd^{-1}]\, dim\langle K_pv\rangle\, dim\langle K_pw\rangle\big)^{1/2} \cdot \xi_{\mathcal{S}_p}(g)$$

*for any $g = k_1adk_2 \in K_pA_p^+\Omega_pK_p$.*

Note that if $G_p(\mathbb{Q}_p)$ is non-compact almost $\mathbb{Q}_p$-simple and simply connected, then any non-trivial irreducible unitary representation is of infinite dimension.

A similar statement has been proven for any local field not of characteristic 2 in [Oh2].

**3.2.** We now recall that the Ramanujan conjecture at a finite prime $p$ asserts that for $G = SL_2$, any non-trivial representation $\rho_p$ in $\hat{G}_p^{Aut}$ (see 2.6) is tempered, that is, for any $SL_2(\mathbb{Z}_p)$-finite unit vectors $v$ and $w$,

$$|\langle\rho_p(g)v,w\rangle| \le (\dim\langle K_pv\rangle \dim\langle K_pw\rangle)^{1/2} \cdot \Xi_{SL_2(\mathbb{Q}_p)}(g) \;\text{ for any } g \in SL_2(\mathbb{Q}_p).$$

Note that $\Xi_{SL_2(\mathbb{Q}_p)}(g) = \Xi_{PGL_2(\mathbb{Q}_p)}[g]$ when $[g]$ denotes the image of $g$ under the natural projection $SL_2(\mathbb{Q}_p) \to PGL_2(\mathbb{Q}_p)$.

It follows from the Gelbart-Jacquet estimate toward the Ramanujan conjecture [GJ] that for any non-trivial $\rho_p \in \hat{G}_p^{Aut}$, the matrix coefficients of $\rho_p$ with respect to $SL_2(\mathbb{Z}_p)$-finite vectors are $L^{4+\epsilon}$-integrable for any $\epsilon > 0$. Then applying [CHH], we obtain that for any $SL_2(\mathbb{Z}_p)$-finite unit vectors $v$ and $w$,

$$|\langle\rho_p(g)v,w\rangle| \le (\dim\langle K_pv\rangle \dim\langle K_pw\rangle)^{1/2} \cdot \Xi^{1/2}_{SL_2(\mathbb{Q}_p)}(g) \;\text{ for any } g \in SL_2(\mathbb{Q}_p).$$

**3.3.** The following $p$-adic analogue of Burger and Sarnak's results on lifting the automorphic bound of a $\mathbb{Q}$-subgroup to the ambient $\mathbb{Q}$-group [BS, Theorem 1.1] is obtained in [CU].

**Theorem** [CU, Theorem 5.1]**.** *Let $G$ be a connected almost simple simply connected $\mathbb{Q}$-group and $H$ a connected semisimple $\mathbb{Q}$-subgroup of $G$. Then for any $p \in R_f$ and for any $\rho_p \in \hat{G}_p^{Aut}$, any irreducible unitary representation of $H(\mathbb{Q}_p)$ weakly contained in $\rho_p|_{H(\mathbb{Q}_p)}$ is contained in $\hat{H}_p^{Aut}$.*

**3.4. Theorem.** *Let $G$ be a connected almost simple simply connected $\mathbb{Q}$-group with $rank_{\mathbb{Q}} G = 1$. Let $p \in R_f$ such that $rank_{\mathbb{Q}_p} G = 1$. Then for any non-trivial representation $\rho_p \in \hat{G}_p^{Aut}$ with $K_p$-finite unit vectors $v$ and $w$,*

$$|\langle\rho_p(g)v,w\rangle| \le \big([K_p : K_p \cap dK_pd^{-1}]\, dim\langle K_pv\rangle\, dim\langle K_pw\rangle\big)^{1/2} \cdot \xi^{1/2}_{\mathcal{S}_p}(g)$$

*for any* $g = k_1 a d k_2 \in K_p A_p^+ \Omega_p K_p$ *and for the strongly orthogonal system* $\mathcal{S}_p = \{\alpha_p\}$ *where* $\alpha_p$ *is the simple root in* $\Phi_p$.

