HARDY-LITTLEWOOD SYSTEM AND REPRESENTATIONS OF INTEGERS BY AN INVARIANT POLYNOMIAL

HEE OH

Abstract. Let f be an integral homogeneous polynomial of degree d, and let $V_m = \{X : f(X) = m\}$ be the level set for each $m \in \mathbb{N}$. For a compact subset Ω in $V_1(\mathbb{R})$, set

$$N_m(f, \Omega) = \# V_m(\mathbb{Z}) \cap m^{1/d}\Omega.$$

We define the notion of Hardy-Littlewood system for the sequence $\{V_m\}$, according as the asymptotic of $N_m(f, \Omega)$ as $m \to \infty$ coincides with the one predicted by Hardy-Littlewood circle method. Using a recent work of Eskin and Oh [EO], we then show for a large family of invariant polynomials f, the level sets $\{V_m\}$ are Hardy-Littlewood. In particular, our results yield a new proof of Siegel mass formula for quadratic forms.

1. Introduction

Let f be an integral homogeneous polynomial in n-variables with degree d. A basic Diophantine problem is to understand the asymptotic number of integer representations of m by f as $m \to \infty$.

For each $m \in \mathbb{N}$, set

$$V_m := \{x \in \mathbb{C}^n : f(x) = m\},$$

and for any compact subset $\Omega \subset V_1(\mathbb{R})$, set

$$N_m(f, \Omega) := \# V_m(\mathbb{Z}) \cap m^{1/d}\Omega.$$

For instance, if

$$\Omega = \{X \in \mathbb{R}^n : f(X) = 1, \ \|X\| \le R\}$$

for a fixed $R > 0$ and for the Euclidean norm $\|\cdot\|$, the number $N_m(f, \Omega)$ is simply the cardinality of the set of integer representations X of m by f, subject to the condition that the norm of X is at most $m^{1/d}R$, i.e.,

$$\#\{X \in \mathbb{Z}^n : f(X) = m, \ \|X\| \le m^{1/d}R\}.$$

When the varieties $V_m(\mathbb{R})$ are not compact, $\# V_m(\mathbb{Z})$ may be infinite. For this reason, it is natural to consider the asymptotic, as $m \to \infty$, of $N_m(f, \Omega)$ for a fixed compact subset $\Omega \subset V_1(\mathbb{R})$. This set up was first considered by Linnik ([Li1],[Li2], see also [Sa]).

Partially supported by DMS-0070544 and DMS-0333397.

When the Hardy-Littlewood circle method can be successfully used to obtain the asymptotic of $N_m(f,\Omega)$, as $m\to\infty$, which usually requires that the number of variables is much larger than the degree of f (cf. [Sc], [Va]), the counting function $N_m(f,\Omega)$ behaves as

$$N_m(f,\Omega)\sim_{m\to\infty} J_m(m^{1/d}\Omega)\cdot\mathfrak{S}_m \tag{1.1}$$

for any nice compact subset $\Omega\subset V_1(\mathbb{R})$. Here $\mathfrak{S}_m$ and J_m, classically called singular series and singular integral respectively, are defined as follows:

$$\mathfrak{S}_m:=\prod_p \lim_{k\to\infty}\frac{\#V_m(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\cdot\dim(V_m)}},$$

and for any nice compact subset $K\subset\mathbb{R}^n$,

$$J_m(K\cap V_m):=\lim_{\epsilon\to 0}\frac{|\{X\in K:|f(X)-m|\le\epsilon\}|}{2\epsilon}$$

where $|\cdot|$ denotes the usual Lebesgue measure on $\mathbb{R}^n$.

While it is easy to check that $J_m(m^{1/d}\Omega)=m^{(n-d)/d}J_1(\Omega)$, the behavior of the singular series $\mathfrak{S}_m$ reflects certain arithmetic complexity of the polynomial f. However when the circle method works, it is usually the case that $\mathfrak{S}_m$ absolutely converges and $\mathfrak{S}_m\gg 1$ (or at least $\mathfrak{S}_m\gg m^{-\epsilon}$) as $m\to\infty$, (cf. [Va]). In particular,

$$J_m(m^{1/d}\Omega)\cdot\mathfrak{S}_m\to\infty \quad \text{as } m\to\infty$$

(assuming $n>d$), which will certainly be the case if the circle method applies.

Even when the Hardy-Littlewood circle method does not directly apply, it is still hoped that for many polynomials f, the asymptotic of $N_m(f,\Omega)$ would be given by (1.1). This is the question we address in this paper, namely, for what kind of polynomial f, the counting function $N_m(f,\Omega)$ as $m\to\infty$ behaves as the Hardy-Littlewood circle method predicts.

Our approach here is to analyze the properties of the level sets $\{V_m\}$, rather than that of a polynomial f. Following the terminology introduced by Schmidt [Sc] and more refined by Borovoi and Rudnick [BR] in the study of integer points of a fixed variety in the sequence of balls of radius going to infinity, we define an analogous notion of Hardy-Littlewood system for a sequence of varieties $\{V_m\}$.

Let V be an affine space with a given $\mathbb{Z}$-structure, and $\{V_m\subset V: m\in\mathbb{N}\}$ be a sequence of non-singular subvarieties defined over $\mathbb{Q}$ such that

$$V_m=\lambda_m V_1$$

for a strictly increasing sequence $\{\lambda_m\}$ of positive real numbers (with $\lambda_1=1$).

For any ring J containing $\mathbb{Z}$, we set $V_m(J)=V_m\cap V(J)$. We denote by $\mathbb{A}$ the adele ring over $\mathbb{Q}$ and $\mathbb{A}_f$ its finite adeles.

A natural substitute for $J_m \cdot \mathfrak{S}_m$ in this generality is given by a Tamagawa measure on $V_m(\mathbb{A})$. For that, we assume that there exists a gauge form, a nowhere zero regular differential form of maximal degree, on each variety V_m, which defines a Tamagawa measure μ_m on $V_m(\mathbb{A})$ (with respect to some convergence factors if needed) [We2].

A compact subset $\Omega \subset V_1(\mathbb{R})$ is said to be *nice* if Ω has non-empty interior with piecewise smooth boundary.

Definition 1.2. *Let $\mathcal{O}$ be a connected component of $V_1(\mathbb{R})$. We call a sequence $\{(V_m : \mathcal{O})\}$* **strongly Hardy-Littlewood** *if for any nice compact subset Ω in $\mathcal{O}$,*

$$\lim_{m\to\infty} \mu_m(\lambda_m\Omega \times \prod_p V_m(\mathbb{Z}_p)) = \infty; \text{ and}$$

$$\#V_m(\mathbb{Z}) \cap \mathbb{R}^+\Omega \sim_{m\to\infty} \mu_m(\lambda_m\Omega \times \prod_p V_m(\mathbb{Z}_p)).$$

Definition 1.3. *We call a sequence $\{(V_m : \mathcal{O})\}$ is* **(relatively) Hardy-Littlewood** *if there exist locally constant functions $\delta_m : V_m(\mathbb{A}) \to \mathbb{R}^+ \cup \{0\}$ constant on connected components of $V_m(\mathbb{R})$, such that for any nice compact subset Ω in $\mathcal{O}$,*

$$\#V_m(\mathbb{Z}) \cap \mathbb{R}^+\Omega \sim_{m\to\infty} \int_{\lambda_m\Omega\times\prod_p V_m(\mathbb{Z}_p)} \delta_m(x)\, d\mu_m$$

with both sides tending to ∞ as $m \to \infty$.

We call δ_m the density function of $V_m(\mathbb{A})$.

Definition 1.4. *A sequence $\{V_m\}$ is called strongly (resp. relatively) Hardy-Littlewood if $\{(V_m : \mathcal{O})\}$ is so for each connected component $\mathcal{O}$ of $V_1(\mathbb{R})$.*

For a sequence of arbitrary varieties V_m, it is of course very hard (almost hopeless) to check whether $\{V_m\}$ is Hardy-Littlewood or not. However when V_m are given as the homogeneous spaces of a semisimple algebraic group G, we are able to formulate certain sufficient conditions for $\{(V_m : \mathcal{O})\}$ to be (relatively or strongly) Hardy-Littlewood. Based on a recent work of Eskin and the author [EO], we then show that a large family of homogeneous spaces $\{(V_m : \mathcal{O})\}$ are Hardy-Littlewood and compute their density functions as well. In particular, there are examples of relatively Hardy-Littlewood sequences but not strongly Hardy-Littlewood (see Example (4.3)).