*Proof.* We can find a connected semisimple $\mathbb{Q}$-split $\mathbb{Q}$-subgroup $H$ of $G$ containing a maximal $\mathbb{Q}$-split torus of $G$ and whose root system is equal to the set of non-multipliable $\mathbb{Q}$-roots of $G$ [BT]. Since $\mathrm{rank}_{\mathbb{Q}}(G) = 1$, $H$ is necessarily isomorphic to $SL_2$. We may assume that for a maximal $\mathbb{Q}$-split torus $A$ of $H$, $A_p = A$ and $SL_2(\mathbb{Z}_p) \subset K_p$. Hence $H \cap \Gamma$ is a congruence subgroup of $H$ and if $g = k_1 a d k_2 \in K_p A_p^+ \Omega_p K_p = G(\mathbb{Q}_p)$,

$$\langle \rho_p(g) v, w\rangle = \langle \rho_p(a)(dk_2 v), (k_1^{-1} w)\rangle.$$

It is not difficult to see that $dk_2 v$ is also $K_p$-finite and the dimension of the subspace $\langle K_p d k_2 v\rangle$ is bounded by $[K_p : K_p \cap dK_p d^{-1}]\dim\langle K_p v\rangle$ (see [Oh2, Lemma 5.6]). Since $a \in H$, by applying Theorem 3.3 with the known bound of the Ramanujan conjecture discussed in 3.2, the claim follows. □

## 4. Proof of main theorems

**4.1. Proof of Theorem 1.1.** Without loss of generality, we may assume that each $K_p$ in section 2.1 is a good maximal compact subgroup in the sense of [Oh2]. By Proposition 2.6, Theorem 3.1, Proposition 3.4, it suffices to set $F_p = \max_{d\in\Omega_p}[K_p : K_p \cap dK_p d^{-1}] \cdot \xi_{\mathcal{S}_p}$ if $p \in R_2$, $F_p = \max_{d\in\Omega_p}[K_p : K_p \cap dK_p d^{-1}] \cdot \xi_{\mathcal{S}_p}^{1/2}$ if $p \in R_1$ and $F_p \equiv 1$ for $p \notin R_1 \cup R_2$. □.

*Remark.* From the proof, the constant $C$ depending on $\Gamma$ in Theorem 1.1 can be made explicit. In fact,

$$C_\Gamma = \prod_{p \in R_f - R_h} \left( \max_{d\in\Omega_p} [K_p : K_p \cap dK_p d^{-1}]\, [K_p : K_p \cap U_p] \right).$$

(1) If $G$ is split over $\mathbb{Q}$ and $\Gamma = G(\mathbb{Z})$, then $K_p = G(\mathbb{Z}_p) = U_p$ and $\Omega_p = \{e\}$ for each $p \in R_f$, and hence $C_\Gamma = 1$.
(2) If $\Gamma'$ is a congruence subgroup of $\Gamma$, then $C_{\Gamma'} \le [\Gamma : \Gamma'] C_\Gamma$. To see this, letting $\Gamma' = G(\mathbb{Q}) \cap (G(\mathbb{R}) \times U_f')$, by a similar argument to the proof of Lemma 2.2, we can obtain a bijection between $\Gamma'\backslash\Gamma$ and $U_f'\backslash U_f$. Hence

$$\frac{C_{\Gamma'}}{C_\Gamma} = [K_f \cap U_f : K_f \cap U_f'] \le [U_f : U_f'] = [\Gamma : \Gamma'].$$

**4.2.** The above constant $C_\Gamma$ increases (exponentially) whenever $\Gamma$ goes deeper in its congruence level, independent of $a$. Therefore fixing $a$, the norms of $T_a$ with respect to a family of congruence subgroups are more sharply estimated in the following statement, which is immediate from 4.1 in view of Remark 2.7.

**Theorem.** *With the same notation as in Theorem 1.1, set*

$$R_{\Gamma,a} = \{p \in R_f \mid a \notin U_p \cup K_p\Omega_p\};\ \textit{and}$$

$$C_{\Gamma,a} = \prod_{p\in R_\Gamma(a)} \left(\max_{d\in\Omega_p}[K_p : dK_pd^{-1}]\,[K_p : K_p \cap U_p]\right).$$

*Then*

$$\|T_a^0\| \le \begin{cases} C_{\Gamma,a}\left(\prod_{p\in R_1\cap R_{\Gamma,a}} \xi_{\mathcal{S}_p}^{1/2}(a)\right)\left(\prod_{p\in R_2\cap R_{\Gamma,a}} \xi_{\mathcal{S}_p}(a)\right) & \textit{if } rank_{\mathbb{Q}}G \ge 1; \\ C_{\Gamma,a}\left(\prod_{p\in R_2\cap R_{\Gamma,a}} \xi_{\mathcal{S}_p}(a)\right) & \textit{if } rank_{\mathbb{Q}}G = 0 \end{cases}$$

We remark that for example, if $U_p \subset K_p$ for all $p \in R_f$, we have $\prod_{p\in R_{\Gamma,a}} \xi_{\mathcal{S}_p}(a) = \prod_{p\in R_f} \xi_{\mathcal{S}_p}(a)$. By a similar argument as in the remark in 4.1, we can show that $C_{\Gamma',a}/C_{\Gamma,a} \le [\Gamma : \Gamma']$ for any congruence subgroup $\Gamma'$ of $\Gamma$. Note that fixing $a \in G(\mathbb{Q})$, $C_{\Gamma,a}$ is bounded by the constant $\prod_{p\in R_f - R_h} \max_{d\in\Omega_p}[K_p : dK_pd^{-1}]$ for any congruence subgroup $\Gamma$ as long as $a \in U_p \cup K_p$ whenever $K_p \neq U_p$. Roughly speaking, $C_{\Gamma,a}$ remains unchanged for a family of congruence subgroups $\Gamma$ as long as the primes defining this family does not overlap with the primes appearing in the prime factorization of $a$.

**4.3. Example.** Let $G = SL_n$, $(n \ge 2)$. Let $N$ be the unipotent radical of a parabolic subgroup of $G$ contained in the lower (or upper) triangular subgroup. Let $J_N = \{(i,j) \mid I_n + E_{ij} \in N\}$ where $E_{ij}$ is the elementary matrix whose only non-zero element is 1 at $(i,j)$-entry and $I_n$ is the identity matrix. Fix a positive integer $k$ and set

$$\Gamma_{q^k} = \{(a_{ij}) \in SL_n(\mathbb{Z}) \mid q^k | a_{ij} \text{ for all } (i,j) \in J_N\}.$$

Then $\Gamma_{q^k} = SL_n(\mathbb{Q}) \cap (SL_n(\mathbb{R}) \times U_f)$ where $U_p = SL_n(\mathbb{Z}_p) = K_p$ for all finite prime $p \neq q$ and $U_q = \{(a_{ij}) \in SL_n(\mathbb{Z}_q) \mid a_{ij} \in q^k\mathbb{Z}_q \text{ for all } (i,j) \in J_N\}$. These $\Gamma_{q^k}$'s are (Hecke) congruence subgroups of $SL_n$. If $a = \mathrm{diag}(m_1, \cdots, m_n) \in SL_n(\mathbb{Q})$ is such that each $m_i$ is relatively prime to $q$, then $a \in U_q$. Note that $R_{\Gamma,a}$ is precisely the set of finite primes which appear in the prime factorization of $m_i$ for some $1 \le i \le n$. Since $q \notin R_{\Gamma,a}$, we have $K_p = U_p$ for all $p \in R_{\Gamma,a}$ and hence $C_{\Gamma_{q^k},a} = 1$ for any $k$ and $N$ (while $C_{\Gamma_{q^k}} = [SL_n(\mathbb{Z}_q) : U_q]$).

Theorem 4.2 then implies that if $n \ge 3$, the norm $\|T_a^0\|$ with respect to any congruence subgroup of the form $\Gamma_{q^k}$ is bounded above by $\prod_{p\in R_{\Gamma,a}} \xi_{\mathcal{S}_p}(a)$, which is equal to $\prod_{p\in R_f} \xi_{\mathcal{S}_p}(a)$ in this case.

**4.4. Proof of Corollary 1.4.** We first state a simple lemma concerning $\deg(a)$.

**Lemma.** *Let $a_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$. If $\deg(a_n) \to \infty$ as $n \to \infty$, then either (i) there exists a strictly increasing sequence of primes $p_n$ tending to $\infty$ such that $a_n \notin K_{p_n}\Omega_{p_n}$ for all $n \in \mathbb{N}$, or (ii) $\{a_n\}$ is unbounded in $G(\mathbb{Q}_p)$ for some prime $p \in R_f$.*

*Proof.* If (i) does not hold, then there exists some prime $q$ such that for all $n \in \mathbb{N}$, $a_n \in K_p\Omega_p$ for each prime $p > q$. Hence for any prime $p > q$ and for any $n \in \mathbb{N}$, $\deg_p(a_n) = U_p\backslash U_p a_n U_p$ is bounded by the number, say, $N$, of cosets in $U_p\backslash(U_pK_p\Omega_pU_p)$. Since both $U_p$ and $U_pK_p\Omega_pU_p$ are open compact subsets, $N$ is finite. Consider the diagonal embedding of $\{a_n\}$ into $\prod_{p\le q} G_p(\mathbb{Q}_p)$. If the image of $\{a_n\}$ under this embedding is bounded, and hence lies in some open compact subset, there exists some $M > 0$ such that $\deg_p(a_n) \le M$ for all $n \in \mathbb{N}$ and $p \le q$. Since $\deg(a_n) = \prod_{p\in R_f} \deg_p(a_n)$ by Lemma 2.2, it contradicts the assumption that $\deg(a_n)$ tends to $\infty$. Hence the image of $\{a_n\}$ in $\prod_{p\le q} G_p(\mathbb{Q}_p)$ is unbounded. It follows that for some prime $p$, the sequence $\{a_n\}$ is unbounded in $G(\mathbb{Q}_p)$, yielding the claim. □