Let V be a linear space with a $\mathbb{Z}$-structure and $\{V_m = \lambda_m V_1\}$ a sequence of subvarieties of V defined over $\mathbb{Q}$. Let $\mathcal{O}$ denote a connected component in $V_1(\mathbb{R})$. We consider the following hypothesis for $(G, \{(V_m : \mathcal{O})\}, H)$:

(A) Let a $\mathbb{Q}$-rational representation $\rho : G \to \mathrm{GL}(V)$ be given where G is a connected simply connected semisimple algebraic group defined over $\mathbb{Q}$ such that $G(\mathbb{R})$ has no compact factor defined over $\mathbb{Q}$. We assume that for each m, $V_m(\mathbb{Q}) \cap \lambda_m\mathcal{O} \neq \emptyset$.

Suppose that V_1 is a homogeneous space of G, so that $V_1 := v_0 G$ for some $v_0 \in V_1(\mathbb{Q})$, and that the stabilizer H of v_0 in G is connected and semisimple.

Under the hypothesis (A), there exists a G-invariant gauge form on each V_m, unique up to a scalar multiple of $\mathbb{Q}^*$, which induces local invariant measures $\mu_{m,p}$ on $V_m(\mathbb{Q}_p)$ for primes p and $\mu_{m,\infty}$ on $V_m(\mathbb{R})$. By [BR, 1.7], the product $\prod_p \mu_{m,p}(V_m(\mathbb{Z}_p))$ converges absolutely. Hence the Tamagawa measure μ_m on $V_m(\mathbb{A})$ is well defined with respect to the convergence factors $\{1\}_p$, and moreover unique by the product formula [We2, 2.3].

Though Definitions 1.2 and 1.3 might depend on the choice of Tamagawa measure in general, they depend only on $\{V_m\}$ under the hypothesis (A).

Remark Since the group G is simply connected, the group $G(\mathbb{R})$ is connected. On the other hand, each $G(\mathbb{R})$-orbit on $V_m(\mathbb{R})$ is open and closed. Hence the $G(\mathbb{R})$-orbits on $V_m(\mathbb{R})$ are precisely connected components of $V_m(\mathbb{R})$. In particular, if $G(\mathbb{R})$ acts transitively on $V_1(\mathbb{R})$, $V_1(\mathbb{R})$ is connected.

Fix any invariant measure $\mu_{G(\mathbb{R})}$ on $G(\mathbb{R})$. For any $v \in V_m(\mathbb{Z})$, denoting by H_v the stabilizer of v in G, we choose the measure μ_v on $H_v(\mathbb{R})$ in the following way. Since the orbit $vG(\mathbb{R})$ is open in $V_m(\mathbb{R})$, via the isomorphism of $vG(\mathbb{R})$ to $H_v(\mathbb{R})\backslash G(\mathbb{R})$ defined by $vg \mapsto \bar{g} = H_v(\mathbb{R})g$, we first get an invariant measure on $H_v(\mathbb{R})\backslash G(\mathbb{R})$ from the restriction of $\mu_{m,\infty}$ to $vG(\mathbb{R})$. Denote this measure again by $\mu_{m,\infty}$.

Then the Haar measure μ_v on $H_v(\mathbb{R})$ is defined so that $(d\mu_{G(\mathbb{R})}, d\mu_{m,\infty}, d\mu_v)$ match together topologically in the sense of [We2, P. 27], that is, for any $f \in L^1(G(\mathbb{R}), d\mu_{G(\mathbb{R})})$, the integral formula

$$(1.5) \qquad \int_{H_v(\mathbb{R})\backslash G(\mathbb{R})} d\mu_{m,\infty}(\bar{g}) \int_{H_v(\mathbb{R})} f(hg)\, d\mu_v(h) = \int_{G(\mathbb{R})} f(g)\, d\mu_{G(\mathbb{R})}(g)$$

holds. For simplicity, we write $d\mu_{G(\mathbb{R})} = d\mu_{m,\infty} \cdot d\mu_v$ if $(d\mu_{G(\mathbb{R})}, d\mu_{m,\infty}, d\mu_v)$ match together topologically.

In fact, for any invariant measure μ_v on $H_v(\mathbb{R})$, the left hand side of (1.5) defines an invariant measure on $G(\mathbb{R})$. It follows that for given $\mu_{m,\infty}$ and $\mu_{G(\mathbb{R})}$, there exists the unique Haar measure μ_v on $H_v(\mathbb{R})$ such that $d\mu_{G(\mathbb{R})} = d\mu_{m,\infty} \cdot d\mu_v$.

Define for each $m \in \mathbb{N}$ and for each $G(\mathbb{R})$-orbit $\mathcal{O}_m$ in $V_m(\mathbb{R})$

$$(1.6) \qquad \omega_m(\mathcal{O}_m) := \sum_{\xi_m \Gamma \subset V_m(\mathbb{Z}) \cap \mathcal{O}_m} \frac{\mathrm{vol}(\Gamma \cap H_{\xi_m} \backslash H_{\xi_m}(\mathbb{R}))}{\mathrm{vol}(\Gamma \backslash G(\mathbb{R}))}$$

where $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup such that $V(\mathbb{Z})\Gamma \subset V(\mathbb{Z})$, the sum is taken over all the Γ-orbits $\xi_m \Gamma$ in $V_m(\mathbb{Z}) \cap \mathcal{O}_m$, and the volumes are computed with respect to $\mu_{G(\mathbb{R})}$ and μ_{ξ_m}.

Since the groups G and H_{ξ_m} are semisimple defined over $\mathbb{Q}$, and $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup, the volumes involved above are finite and the sum is taken over a finite set by theorems of Borel and Harish-Chandra [BH]. Note that $\omega_m(\mathcal{O}_m)=0$ if and only if $V_m(\mathbb{Z})\cap\mathcal{O}_m=\emptyset$.

In fact the definition of $\omega_m(\mathcal{O}_m)$ is independent of the choice of an arithmetic subgroup Γ(see Proposition 2.3).

We now consider the following hypothesis for $(G,\{(V_m:\mathcal{O})\},H)$:

(B) For any nice compact subset Ω in $\mathcal{O}$,

$$\lim_{m\to\infty}\omega_m(\lambda_m\mathcal{O})\cdot\mathrm{vol}(\lambda_m\Omega)=\infty \quad\text{ and }\quad \#V_m(\mathbb{Z})\cap\mathbb{R}^+\Omega\sim_{m\to\infty}\omega_m(\lambda_m\mathcal{O})\cdot\mathrm{vol}(\lambda_m\Omega).$$

This hypothesis in particular implies that $V_m(\mathbb{Z})\cap\lambda_m\mathcal{O}\neq\emptyset$ for all sufficiently big m.

Theorem 1.7. *Let G, $\{(V_m:\mathcal{O})\}$ and H satisfy the conditions* (A) *and* (B)*. Then $\{(V_m:\mathcal{O})\}$ is Hardy-Littlewood where the density function δ_m is constant on each $G(\mathbb{A})$-orbit $\mathcal{O}^m_{\mathbb{A}}$ in $V_m(\mathbb{A})$ and defined by*

$$\delta_m(\mathcal{O}^m_{\mathbb{A}})=\begin{cases}\#C(H) & \text{if } \kappa(\mathcal{O}^m_{\mathbb{A}})=0\\ 0 & \text{otherwise}\end{cases}\qquad \text{for each } m\in\mathbb{N}.$$

Here $C(H)$ denotes (the torsion subgroup of) the group of coinvariants of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *in $\pi_1(H)$ and $\kappa(\mathcal{O}^m_{\mathbb{A}})\in C(H)$ denotes the Kottwitz invariant of $\mathcal{O}^m_{\mathbb{A}}$.*

Remark

- Note that for any $m\in\mathbb{N}$, the density function δ_m takes values in the set $\{0,\#C(H)\}$ of exactly two numbers.
- For a $G(\mathbb{A})$-orbit $\mathcal{O}^m_{\mathbb{A}}$, we have $\kappa(\mathcal{O}^m_{\mathbb{A}})=0$ if and only if $\mathcal{O}^m_{\mathbb{A}}$ contains a $\mathbb{Q}$-rational point [BR, Theorem 0.4]. We refer to [BR] for a more detailed definition of $\kappa(\mathcal{O}^m_{\mathbb{A}})$ as well as of $C(H)$ and $\pi_1(H)$.