To deduce Corollary 1.4 from Theorem 1.1, for each $p \in R_f$, let $\mathcal{S}_p$ be a large strongly orthogonal system of $\Phi_p$, for instance $\mathcal{S}_p = \{\gamma_p\}$ for the highest root $\gamma_p$ in $\Phi_p$. For the case (i) in the above lemma, we have for any $\epsilon > 0$, $\prod_{p\in R_f} \xi_{\mathcal{S}_p}(a_n) \le \xi_{\mathcal{S}_{p_n}}(a_n) \le d\cdot p_n^{-1/2+\epsilon}$ for some constant $d$ independent of $n$. This follows from the fact that if $\mathcal{S}_p$ is large and $a \notin K_p\Omega_p$, then

$$\xi_{\mathcal{S}_p}(a) \le \Xi_{PGL_2(\mathbb{Q}_p)}\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}$$

(cf. [Oh2]). Hence

$$\lim_{n\to\infty} \prod_{p\in R_f} \xi_{\mathcal{S}_p}(a_n) = 0.$$

Now for the case (ii), since $\{a_n\}$ is unbounded in $G(\mathbb{Q}_p)$, the $A_p^+$-part, say $A_p^+(a_n)$, of $a_n$ in the Cartan decomposition $K_pA_p^+\Omega_pK_p$ tends to $\infty$ as $n\to\infty$. Since $\mathcal{S}_p$ is large, it implies that $\prod_{\alpha\in\mathcal{S}_p} \alpha(A^+(a_n)) \to \infty$ and hence

$$\lim_{n\to\infty} \xi_{\mathcal{S}_p}(a_n) = 0.$$

Since $\prod_{p\in R_f} \xi_{\mathcal{S}_p}(a_n) \le \xi_{\mathcal{S}_p}(a_n)$, we obtain $\lim_{n\to\infty}\prod_{p\in R_f} \xi_{\mathcal{S}_p}(a_n) = 0$. Hence Theorem 1.1 implies Corollary 1.4.

**4.5.** Theorem 1.6 can be easily deduced from Theorem 1.1. In fact, let $G$ be a connected semisimple Lie group and $\Gamma$ a lattice in $G$. Let $\mu$ denote the normalized Haar measure on $\Gamma\backslash G$. For a finite subset $E \subset \Gamma\backslash G$ invariant by $\Gamma$ on the right, we define an operator $T_E$ on $L^2(\Gamma\backslash G)$ by

$$T_Ef(x) = \frac{1}{|E|}\sum_{y\in E} f(yx).$$

Then in the same setting as in the introduction, the Hecke operator $T_a$ is equal to $T_{\Gamma a \Gamma}$. Namely the following proposition can be proved in a standard way (cf. [CU, Proposition 8.1]).

**Proposition.** *Let $E_n \subset \Gamma\backslash G$ be a sequence of finite subsets invariant by $\Gamma$ on the right such that for any $f \in L^2(\Gamma\backslash G)$,*

$$\lim_{n\to\infty} \|T_{E_n} f - \int_{\Gamma\backslash G} f(x) d\mu(x)\| = 0.$$

*Then for any continuous function $f$ on $\Gamma\backslash G$ with compact support and for any $x \in \Gamma\backslash G$,*

$$\lim_{n\to\infty} T_{E_n} f(x) = \int_{\Gamma\backslash G} f(x) d\mu(x).$$

**4.6.** The following theorem now provides a passage from an $L^2$-norm estimate of the Hecke operator $T_a$ to an $L^\infty$-norm estimate of $T_a$.

**Theorem** [CU, Proposition 8.2]**.** *Let $G$ be a connected semisimple algebraic $\mathbb{Q}$-group and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. Let $f$ be any smooth function on $\Gamma\backslash G(\mathbb{R})$ with compact support. Then for any $x \in \Gamma\backslash G(\mathbb{R})$,*

$$|T_a f(x) - \int_{\Gamma\backslash G(\mathbb{R})} f(x) d\mu(x)| \le C \cdot \|T_a f - \int_{\Gamma\backslash G(\mathbb{R})} f(x) d\mu(x)\|$$

*for some constant $C$ (depending on $f$). Here $C$ can be taken uniformly over compact subsets of $\Gamma\backslash G(\mathbb{R})$.*

The above statement holds for a more general class of functions. Let $G(\mathbb{R}) = K \exp \mathfrak{p}$ be the Cartan decompositions of $G(\mathbb{R})$. Denote by $\Omega_G$ and by $\Omega_K$ the Casimir operators of $G(\mathbb{R})$ and $K$ respectively. Then $D = \Omega_G - 2\Omega_K$ is elliptic as a differential operator on $G(\mathbb{R})$. Let $m$ be an integer $\ge \frac{1}{2}(\dim(G(\mathbb{R})) + 1)$. Then the above theorem holds for any $f \in L^2(\Gamma\backslash G(\mathbb{R}))$ such that $f * D^m \in L^2(\Gamma\backslash G(\mathbb{R}))$ [CU].

Now Proposition 4.5 and Theorem 4.6, combined with Theorem 1.1, imply Theorem 1.6 and 1.7.

**4.7.** We give some explanation about the remark (3) following Theorem 1.7. Let $G$ be $GL_n$ $(n \ge 3)$ and $GSp_{2n}$ $(n \ge 2)$ and $\Gamma = G(\mathbb{Z})$. Then $\Gamma = G(\mathbb{Q}) \cap (G(\mathbb{R}) \times U_f)$ where $U_f = \prod_{p \in R_f} G(\mathbb{Z}_p)$. Let $Z$ denote the center of $G$. It then follows from the strong approximation of the derived group $SL_n$ and $Sp_{2n}$ respectively that the spaces $(Z(\mathbb{R})\Gamma)\backslash G(\mathbb{R})$ and $(Z(\mathbb{A})GL_n(\mathbb{Q}))\backslash G(\mathbb{A})/U_f$ can be identified, and the map $\phi$ defined in 2.1 provides an isometry between the spaces of $L^2$-functions on these two spaces.

In particular, note that $(Z(\mathbb{R})\Gamma)\backslash GL_n(\mathbb{R})$ can be identified with $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$. The statements made in the rest of section 2 then hold for the Hecke operators $T_a^0$, $a \in G(\mathbb{Q})$, acting on the subspace $L_0^2((Z(\mathbb{R})\Gamma)\backslash G(\mathbb{R}))$. Note that Theorem 3.1 holds for $G$. Hence the proof of Theorem 1.1 given in 4.1 can be carried over to $G$ in the same way. As for Corollary 1.4, we only need to note that Lemma 4.4 for $G$ can be stated as follows: if $\deg(a_n) \to \infty$ and $a_n$ is bounded in $Z(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ for all $p \in R_f$, then $a_n \notin K_{p_n}\Omega_{p_n}Z(\mathbb{Q}_{p_n})$ for a sequence $p_n$ tending to $\infty$. Finally Proposition 4.5 and Theorem 4.6, hold for $Z(\mathbb{R})\Gamma\backslash G(\mathbb{R})$ for any connected reductive $\mathbb{Q}$-group $G$ and any arithmetic subgroup $\Gamma$ (see [CU]). In the case for $G = GL_2 = GSp_2$, we only need to note that Theorem 3.4 holds for $GL_2$ by the above argument.

## 5. Examples and Theorem 1.5

**5.1.** Let $G = SL_n$ or $GL_n$ for $n \geq 3$ and $\Gamma = G(\mathbb{Z})$. Let $a = \mathrm{diag}(a_1, \cdots, a_n) \in G$ with $a_i \geq a_{i+1} > 0$ for each $1 \leq i \leq n-1$. For each $i = 1, \cdots, [n/2]$, set

$$\gamma_i(a) = \frac{a_i}{a_{n+1-i}}.$$

Then $\mathcal{S} = \{\gamma_i \mid 1 \leq i \leq [n/2]\}$ is a maximal strongly orthogonal system of $\Phi(G, A)$ with $A$ as in the introduction [Oh1, Proposition 2.4]. Recall the notation $A^+$ from the introduction if $G = SL_n$. For $G = GL_n$, set

$$A^+ = \{\mathrm{diag}(a_1, \cdots, a_n) \mid a_i \in \mathbb{N},\ a_{i+1}|a_i \text{ for each } 1 \leq i \leq n-1,\ a_n = 1\}.$$