If H is simply connected then $\pi_1(H)=0$; hence κ is identically 0 on each $V_m(\mathbb{A})$ and $\#C(H)=1$. Therefore we deduce:

Corollary 1.8. *If H is simply connected in addition, then $\{(V_m:\mathcal{O})\}$ is strongly Hardy-Littlewood.*

In [EO], the condition (B) is proven under certain hypothesis on $\{(V_m:\mathcal{O})\}$. For instance, we deduce the following:

Theorem 1.9. *Let G, $\{(V_m:\mathcal{O})\}$ and H satisfy the condition* (A)*. Suppose that for some $v\in\mathcal{O}$, the identity component $H_v(\mathbb{R})^0$ has no compact factors and is a proper maximal connected closed subgroup of $G(\mathbb{R})^0$. Let $I\subset\mathbb{N}$ be such that $V_m(\mathbb{Z})\cap\lambda_m\mathcal{O}\neq\emptyset$ for all $m\in I$ and for any $m_0\in I$,*

$$\#\{m\in I:\mathcal{O}\cap\lambda_m^{-1}V_m(\mathbb{Z})=\mathcal{O}\cap\lambda_{m_0}^{-1}V_{m_0}(\mathbb{Z})\}<\infty. \tag{1.10}$$

Then the sequence $\{(V_m : \mathcal{O}) : m \in I\}$ *is Hardy-Littlewood with density function* δ_m *defined as in Theorem 1.7.*

In particular, if H *is simply connected, then* $\{(V_m : \mathcal{O}) : m \in I\}$ *is strongly Hardy-Littlewood.*

If for every connected component $\mathcal{O}$ in $V_1(\mathbb{R})$, all the assumptions of the above theorem hold except for (1.10), then it follows from [BR, Theorem 0.6] and [EMS, Theorem 1.11] that each variety V_m is also Hardy-Littlewood in the sense of Borovoi-Rudnick [BR].

In general, without the maximality assumption on H_v, we may replace the assumption (1.10) by the non-focusing condition in the sense of [EO]. However it seems very hard to check the non-focusing condition for a general group H_v.

Still Theorem 1.9 includes a large family of homogeneous spaces, for example, the cases when G is a connected, $\mathbb{Q}$-simple, simply connected algebraic group and $V_1 = G/H$ is an affine symmetric space of G where H is connected semisimple without any $\mathbb{R}$-anisotropic factors. It is so, since $H(\mathbb{R})^0$ is then a maximal connected closed subgroup of $G(\mathbb{R})^0$ (see [Bo2, Lemma 8.0]).

When the varieties V_m are indeed given as the level sets of a fixed invariant polynomial, it is not hard to see that our expectation for a strongly Hardy Littlewood sequence $\{V_m\}$ in terms of the Tamagawa measures μ_m on $V_m(\mathbb{A})$ does coincide with the classical Hardy Littlewood expectation discussed in the beginning of this paper in view of the uniqueness of the Tamagawa measure (see Lemma 3.8).

Hence we obtain the following from Theorem 1.9 and Lemma 3.8:

Theorem 1.11. *Suppose that* f *is an invariant homogeneous integral polynomial of* ρ, *i.e.,*

$$f(vg) = f(v) \quad \text{for all } v \in V \text{ and } g \in G.$$

If $\{(V_m : \mathcal{O})\}$ *satisfies the assumptions in Theorem 1.9 and* H *is simply connected, then for any nice compact subset* Ω *in* $\mathcal{O}$,

$$N_m(f, \Omega) \sim_{m\to\infty} J_m(m^{1/d}\Omega) \cdot \mathfrak{S}_m$$

with both sides going to ∞ *as* $m \to \infty$.

In the special cases when $V_1(\mathbb{R})$ is a homogeneous space of $G(\mathbb{R})$ and H is simply connected, Theorem 1.7 is basically a consequence of the following identity:

$$\omega_m(V_m(\mathbb{R})) = \prod_p \mu_{m,p}(V_m(\mathbb{Z}_p)) = \mathfrak{S}_m \tag{1.12}$$

Once we set up what arithmetic subgroup Γ is to be used in the definition of $\omega_m(V_m(\mathbb{R}))$, this is essentially proved in [BR, Theorem 4.2], which is in turn based on the methods of Weil [We1] combined with the calculation of Tamagawa number of connected groups due to Ono, Sansuc, and Kottwitz. The identity (1.12) for a quadratic form is precisely Siegel's mass formula for V_m [Si]. In section 5, we discuss yet another proof of the mass formula using the work in [EO], in the light of the work of Eskin, Rudnick and Sarnak [ERS].

In section 4, we discuss several explicit examples of Hardy-Littlewood sequence of level sets.

Lastly we mention that it is heuristically expected that the order of magnitude of $N_m(f,\Omega)$ would be $m^{(n-d)/d}$ for n being the number of variables of f and d being the degree of f. The reason is that for a fixed $R>0$, the values of f in the ball $B_{m^{1/d}R}$ of radius $m^{1/d}R$ lie in the interval $[-c_1 m, c_2 m]$ where constants $c_1, c_2 > 0$ are independent of m. Since the number of integral points in the ball $B_{m^{1/d}R}$ is of order $m^{n/d}$ and it is expected that each value between $[-c_1 m, c_2 m]$ is assumed roughly equally often, this gives that f takes value m in $B_{m^{1/d}R}$ roughly $m^{(n-d)/d}$ times, or equivalently the cardinality $N_m(f, B_R \cap V_1)$ has order $m^{(n-d)/d}$.

This indicates that for a strongly Hardy-Littlewood sequence $\{V_m\}$, the archimedean part of the asymptotic $N_m(f,\Omega)$, as $m\to\infty$, given by the singular integral $J_m(m^{1/d}\Omega)$ does coincide with the heuristic expectation in its order of magnitude. However even when each singular series $\mathfrak{S}_m$ absolutely converges, it may make a non-trivial contribution to the order of magnitude of $N_m(f,\Omega)$ as $m\to\infty$ as well, for instance, see the determinant example in section 4.

Acknowledgements I would like to thank the referee for helpful remarks.

2. The weighted class number $\omega_m(\mathcal{O}_m)$

In this section, we fix a connected component $\mathcal{O}$ in $V_1(\mathbb{R})$ and assume that $(G, \{(V_m : \mathcal{O})\}, H)$ is given as in the hypothesis (A). However we do not need the full strength of the hypothesis (A) but only assume that G and H are connected algebraic $\mathbb{Q}$-groups with no non-trivial $\mathbb{Q}$-characters, so that $G\cap\Gamma$ and $H\cap\Gamma$ have finite co-volumes in $G(\mathbb{R})$ and $H(\mathbb{R})$ respectively for any arithmetic subgroup Γ of G contained in $G(\mathbb{Q})$.

Let $\Gamma\subset G(\mathbb{Q})$ be an arithmetic subgroup such that $V(\mathbb{Z})\Gamma\subset V(\mathbb{Z})$. Fix a nice compact subset Ω in $\mathcal{O}$. For each $\xi_m \in V_m(\mathbb{Z})\cap\lambda_m\mathcal{O}$, we define the function f_{ξ_m} on $G(\mathbb{R})$ by

$$f_{\xi_m}(g) := \sum_{\gamma\in H_{\xi_m}\cap\Gamma\backslash\Gamma} \chi_{\lambda_m\Omega}(\xi_m\gamma g) \tag{2.1}$$

where $\chi_{\lambda_m\Omega}$ denotes the characteristic function of the set $\lambda_m\Omega$. Note that f_{ξ_m} is left Γ-invariant and hence can be considered as a function on $\Gamma\backslash G(\mathbb{R})$.