For $G = SL_n$, every double coset $\Gamma g \Gamma$ ($g \in G(\mathbb{Q})$) has a representative of the form $\mathrm{diag}(a_1, \cdots, a_n) \in A^+$. For $G = GL_n$, every double coset $\Gamma g \Gamma$ ($g \in G(\mathbb{Q})$) has a representative of the form $c\,a$ for $c \in Z(\mathbb{Q})$ and $a = \mathrm{diag}(a_1, \cdots, a_n) \in A^+$ (cf. [Fr]). Hence the Hecke operators $T_g$ and $T_{\mathrm{diag}(a_1,\cdots,a_n)}$ coincide. By an estimate of $\xi_{\mathcal{S}}$ given in 3.1, we have that for any $\epsilon > 0$, there exists a constant $C > 1$ such that

$$\prod_{p \in R_f} \prod_{i=1}^{[n/2]} \left( \frac{|a_i|_p}{|a_{n+1-i}|_p} \right)^{1/2} \leq \prod_{p \in R_f} \xi_{\mathcal{S}}(a) \leq C \cdot \prod_{p \in R_f} \prod_{i=1}^{[n/2]} \left( \frac{|a_i|_p}{|a_{n+1-i}|_p} \right)^{1/2-\epsilon}$$

for any $a \in A^+$.

Hence Theorem 1.1 yields the following: for any $\epsilon > 0$, there exists a constant $C$ (depending only on $\epsilon$) such that

$$\|T_a^0\| \leq C \prod_{i=1}^{[n/2]} \left( \frac{a_i}{a_{n+1-i}} \right)^{-1/2+\epsilon}$$

for all $a \in A^+$. Here if $G = GL_n$, $T_a$ is considered as an operator on $L^2(Z(\mathbb{R})\Gamma\backslash G)$. For $G = GL_n$ and all $a_i$ square-free, the above inequality was obtained in [CU] using the description of the residual spectrum for $GL_n$ of Moeglin and Waldspurger.

**5.2. Proof of Corollary 1.9.** Assume that $m = p^r$ for some positive integer $r$. Set

$$D(m) = \{(k_1, \cdots, k_n) \in \mathbb{Z}^n \mid k_i \geq k_{i+1} \text{ for each } 1 \leq i \leq n-1, k_n = 1, \sum_{i=1}^{n} k_i = r\}.$$

For each $(k_1, \cdots, k_n) \in D(m)$, we set $a(k_1, \cdots, k_n) := \operatorname{diag}(p^{k_1}, \cdots, p^{k_n})$. Then

$$X_{\bar{\Lambda}}(m) = \amalg_{(k_1,\cdots,k_n)\in D(m)} T_{a(k_1,\cdots,k_n)}(\bar{\Lambda}).$$

Hence

$$|X_{\bar{\Lambda}}(m)| = \sum_{(k_1,\cdots,k_n)\in D(m)} \deg(a(k_1, \cdots, k_n)).$$

Using the degree formula in Proposition 7.4 in [Gr], we obtain that for each $(k_1, \cdots, k_n) \in D(m)$,

$$\deg(a(k_1, \cdots, k_n)) = p^{\sum_{i=1}^{[n/2]}((k_i - k_{n+1-i})(n-(2i-1))} \left(1 + O(p^{-1})\right)$$

where the big O depends only on $n$. On the other hand for a maximal strongly orthogonal system $\mathcal{S}$ in 5.1, we have that for any $\epsilon > 0$,

$$p^{-1/2\left(\sum_{i=1}^{[n/2]}(k_i - k_{n+1-i})\right)} \leq \prod_{p\in R_f} \xi_{\mathcal{S}}(a(k_1, \cdots, k_n)) \leq C(\epsilon) \cdot p^{-1/2\left(\sum_{i=1}^{[n/2]}(k_i - k_{n+1-i})\right)(1+\epsilon)}.$$

Note that $\sum_{i=1}^{[n/2]}(k_i - k_{n+1-i}) \leq r$. Then

$$\sum_{i=1}^{[n/2]}(n-(2i-1))(k_i - k_{n+1-i}) \leq (n-1)k_1 + (n-3)(r-k_1) = (n-1)r - 2(r-k_1) \leq (n-1)r.$$

For simplicity, let $a_m = a(r, 1, \cdots, 1)$, which is equal to $(m, 1, \cdots, 1)$. Then

$$\deg(a_m) = p^{(n-1)r}(1 + O(p^{-1}))$$

and for any $\epsilon > 0$, there exists a constant $C_\epsilon$ such that

$$p^{(n-3/2)r(1-\epsilon)}(1 + O(p^{-1})) \leq \deg(a_m) \prod_{p\in R_f} \xi_{\mathcal{S}}(a_m) \leq C_\epsilon \cdot p^{(n-3/2)(r+\epsilon)}(1 + O(p^{-1})).$$