Lemma 2.2. *We have*

$$\int_{\Gamma\backslash G(\mathbb{R})} f_{\xi_m}\, d\mu_{G(\mathbb{R})} = \mu_{m,\infty}(\lambda_m\Omega)\cdot \mu_{\xi_m}(H_{\xi_m}\cap\Gamma\backslash H_{\xi_m}(\mathbb{R})).$$

Proof. From the assumption that $d\mu_{G(\mathbb{R})} = d\mu_{m,\infty}\cdot d\mu_{\xi_m}$, we have

$$\begin{aligned}\int_{\Gamma\backslash G(\mathbb{R})} f_{\xi_m}\, d\mu_{G(\mathbb{R})} &= \int_{\Gamma\backslash G(\mathbb{R})} \sum_{\gamma\in H_{\xi_m}\cap\Gamma\backslash\Gamma} \chi_{\lambda_m\Omega}(\xi_m\gamma g)\, d\mu_{G(\mathbb{R})}(g)\\ &= \int_{H_{\xi_m}\cap\Gamma\backslash G(\mathbb{R})} \chi_{\lambda_m\Omega}(\xi_m g)\, d\mu_{G(\mathbb{R})}(g)\\ &= \int_{H_{\xi_m}(\mathbb{R})\backslash G(\mathbb{R})} \chi_{\lambda_m\Omega}(\xi_m g) \int_{H_{\xi_m}\cap\Gamma\backslash H_{\xi_m}(\mathbb{R})} d\mu_{\xi_m} d\mu_{m,\infty}(\bar{g})\\ &= \mu_{m,\infty}(\lambda_m\Omega)\cdot\mu_{\xi_m}(H_{\xi_m}\cap\Gamma\backslash H_{\xi_m}(\mathbb{R})).\end{aligned}$$

□

Proposition 2.3. *Let X be the set of all arithmetic subgroups $\Gamma\subset G(\mathbb{Q})$ preserving $V(\mathbb{Z})$. For any $G(\mathbb{R})$-orbit $\mathcal{O}_m$ in $V_m(\mathbb{R})$, the number $\omega_m(\mathcal{O}_m)$ defined in (1.6) is independent of the choice of $\Gamma\in X$.*

Proof. First $\mathcal{O} := \lambda_m^{-1}\mathcal{O}_m$ is a connected component of $V_1(\mathbb{R})$. If $\mathcal{O}_m\cap V_m(\mathbb{Z})$ is empty, then $\omega(\mathcal{O}_m)=0$. Hence we may assume that $\mathcal{O}_m$ contains an integer point. Set

$$h_m(\Gamma,\mathcal{O}_m) := \sum_{\xi_m\Gamma\subset\mathcal{O}_m\cap V_m(\mathbb{Z})} \mathrm{vol}(H_{\xi_m}\cap\Gamma\backslash H_{\xi_m}(\mathbb{R}))$$

where the volume is computed with respect to μ_{ξ_m}. Fixing a nice compact subset Ω in $\mathcal{O}$, we define a function F_m on $G(\mathbb{R})$ by

$$F_m(g) = \sum_{x\in\mathcal{O}_m\cap V_m(\mathbb{Z})} \chi_{\lambda_m\Omega}(xg) \quad \text{for any } g\in G(\mathbb{R}).$$

For any $\Gamma\in X$, since $(\mathcal{O}_m\cap V_m(\mathbb{Z}))\Gamma\subset(\mathcal{O}_m\cap V_m(\mathbb{Z}))$, the function F_m can be considered as a function on $\Gamma\backslash G(\mathbb{R})$, and hence we may write

$$F_m(g) = \sum_{\xi_m\Gamma\subset\mathcal{O}_m\cap V_m(\mathbb{Z})} f_{\xi_m}(g)$$

where f_{ξ_m} is defined as in (2.1). Then by Lemma 2.2, we have for any $\Gamma\in X$,

$$\begin{aligned}\int_{\Gamma\backslash G(\mathbb{R})} F_m\, d\mu_{G(\mathbb{R})} &= \sum_{\xi_m\Gamma\subset\mathcal{O}_m\cap V_m(\mathbb{Z})} \mathrm{vol}(H_{\xi_m}\cap\Gamma\backslash H_{\xi_m}(\mathbb{R}))\cdot\mu_{m,\infty}(\lambda_m\Omega)\\ &= h_m(\Gamma,\mathcal{O}_m)\cdot a\end{aligned}$$

(here we put $a := \mu_{m,\infty}(\lambda_m\Omega)$ for the sake of simplicity). Since any two arithmetic subgroups of G contained in $G(\mathbb{Q})$ are commensurable with each other (cf. [Bo1]), it is clear that for any $\Gamma_1, \Gamma_2 \in X$, $\Gamma_1 \cap \Gamma_2 \in X$ and hence

$$\int_{\Gamma_1\cap\Gamma_2\backslash G(\mathbb{R})} F_m \, d\mu_{G(\mathbb{R})} = h_m(\Gamma_1 \cap \Gamma_2, \mathcal{O}_m) \cdot a$$

which is, for each $i = 1, 2$, again equal to

$$[\Gamma_i : \Gamma_1 \cap \Gamma_2] \cdot \int_{\Gamma_i\backslash G(\mathbb{R})} F_m \, d\mu_{G(\mathbb{R})} = [\Gamma_i : \Gamma_1 \cap \Gamma_2] \cdot h_m(\Gamma_i, \mathcal{O}_m) \cdot a.$$

For both $i = 1, 2$, it yields

$$h_m(\Gamma_1 \cap \Gamma_2, \mathcal{O}_m) = [\Gamma_i : \Gamma_1 \cap \Gamma_2] \cdot h_m(\Gamma_i, \mathcal{O}_m).$$

Since $\mathrm{vol}(\Gamma_1 \cap \Gamma_2\backslash G(\mathbb{R})) = [\Gamma_i : \Gamma_1 \cap \Gamma_2] \cdot \mathrm{vol}(\Gamma_i\backslash G(\mathbb{R}))$, we have

$$\frac{h_m(\Gamma_i, \mathcal{O}_m)}{\mathrm{vol}(\Gamma_i\backslash G(\mathbb{R}))} = \frac{h_m(\Gamma_1 \cap \Gamma_2, \mathcal{O}_m)}{\mathrm{vol}(\Gamma_1 \cap \Gamma_2\backslash G(\mathbb{R}))}.$$

This proves the claim. □

Lemma 2.4. *Fix any $v \in \mathcal{O} \cap V_1(\mathbb{Z})$. For any nice compact subset Ω in $\mathcal{O}$ and any $\xi_m \in \lambda_m\mathcal{O} \cap V_m(\mathbb{Z})$, we have*

$$\mu_{\xi_m}(H_{\xi_m} \cap \Gamma\backslash H_{\xi_m}(\mathbb{R})) \cdot \mu_{m,\infty}(\lambda_m\Omega) = \mu_v(g_{\xi_m}^{-1}\Gamma g_{\xi_m} \cap H_v\backslash H_v(\mathbb{R})) \cdot \mu_{1,\infty}(\Omega)$$

where $g_{\xi_m} \in G(\mathbb{R})$ is such that $\lambda_m v = \xi_m g_{\xi_m}$.

In particular,

$$\omega_m(\lambda_m\mathcal{O}) \cdot \mu_{m,\infty}(\lambda_m\Omega) = \sum_{\xi_m\Gamma\subset V_m(\mathbb{Z})\cap\lambda_m\mathcal{O}} \frac{\mu_v(g_{\xi_m}^{-1}\Gamma g_{\xi_m} \cap H_v\backslash H_v(\mathbb{R}))}{\mu_{G(\mathbb{R})}(\Gamma\backslash G(\mathbb{R}))} \cdot \mu_{1,\infty}(\Omega). \tag{2.5}$$

Proof. Observe that the function f_{ξ_m} defined in (2.1) can also be written as

$$f_{\xi_m}(g) = \sum_{\gamma\in H_{\xi_m}\cap\Gamma\backslash\Gamma} \chi_\Omega(\lambda_m^{-1}\xi_m\gamma g).$$

Since $d\mu_{G(\mathbb{R})} = d\mu_v \cdot d\mu_{1,\infty}$, we have

$$\begin{aligned}
\int_{\Gamma\backslash G(\mathbb{R})} f_{\xi_m} \, d\mu_{G(\mathbb{R})} &= \int_{H_{\xi_m}\cap\Gamma\backslash G(\mathbb{R})} \chi_\Omega(\lambda_m^{-1}\xi_m g) \, d\mu_{G(\mathbb{R})}(g) \\
&= \int_{H_v\cap g_{\xi_m}^{-1}\Gamma g_{\xi_m}\backslash G(\mathbb{R})} \chi_\Omega(\lambda_m^{-1}\xi_m g_{\xi_m} g) \, d\mu_{G(\mathbb{R})}(g) \\
&= \int_{H_v(\mathbb{R})\backslash G(\mathbb{R})} \chi_\Omega(vg) \int_{H_v\cap g_{\xi_m}^{-1}\Gamma g_{\xi_m}\backslash H_v(\mathbb{R})} d\mu_v d\mu_{1,\infty}(\bar{g}) \\
&= \mu_{1,\infty}(\Omega) \cdot \mu_v(g_{\xi_m}^{-1}\Gamma g_{\xi_m} \cap H_v\backslash H_v(\mathbb{R})) \qquad (2.6)
\end{aligned}$$

The claim follows by comparing (2.6) and Lemma 2.2. $\square$

3. Hardy-Littlewood systems: Proof of Theorems 1.7 and 1.9

In this section, we assume that $G, \{(V_m : \mathcal{O})\}$ and H satisfy the hypothesis (A). Set $\mathcal{O}_m = \lambda_m \mathcal{O}$.

Fix a gauge form μ_G on G and for each $\xi_m \in V_m(\mathbb{Z}) \cap \lambda_m \mathcal{O}$, consider the gauge form $\mu_{H_{\xi_m}}$ on H_{ξ_m} so that the gauge forms $(\mu_G, \mu_{H_{\xi_m}}, \mu_m)$ match together algebraically in the sense of [We2, P. 26]. If $\mu_{G,\infty}$ and $\mu_{H_{\xi_m},\infty}$ denote the invariant measures on $G(\mathbb{R})$ and $H_{\xi_m}(\mathbb{R})$ induced from μ_G and $\mu_{H_{\xi_m}}$ respectively, then $d\mu_{m,\infty} \cdot d\mu_{H_{\xi_m},\infty} = d\mu_{G,\infty}$.

For a $G(\mathbb{A})$-orbit $\mathcal{O}^m_{\mathbb{A}}$ in $V_m(\mathbb{A})$, the notation $V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}$ is to be understood as the set $\{x \in V_m(\mathbb{Z}) : (x) \in \mathcal{O}^m_{\mathbb{A}}\}$ where (x) denotes the diagonal embedding of x into $V_m(\mathbb{A})$. In particular, the set $V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}$ lies in $V_m(\mathbb{R})$, rather than in $V_m(\mathbb{A})$. Notice that $V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}$ is contained in a single $G(\mathbb{R})$-orbit of $V_m(\mathbb{R})$.

For a $G(\mathbb{A})$-orbit $\mathcal{O}^m_{\mathbb{A}}$, we denote by $\mathcal{O}^m_{\mathbb{A}_\infty}$ its infinite component and by $\mathcal{O}^m_{\mathbb{A}_f}$ its finite component so that $\mathcal{O}^m_{\mathbb{A}}$ can be written as $\mathcal{O}^m_{\mathbb{A}_\infty} \times \mathcal{O}^m_{\mathbb{A}_f}$.

Theorem 3.1. *We have*

$$\omega_m(\mathcal{O}_m) = \#C(H) \sum_{\{\mathcal{O}^m_{\mathbb{A}} : (V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}) \cap \mathcal{O}_m \neq \emptyset\}} \mu_{m,f}\Big(\prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}^m_{\mathbb{A}_f}\Big)$$

where the sum is taken over all $G(\mathbb{A})$-orbits $\mathcal{O}^m_{\mathbb{A}}$ in $V_m(\mathbb{A})$ such that $(V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}) \cap \mathcal{O}_m \neq \emptyset$ and $\mu_{m,f} = \prod_p \mu_{m,p}$.

Proof. Letting K_f be an open compact subgroup of $G(\mathbb{A}_f)$ such that $\prod_p V(\mathbb{Z}_p)\, K_f = \prod_p V(\mathbb{Z}_p)$, set Γ to be the congruence subgroup defined by

$$\Gamma := G(\mathbb{Q}) \cap (G(\mathbb{R}) \times K_f).$$

Since $V(\mathbb{Z}) = V(\mathbb{Q}) \cap (V(\mathbb{R}) \times \prod_p V(\mathbb{Z}_p))$, we have $V(\mathbb{Z})\Gamma = V(\mathbb{Z})$. By Proposition 2.3, we may use this Γ in our definition of $\omega_m(\mathcal{O}_m)$. Hence if we set

$$\omega_{\mathcal{O}^m_{\mathbb{A}}} = \sum_{\xi_m \Gamma \subset V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}} \frac{\mu_{H_{\xi_m},\infty}(\Gamma \cap H_{\xi_m} \backslash H_{\xi_m}(\mathbb{R}))}{\mu_{G,\infty}(\Gamma \backslash G(\mathbb{R}))},$$

we may write

$$w_m(\mathcal{O}_m) = \sum_{\{\mathcal{O}^m_{\mathbb{A}} : (V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}) \cap \mathcal{O}_m \neq \emptyset\}} \omega_{\mathcal{O}^m_{\mathbb{A}}}. \tag{3.2}$$

On the other hand, for each $G(\mathbb{A})$-orbit $\mathcal{O}^m_{\mathbb{A}}$ with $(V_m(\mathbb{Z}) \cap \mathcal{O}^m_{\mathbb{A}}) \cap \mathcal{O}_m \neq \emptyset$, we have $\kappa(\mathcal{O}^m_{\mathbb{A}}) = 0$ [BR, Theorem 0.4], and hence we may apply [BR, Theorem 4.2] to obtain

$$\omega_{\mathcal{O}^m_{\mathbb{A}}} = \#C(H_{\xi_m}) \cdot \mu_{m,f}\Big(\prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}^m_{\mathbb{A}_f}\Big).$$

Since $C(H)$ is computed in terms of the Galois module $\pi_1(H)$, it is invariant under inner twisting. Since $\lambda_m^{-1}\xi_m \in V_1$ and G acts transitively on V_1, each H_{ξ_m} is an inner form of H and hence $C(H_{\xi_m}) = C(H)$. This proves the claim. □

Lemma 3.3. *For any $G(\mathbb{A})$-orbit $\mathcal{O}_{\mathbb{A}}^m$ in $V_m(\mathbb{A})$, the following two are equivalent:*

1. $(\mathcal{O}_{\mathbb{A}}^m \cap V_m(\mathbb{Z})) \cap \mathcal{O}_m \neq \emptyset$;
2. $\kappa(\mathcal{O}_{\mathbb{A}}^m) = 0$, $\mathcal{O}_{\mathbb{A}_\infty}^m = \mathcal{O}_m$ *and* $\prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}_{\mathbb{A}_f}^m \neq \emptyset$.

Proof. The direction that (1) implies (2) is easy to check. Now assume (2). Since $\kappa(\mathcal{O}_{\mathbb{A}}^m) = 0$, there exists a vector $x_0 \in V_m(\mathbb{Q}) \cap \mathcal{O}_{\mathbb{A}}^m$. Since G is semisimple and simply connected with no $\mathbb{R}$-anisotropic factor defined over $\mathbb{Q}$, G satisfies the strong approximation property (cf. [PR, Theorem 7.12]). Hence if M_f is a compact open subgroup of $G(\mathbb{A}_f)$ preserving $\prod_p V_m(\mathbb{Z}_p)$, then $G(\mathbb{Q})(G(\mathbb{R}) \times M_f) = G(\mathbb{A})$; and $x_0 G(\mathbb{Q})(G(\mathbb{R}) \times M_f) = \mathcal{O}_{\mathbb{A}}^m$. Pick a vector $x \in \prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}_{\mathbb{A}_f}^m$, which exists by the assumption and write $x = x_0 g k$ where $g \in G(\mathbb{Q})$ and $k \in M_f$. Since $x_0 g \in V_m(\mathbb{Q}) \cap x_0 G(\mathbb{R})$, $xk^{-1} \in \prod_p V_m(\mathbb{Z}_p) \cap x_0 G(\mathbb{A}_f)$, and the each p-adic component of xk^{-1} is equal to $x_0 g$, we have $x_0 g \in V_m(\mathbb{Z})$ as well as the diagonal embedding of $x_0 g$ into $V_m(\mathbb{A})$ is contained in $\mathcal{O}_{\mathbb{A}}^m = x_0 G(\mathbb{A})$, proving the claim. □

Proposition 3.4. *Let δ_m be the function on $V_m(\mathbb{A})$ defined in Theorem 1.7. Then for any nice compact subset Ω in $\mathcal{O}$, we have*

$$\int_{\lambda_m\Omega \times \prod_p V_m(\mathbb{Z}_p)} \delta_m(x)\, d\mu_m = \omega_m(\mathcal{O}_m) \cdot \mu_{m,\infty}(\lambda_m \Omega). \tag{3.5}$$