On the other hand, if $(k_1, \cdots, k_n)$ is an element in $D(m)$ different from $(r, 1 \cdots, 1)$, then $r - k_1 \geq 1$ and hence

$$\sum_{i=1}^{[n/2]}(n-(2i-1))(k_i - k_{n+1-i}) \leq (n-1)r - 2.$$

Hence for all $(k_1, \cdots, k_n) \in D(m) - \{(r, 1, \cdots, 1)\}$,

$$\deg(a(k_1, \cdots, k_n)) \le p^{(n-1)r-2}(1 + O(p^{-1})).$$

Since the exponent of $p$ in $\deg(a(k_1, \cdots, k_n)) \cdot \prod_{p \in R_f} \xi_{\mathcal{S}}(a(k_1, \cdots, k_n))$ is $\sum_{i=1}^{[n/2]}(n - (2i-1) - 1/2)(k_i - k_{n+1-i})$ up to an $\epsilon$-factor, we also have

$$\deg(a(k_1, \cdots, k_n)) \cdot \prod_{p \in R_f} \xi_{\mathcal{S}}(a(k_1, \cdots, k_n)) \le C_\epsilon \cdot p^{(n-3/2)r(1+\epsilon)-2}.$$

Since $|D(m)| \le (\log m)^n$, the above inequalities imply that the leading exponent of $m$ in the fraction

$$\frac{\sum_{(k_1, \cdots, k_n) \in D(m)} \deg(a(k_1, \cdots, k_n)) \cdot \prod_{p \in R_f} \xi_{\mathcal{S}}(a(k_1, \cdots, k_n))}{\sum_{(k_1, \cdots, k_n) \in D(m)} \deg(a(k_1, \cdots, k_n))}$$

essentially comes from the term

$$\frac{\deg(a_m) \prod_{p \in R_f} \xi_{\mathcal{S}}(a_m)}{\deg(a_m)} = \prod_{p \in R_f} \xi_{\mathcal{S}}(a_m).$$

Since for any $\epsilon > 0$,

$$p^{-r/2} \le \prod_{p \in R_f} \xi_{\mathcal{S}}(a_m) \le C_\epsilon \cdot p^{-r/2(1-\epsilon)},$$

and $p^{-r/2} = m^{-1/2}$, the corollary is proved for $m = p^r$.

It is not hard to see the above process can be generalized to an arbitrary positive integer $m$. Hence Corollary 1.9 follows from Theorem 1.7.

**5.3.** Let $G = Sp_{2n}$ or $GSp_{2n}$ for $n \ge 2$ and $\Gamma = G(\mathbb{Z})$. The group $G$ will be defined by the bi-linear from $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. Let $a = \mathrm{diag}(a_1, \cdots, a_n, b_n, \cdots, b_1) \in G(\mathbb{Q})$ with $a_i \ge 1$ and $b_i > 0$ for each $1 \le i \le n$. For each $i = 1, \cdots, n$, set

$$\gamma_i(a) = \frac{a_i}{b_i}.$$

Then $\mathcal{S} = \{\gamma_1, \cdots, \gamma_n\}$ is a maximal strongly orthogonal system of $\Phi(G, A)$ [Oh1, Proposition 2.3]. Recall $A^+$ from the introduction for $G = Sp_{2n}$. For $G = GSp_{2n}$, set

$$A^+ = \{\mathrm{diag}(a_1, \cdots, a_n, \frac{c}{a_n}, \cdots, \frac{c}{a_1}) \mid c \in \mathbb{Z},\ a_i \in \mathbb{N},\ a_{i+1} | a_i \text{ for each } 1 \le i \le n,\ a_1 | c, c | a_n^2\}.$$

Then any double coset $\Gamma g \Gamma$ ($g \in G(\mathbb{Q})$) has a representative $d\,a$ for $d \in Z(\mathbb{Q})$ and $a = \operatorname{diag}(a_1, \cdots, a_n, \frac{c}{a_n}, \cdots, \frac{c}{a_1}) \in A^+$ [Fr]. Then the Hecke operators $T_g$ and $T_a$ coincide. By an estimate of $\xi_{\mathcal{S}}$ given in 3.1, we have that for any $\epsilon > 0$,

$$\prod_{p \in R_f} \prod_{i=1}^{n} \left( \frac{|a_i|_p^2}{|c|_p} \right)^{1/2} \leq \prod_{p \in R_f} \xi_{\mathcal{S}}(a) \leq C(\epsilon) \cdot \prod_{p \in R_f} \prod_{i=1}^{n} \left( \frac{|a_i|_p^2}{|c|_p} \right)^{1/2-\epsilon}$$

for any $a \in A^+$. Hence Theorem 1.1 yields the following: for any $\epsilon > 0$, there exists a constant $C$ (depending on $\epsilon$) such that

$$\|T_a^0\| \leq C \prod_{i=1}^{n} \left( \frac{a_i^2}{|c|} \right)^{-1/2+\epsilon}.$$

Here if $G = GSp_{2n}$, $T_a$ is considered as an operator on $L^2(Z(\mathbb{R})\Gamma\backslash G(\mathbb{R}))$. For $G = GSp_{2n}$ and $a_i = p = c$ for all $i$, this inequality was obtained in [CU] assuming the Ramanujan conjecture for $SL_2$.