Proof. By the definition of δ_m,

$$\int_{\lambda_m\Omega \times \prod_p V_m(\mathbb{Z}_p)} \delta_m(x)\, d\mu_m = \#C(H) \cdot \sum_{\kappa(\mathcal{O}_{\mathbb{A}}^m)=0} \mu_m((\lambda_m\Omega \times \prod_p V_m(\mathbb{Z}_p)) \cap \mathcal{O}_{\mathbb{A}}^m).$$

In order that $\mu_m((\lambda_m\Omega \times \prod_p V_m(\mathbb{Z}_p)) \cap \mathcal{O}_{\mathbb{A}}^m)$ be non-zero, it is necessary that $\mathcal{O}_{\mathbb{A}_\infty}^m = \mathcal{O}_m$ and $\prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}_{\mathbb{A}_f}^m \neq \emptyset$. Hence the above is equal to

$$\#C(H) \cdot \sum \mu_{m,\infty}(\lambda_m\Omega) \cdot \mu_{m,f}(\prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}_{\mathbb{A}_f}^m)$$

where the sum is taken over the $G(\mathbb{A})$-orbits $\mathcal{O}_{\mathbb{A}}^m$ such that $\kappa(\mathcal{O}_{\mathbb{A}}^m) = 0$, $\mathcal{O}_{\mathbb{A}_\infty}^m = \mathcal{O}_m$ and $\prod_p V_m(\mathbb{Z}_p) \cap \mathcal{O}_{\mathbb{A}_f}^m \neq \emptyset$. By Lemma 3.3, we may equivalently take the last sum over the $G(\mathbb{A})$-orbits $\mathcal{O}_{\mathbb{A}}^m$ such that $(\mathcal{O}_{\mathbb{A}}^m \cap V_m(\mathbb{Z})) \cap \mathcal{O}_m \neq \emptyset$. It remains to apply Theorem 3.1 to finish the proof. □

Proof of Theorem 1.7 Consider the function δ_m defined as in the statement of the theorem. Since the orbits of $G(\mathbb{A})$ in $V_m(\mathbb{A})$ are open (see [BR, Lemma 1.6.4]), δ_m is certainly a locally constant function on $V_m(\mathbb{A})$.

Since the hypothesis (A) and (B) give that for any nice compact subset $\Omega \subset \mathcal{O}$,

$$\# V_m(\mathbb{Z}) \cap \mathbb{R}^+\Omega \sim \omega_m(\mathcal{O}_m) \cdot \mu_{m,\infty}(\lambda_m \Omega),$$

with both sides tending to ∞ as $m \to \infty$, it remains to apply the identity (3.5).

In order to prove Theorem 1.9, we recall the following result (here we are assuming that $(G, \{V_m : \mathcal{O}), H)$ satisfies the hypothesis (A)):

Theorem 3.6 (EO, Theorem 1.2)**.** *Assume that $V_m(\mathbb{Z}) \cap \lambda_m \mathcal{O} \neq \emptyset$ and that $H_v^0(\mathbb{R})$ is a proper maximal connected closed subgroup of $G^0(\mathbb{R})$ for some $v \in \mathcal{O} \cap V_1(\mathbb{Z})$. Suppose that one of the following equivalent condition holds:*

- *For each $m_0 \in \mathbb{N}$,*
$$\#\{m \in \mathbb{N} : \mathcal{O} \cap \lambda_{m_0}^{-1} V_{m_0}(\mathbb{Z}) = \mathcal{O} \cap \lambda_m^{-1} V_m(\mathbb{Z})\} < \infty; \qquad or$$
- $$\lim_{m\to\infty} \frac{\sum_{\xi_m \Gamma \subset \mathcal{O}_m \cap V_m(\mathbb{Z})} \mu_v(g_{\xi_m}^{-1} \Gamma g_{\xi_m} \cap H_v \backslash H_v(\mathbb{R}))}{\mu_{G(\mathbb{R})}(\Gamma \backslash G(\mathbb{R}))} = \infty$$
  where $g_{\xi_m} \in G(\mathbb{R})$ is such that $\xi_m g_{\xi_m} = \lambda_m v$.

Then for any nice compact subset Ω in $\mathcal{O}$,

$$\# V_m(\mathbb{Z}) \cap \mathbb{R}^+\Omega \sim_{m\to\infty} \sum_{\xi_m \Gamma \subset V_m(\mathbb{Z}) \cap \mathcal{O}_m} \frac{\mathrm{vol}(g_{\xi_m}^{-1} \Gamma g_{\xi_m} \cap H_v \backslash H_v(\mathbb{R}))}{\mathrm{vol}(\Gamma \backslash G(\mathbb{R}))} \cdot \mathrm{vol}(\Omega). \tag{3.7}$$

Remark In fact Theorem 1.2 in [EO] is stated under the assumption that $H_v(\mathbb{R})$ is a proper maximal real algebraic subgroup of $G(\mathbb{R})$. However in the proof, this assumption was used only to ensure that H_v is not contained in any proper parabolic subgroup of G and that $H_v(\mathbb{R})$ has a finite index in the normalizer of $H_v(\mathbb{R})$ in $G(\mathbb{R})$. These two conditions are clearly satisfied under the assumption of Theorem 3.6.

Proof of Theorem 1.9 By (2.5), the right hand side of (3.7) is equal to $\omega_m(\mathcal{O}_m) \cdot \mathrm{vol}(\lambda_m \Omega)$. Hence Theorem 3.6 yields that the hypothesis (B) is satisfied for $(G, \{(V_m : \mathcal{O})\}, H)$ under the assumptions of Theorem 1.9. Hence Theorem 1.9 follows from Theorem 1.7.

Suppose that $f : V = \mathbb{C}^n \to \mathbb{C}$ and $\{V_m\}$ are as in Theorem 1.11. There exists a differential form η on $\mathbb{C}^n$ of degree $n-1$ such that $\eta \wedge f^*(dx) = dx_1 \wedge \cdots \wedge dx_n$. Then the restriction of η to V_m defines a gauge form, say μ_m, on V_m. Using this construction, we have (cf. [BR, Lemmas 1.8.1-2]):

Lemma 3.8. *Then for any nice compact subset $\Omega \subset V_1(\mathbb{R})$,*

$$\mu_{m,\infty}(m^{1/d}\Omega) = J_m(m^{1/d}\Omega); \quad \prod_p \mu_{m,p}(V_m(\mathbb{Z}_p)) = \mathfrak{S}_m$$

for $d = \deg(f)$.

Since the Tamagawa measure on $V_m(\mathbb{A})$ is unique, Theorem 1.11 follows from the above lemma and Theorem 1.9.

If H is simply connected, $G(\mathbb{A}_f)$ is known to act transitively on $V_m(\mathbb{A}_f)$ (cf. [BR, Proof of Theorem 3.2]). Hence Lemma 3.3 implies the following:

Lemma 3.9. *Assume that $G(\mathbb{R})$ acts transitively on $V_1(\mathbb{R})$. Suppose that the hypothesis (A) holds and that H is simply connected. If $\kappa(\mathcal{O}_{\mathbb{A}}^m) = 0$, then*

$$V_m(\mathbb{Z}) \cap \mathcal{O}_{\mathbb{A}}^m \neq \emptyset.$$

In particular, if $V_m(\mathbb{Q}) \neq \emptyset$, then $V_m(\mathbb{Z}) \neq \emptyset$.

4. Examples

4.1. **Determinant.** Let M_n denote the space of n by n matrices ($n \geq 2$), and set

$$V_m = \{X \in \mathrm{M}_n : \det(X) = m\}.$$

Then the group $G = \mathrm{SL}_n \times \mathrm{SL}_n$ acts on M_n by

$$X \mapsto A^{-1}XB \quad \text{for } (A, B) \in G.$$

Clearly we have $V_1 = I_n G$ (here I_n denotes the $n \times n$ identity matrix), $V_1(\mathbb{R}) = I_n G(\mathbb{R})$ and $V_m = m^{1/n} V_1$ for each $m \in \mathbb{N}$.

The stabilizer H of I_n in G is given by $H = \{(A, A) : A \in \mathrm{SL}_n\}$.

Note that both G and H are connected semisimple simply connected algebraic groups whose group of real points have no compact factors. Moreover $H(\mathbb{R})$ is a maximal connected closed subgroup of $G(\mathbb{R})$.

Since $\mathrm{diag}(m, I_{n-1}) \in V_m(\mathbb{Z})$, we have $V_m(\mathbb{Z}) \neq \emptyset$ for each $m \in \mathbb{N}$.

We now claim that for a fixed $m_0 \in \mathbb{N}$,

$$\#\{m \in \mathbb{N} : m_0^{-1/n} V_{m_0}(\mathbb{Z}) = m^{-1/n} V_m(\mathbb{Z})\} < \infty.$$

Suppose that $m_0^{-1/n} V_{m_0}(\mathbb{Z}) = m^{-1/n} V_m(\mathbb{Z})$. Since $\mathrm{diag}(m, I_{n-1}) \in V_m(\mathbb{Z})$, we must have $m^{-1/n} = m_0^{-1/n} k$ for some positive integer k. Hence $m = m_0 k^{-n} \leq m_0$, yielding that

$$\#\{m \in \mathbb{N} : m_0^{-1/n} V_{m_0}(\mathbb{Z}) = m^{-1/n} V_m(\mathbb{Z})\} \leq m_0 < \infty.$$

Therefore by Theorem 1.9, $\{V_m : m \in \mathbb{N}\}$ is strongly Hardy-Littlewood and by Theorem 1.11, for any nice compact subset $\Omega \subset V_1(\mathbb{R})$,

$$N_m(\det, \Omega) \sim_{m \to \infty} m^{n-1} \mathfrak{S}_m J_1(\Omega)$$

where J_m and $\mathfrak{S}_m$ are defined as in the introduction for $f = \det$.

This was shown by Linnik and Skubenko (see [Li1, Lemma V. 11.1], [LS] and [Sk, Theorem 1]).

On the other hand, $\omega_m \cdot \mathrm{vol}(m^{1/n}\Omega)$ in Condition (B) can be computed explicitly in this case, using Hecke operators ([Sa, (3.6)], also see [COU, (5.1)] and [GO, (1.5)] for details):

$$\omega_m(V_m(\mathbb{R})) \cdot \mathrm{vol}(m^{1/n}\Omega) = \prod_i \frac{(p_i^{k_i+1}-1)\cdots(p_i^{k_i+n-1}-1)}{(p_i-1)\cdots(p_i^{n-1}-1)} J_1(\Omega)$$

when $m = \prod_i p_i^{k_i}$ is the prime factorization of m.

Since it follows from Proposition 3.4 and Lemma 3.8 that

$$m^{n-1}J_1(\Omega)\mathfrak{S}_m = \omega_m(V_m(\mathbb{R})) \cdot \mathrm{vol}(m^{1/n}\Omega)$$

we deduce that

$$\mathfrak{S}_m = m^{-n+1}\prod_i \frac{(p_i^{k_i+1}-1)\cdots(p_i^{k_i+n-1}-1)}{(p_i-1)\cdots(p_i^{n-1}-1)}.$$

Hence

$$\limsup_{m\in\mathbb{N}} \mathfrak{S}_m \geq \limsup \prod_i \frac{p_i^n-1}{p_i(p_i-1)} \geq \limsup \prod_i (1+p_i^{-1}) = \infty.$$

4.2. **Pfaffian.** Let $V = \{X \in \mathrm{M}_{2n} : X^t = -X\}$ $(n \geq 2)$ and for each $m \in \mathbb{N}$, set

$$V_m := \{X \in V : \mathrm{Pff}(X) = m\}$$

where $\mathrm{Pff}(X)$ denotes the Pfaffian of a skew symmetric matrix X so that $\mathrm{Pff}(X)^2 = \det(X)$ and $\mathrm{Pff}(v_0) = 1$ where

$$v_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Note that $\dim V = n(2n-1)$ and Pff is an integral homogeneous polynomial of degree n. Then the group $G = \mathrm{SL}_{2n}$ acts on V by

$$X \mapsto g^t X g \quad \text{ for } g \in G.$$

It is easy to check that G acts transitively on V_1, $V_1(\mathbb{R}) = v_0 G(\mathbb{R})$ and $V_m = m^{1/n}V_1$ for each $m \in \mathbb{N}$.

Now the stabilizer H of v_0 is the symplectic group Sp_{2n}, and hence it satisfies the assumption for H in Theorem 1.9 and is simply connected. Since

$$\begin{pmatrix} & & m & 0 \\ & & 0 & I_{n-1} \\ -m & 0 & & \\ 0 & -I_{n-1} & & \end{pmatrix} \in V_m(\mathbb{Z})$$

$V_m(\mathbb{Z}) \neq \emptyset$ for each $m \in \mathbb{N}$. The finiteness condition (1.10) can be checked similarly to the previous example. Therefore by Theorem 1.11, for any nice compact subset $\Omega \subset V_1(\mathbb{R})$, as $m \to \infty$,

$$N_m(\mathrm{Pff}, \Omega) \sim J_m(m^{1/n}\Omega) \cdot \mathfrak{S}_m = m^{2n-2}\mathfrak{S}_m J_1(\Omega).$$

4.3. **Determinant of symmetric matrices.** Let $V = \{X \in \mathrm{M}_n : X^t = X\}$ $(n \geq 2)$ and

$$V_m := \{X \in V : \det(X) = -m\} \quad \text{for each } m \in \mathbb{N}.$$

The group $G = \mathrm{SL}_n$ acts on V by

$$X \mapsto g^t X g \quad \text{for } g \in G.$$

Fix a positive odd integer $k < n$ and set $v_0 = \mathrm{diag}(-I_k, I_{n-k})$ and $\mathcal{O} = v_0 G(\mathbb{R})$. The stabilizer H of v_0 is equal to $\mathrm{SO}(k, n-k)$. Provided $n \geq 3$, the identity component of H is a non-compact simple maximal connected closed group of $\mathrm{SL}_n(\mathbb{R})$.

We also have $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$, and hence $\#C(H) = 2$. The same argument as in the first example shows that the sequence $\{(V_m : \mathcal{O}) : m \in \mathbb{N}\}$ satisfies the assumption (1.10) and for all $m \in \mathbb{N}$, $V_m(\mathbb{Z}) \cap m^{1/n}\mathcal{O} \neq \emptyset$ since it contains $\mathrm{diag}(-m, -I_{k-1}, I_{n-k})$.

Therefore by Theorem 1.9, the sequence $\{(V_m : \mathcal{O}) : m \in \mathbb{N}\}$ is Hardy-Littlewood, but not strongly Hardy-Littlewood, with the density function taking exactly two values 0 and 2.

4.4. **Quadratic forms of signature** (p, q)**.** Let Q be an integral quadratic form in n variables with signature (p, q) with $p + q = n \geq 4$, $p \geq 2$ and $q \geq 1$. For each $m \in \mathbb{N}$, set

$$V_m = \{X \in \mathbb{C}^n : Q(X) = m\}.$$

Let $G = \mathrm{Spin}(p, q)$. Then G acts on $\mathbb{C}^n$ via the projection $G \to \mathrm{SO}(p, q)$ and the standard action of $\mathrm{SO}(p, q)$ on $\mathbb{C}^n$. Each $V_m(\mathbb{R})$ is a symmetric homogeneous space of $G(\mathbb{R})$ and $V_m = \sqrt{m}V_1$. The stabilizer H of a vector of V_1 is isomorphic over $\mathbb{R}$ to $\mathrm{Spin}(p-1, q)$. Hence H is simply connected and $H(\mathbb{R})$ is a non-compact maximal closed subgroup of $G(\mathbb{R})$.

Recall that an integer m is called a fundamental discriminant if and only if m is either a square-free integer congruent to 1 mod 4 or 4 times of a square-free integer which is 2 or 3 mod 4.

Note that any indefinite integral quadratic form in at least 4 variables represents any fundamental discriminant over $\mathbb{Q}$ by Hasse-Minkowski theorem (cf. [Se, Theorem 8 in IV.3.2]) and [Se, Corollary in IV. 2.2], since an indefinite quadratic form represents any real number over $\mathbb{R}$. That is, for any fundamental discriminant m, $V_m(\mathbb{Q}) \neq \emptyset$ and hence by Lemma 3.9, we have $V_m(\mathbb{Z}) \neq \emptyset$. Moreover it can be easily seen that the sets $m^{-1/2}V_m(\mathbb{Q})$ are mutually disjoint among the fundamental discriminants $m \in \mathbb{N}$ (cf.