**5.4.** In this subsection, let $G$ be as in 5.1 or 5.3. Let $P_0$ be the maximal parabolic subgroup of $G$ which stabilizes the line containing the standard vector $e_1$. The following theorem with [Oh2, Theorem 1.1] (see Theorem 3.1 above) implies that the unitarily induced representation $\operatorname{Ind}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(1)$ from the trivial representation 1 of $P_0(\mathbb{Q}_p)$ to $G(\mathbb{Q}_p)$ provides the slowest decay in the spherical unitary dual of $G(\mathbb{Q}_p)$.

**Theorem** [Oh2, Theorem 1.3]**.** *Let $p \in R_f$ and $\mathcal{S}$ a maximal strongly orthogonal system of $\Phi_p$. Let $v$ be a $K_p$-invariant unit vector of $\operatorname{Ind}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(1)$. Then for any $\epsilon > 0$, there exists a constant $C$ depending on $\epsilon$ such that*

$$C \cdot \xi_{\mathcal{S}}^{1+\epsilon}(g) \leq \langle \operatorname{Ind}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(1)(g)v, v\rangle \leq \xi_{\mathcal{S}}(g) \quad \textit{for any } g \in G(\mathbb{Q}_p).$$

Recall that $U_f = \prod_{p \in R_f} G(\mathbb{Z}_p)$. For $G = SL_n$ or $Sp_{2n}$, set $X = G(\mathbb{Z})\backslash G(\mathbb{R})$ and for $G = GL_n$ or $GSp_{2n}$, set $X = Z(\mathbb{R})G(\mathbb{Z})\backslash G(\mathbb{R})$. The spectral decomposition theorem says the following (see, e.g., [Ar]):

$$L^2(X) = \bigoplus_{[P]} \bigoplus_{\tau} \int_{\operatorname{Im} X_{M_P}^G} \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tau \otimes s \otimes 1)^{U_f}\, d\mu_\tau(s)$$

where $[P]$ ranges over the associated class of standard parabolic subgroups containing $B$, $\tau$ ranges over irreducible representations occurring in the discrete part of regular representation of the Levi subgroups $M_P$ on $L^2(Z(M_P)(\mathbb{R})M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))$ and $s$ ranges over the subset $\operatorname{Im}(X_{M_P}^G) \subset \operatorname{Im}(X_{M_P})$ of all unitary unramified characters of $M_P(\mathbb{A})$.

The notation $\tau \otimes s \otimes 1$ denotes the representation extended trivially to $P(\mathbb{A})$ from the representation $\tau \otimes s$ on $M_P(\mathbb{A})$.

Set $\pi(1,s) = \mathrm{Ind}_{P_0(\mathbb{A})}^{G(\mathbb{A})}(1 \otimes s \otimes 1)^{U_f}$. If $s = \otimes_{p \in R_f} s_p$, then

$$\int_{\mathrm{Im} X^G_{M_{P_0}}} \pi(1,s) d\mu_\tau(s) = \otimes_{p \in R} \int_{\mathrm{Im} X^G_{M_{P_0}}} \mathrm{Ind}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} (1 \otimes s_p \otimes 1)^{G(\mathbb{Z}_p)} d\mu_{\tau_p}(s_p).$$

Then for a $U_f$-invariant unit vector $v_s = \otimes_{p \in R_f} v_{sp}$ of $\pi(1,s)$, we have

$$\int_{\mathrm{Im} X^G_{M_{P_0}}} \langle \pi(1,s)(a) v_s, v_s \rangle d\mu_\tau(s) = \prod_{p \in R_f} \int_{\mathrm{Im} X^{G(\mathbb{Q}_p)}_{M_{P_0}}} \langle \mathrm{Ind}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} (1 \otimes s_p \otimes 1)(a) v_{sp}, v_{sp} \rangle d\mu_{\tau_p}(s_p).$$

Since each $s_p$ is a unitary character, the above theorem 5.4 is applied for each $s$ and $p$. Hence Theorem 1.5 follows.

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