[EO, Remark preceding Theorem 7.1]). Hence applying Theorem 1.11, as $m \to \infty$ along fundamental discriminants,

$$N_m(Q,\Omega) \sim m^{(n-2)/2}\mathfrak{S}_m J_1(\Omega)$$

for any nice compact subset $\Omega \subset V_1(\mathbb{R})$. It is known to experts that this can be shown by the Hardy-Littlewood circle method for $n \geq 5$ and by the Kloosterman sum method for $n = 4$.

5. Remark on Siegel's mass formula

Let Q and V_m be as in the last subsection 4.4.

Fix an integer m with $V_m(\mathbb{Z}) \neq \emptyset$. Eskin, Rudnick and Sarnak [ERS] gave a new proof of Siegel's mass formula for the quadric V_m [Si]:

$$\sum_{\xi_m \Gamma \subset V_m(\mathbb{Z})} \frac{\mathrm{vol}(\Gamma \cap H_{\xi_m} \backslash H_{\xi_m}(\mathbb{R}))}{\mathrm{vol}(\Gamma\backslash G(\mathbb{R}))} = \prod_p \lim_{k\to\infty} \frac{\# V_m(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\cdot \dim(V_m)}}, \tag{5.1}$$

or in our notation,

$$\omega_m(V_m(\mathbb{R})) = \mathfrak{S}_m.$$

In the following, we set $\omega_m = \omega_m(V_m(\mathbb{R}))$ for simplicity.

They deduced this formula from two different ways of computing the limit

$$\lim_{T\to\infty} \frac{\# V_m(\mathbb{Z}) \cap B_T}{\mathrm{vol}(B_T)}$$

where B_T is a ball of radius T with respect to the Euclidean norm, the right hand side of (5.1) given by Hardy-Littlewood circle method for $p + q \geq 5$ and Kloosterman's method of levelling for $p + q = 4$ [Es], and the left hand side given by the orbit counting method of [DRS] (or [EM], [EMS]).

They also explained how to show the Tamagawa number $\tau(G)$ is 2 for any special orthogonal group G.

We discuss a different way of deducing this fact. As we mentioned in the previous section, Hardy-Littlewood method together with Kloosterman's method (needed for $p + q = 4$) yields that

$$N_m(Q,\Omega) \sim \mathfrak{S}_m J_m(\sqrt{m}\Omega) \tag{5.2}$$

with both sides tending to ∞ as $m \to \infty$.

On the other hand, by [EO, Theorem 1.2] (see Theorem 3.6) and (2.5), we have

$$N_m(Q,\Omega) \sim \omega_m \cdot J_m(\sqrt{m}\Omega) \tag{5.3}$$

as $m \to \infty$ along fundamental discriminants (noting that $V_m(\mathbb{Z}) \neq \emptyset$, by the discussion in section 4.4).

Hence from (5.2) and (5.3), we deduce that there exists an infinite subsequence m_i such that

$$\lim_{m_i\to\infty} \frac{\omega_{m_i}}{\mathfrak{S}_{m_i}} = 1. \tag{5.4}$$

On the other hand, Weil [We1] (see [ERS]) showed that

$$\frac{\tau(\mathrm{SO}(Q))}{2} \cdot \omega_m = \mathfrak{S}_m \tag{5.5}$$

for any $m \in \mathbb{N}$ with $V_m(\mathbb{Z}) \neq \emptyset$.

Hence comparing (5.4) and (5.5), we deduce

$$\tau(\mathrm{SO}(Q)) = 2.$$

Now the argument in section 3 of [ERS] shows that this is sufficient to prove that the Tamagawa measure of any special orthogonal group is equal to 2, or equivalently to show the general mass formula by [We1].

References

[Bo1] A. Borel *Introduction aux Groupes Arithmétiques*, Publ. de. l'Inst. Math. del l'Univ. de Strasbourg XV, Hermann, Paris, 1969

[Bo2] A. Borel *Values of indefinite quadratic forms at integral points and flows on spaces of lattices*, Bull. AMS, Vol 32, 1995, pp. 184–204.

[BH] A. Borel and Harish-Chandra *Arithmetic subgroups of algebraic groups*, Ann. Math., Vol 75, 1962, pp. 485–535.

[BR] M. Borovoi and Z. Rudnick *Hardy-Littlewood varieties and semisimple groups*, Invent. Math., Vol 119, 1995, pp. 37–66.

[COU] L. Clozel, H. Oh and E. Ullmo *Hecke operators and equidistribution of Hecke points*, Invent. Math., Vol 144, 2001, pp. 327–351.

[Da] H. Davenport *Analytic methods for Diophantine Equations and Diophantine Inequalities*, Ann Arbor Publishers, Ann Arbor, Mich. 1962

[DRS] W. Duke, Z. Rudnick and P. Sarnak *Density of integer points on affine homogeneous varieties*, Duke Math. J. Vol 71, 1993, 143-179

[EM] A. Eskin and C. McMullen *Mixing, counting and equidistribution in Lie groups*, Duke Math. J. Vol 71, 1993, 181–209.

[EMS] A. Eskin, S. Mozes and N. Shah *Unipotent flows and counting lattice points on homogeneous varieties*, Ann. Math, Vol 143, 1996, pp. 149–159.

[EO] A. Eskin and H. Oh *Integer points on a family of homogeneous varieties and unipotent flows*, Preprint (2002, Feb)

[ERS] A. Eskin, Z. Rudnick and P. Sarnak *A proof of Siegel's weight formula*, Int. Math. Res. Notices, Vol 5, 1991, pp. 65–69

[Es] T. Estermann *A new applications of the Hardy-Littlewood-Kloosterman method*, Proc. London. Math. Soc. Vol 12, 1962, pp. 425–444

[GO] W. T. Gan and H. Oh *Equi-distribution of integer points on a family of homogeneous varieties: a problem of Linnik*, Compositio Math. Vol 136, 2003, pp. 323–352

[Li1] Y. V. Linnik *Ergodic properties of algebraic fields*, Ergebnisse der Mathematik und Grenzgebiete, Band 45, Springer-Verlag New York, 1968

[Li2] Y. V. Linnik *Additive problems and eigenvalues of the modular operators*, Proc. Int. Cong. Math. Stockholm, 1962, pp. 270–284.

[LS] Y. V. Linnik and B. F. Skubenko *Asymptotic distribution of integral matrices of third order*, Vestnik Leningrad. Univ. Ser. Mat. Mekh. Astron., Vol. 19, 1964, pp. 25–36 (Russian)

[PR] V. Platonov and A. Rapinchuk *Algebraic groups and Number theory*, Academic Press, New York, 1994

[Sa] P. Sarnak *Diophantine problems and linear groups*, Proc. Int. Cong. Math. 1990, Vol 1, pp. 459–471.

[Sc] W. M. Schmidt *The density of integer points on homogeneous varieties*, Acta. Math. 1985, Vol 154, pp. 243–296.

[Se] J. P. Serre *A course in arithmetic*, Springer 1973, GTM 7.

[Si] C. L. Siegel *On the theory of indefinite quadratic forms*, Ann. of Math. 1944, Vol 45, pp. 577-622.

[Sk] B. F. Skubenko *The distribution of integer matrices and calculation of the volume of a fundamental domain of a unimodular group of matrices*, Trudy Math. Inst. Steklov. 1965, Vol 80, pp. 129-144 (Russian). English Translation: Algebraic number theory and representations, (D. K. Faddeev ed.), Proc. Inst. Steklov of Math. No 80 (1965) Amer. Math. Soc. Providence, RI, 1968, pp. 147-163

[Va] R. C. Vaughan *The Hardy-Littlewood method*, 2nd ed. Cambridge Tracts in Math. 1997, Vol 125, Cambridge-New York

[We1] A. Weil *Sur la théorie des formes quadratiques*, in Collo. Théorie des Groupes Algébriques, Bruxelles, 1962, pp. 9–22

[We2] A. Weil *Adeles and algebraic groups*, Boston, Birkhäuser, 1982

Mathematics Department, Princeton University, Princeton, NJ 08544, School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, Mathematics 253-37, Caltech, Pasadena, CA 91125

E-mail address: `heeoh@its.caltech.edu`