NEW EXAMPLES OF COMPACT CLIFFORD-KLEIN FORMS OF HOMOGENEOUS SPACES OF $\mathrm{SO}(2,n)$

HEE OH AND DAVE WITTE

Abstract. A homogeneous space G/H is said to have a compact Clifford-Klein form if there exists a discrete subgroup Γ of G that acts properly on G/H such that the quotient space $\Gamma\backslash G/H$ is compact. When n is even, we find every closed connected subgroup H of $G = \mathrm{SO}(2,n)$, such that G/H has a compact Clifford-Klein form, but our classification is not quite complete when n is odd. The work reveals new examples of homogeneous spaces of $\mathrm{SO}(2,n)$ that have compact Clifford-Klein forms, if n is even. Furthermore, we show that if H is a closed, connected subgroup of $G = \mathrm{SL}(3,\mathbb{R})$, and neither H nor G/H is compact, then G/H does not have a compact Clifford-Klein form, and we also study non-compact Clifford-Klein forms of finite volume.

1. Introduction

Let G be a Zariski-connected semisimple linear Lie group.

1.1. **Definition.** Let H be a closed connected subgroup of G. We say that the homogeneous space G/H has a *compact Clifford-Klein form* if there is a discrete subgroup Γ of G such that

- Γ acts properly on G/H; and
- $\Gamma\backslash G/H$ is compact.

A basic question in geometry is to determine which homogeneous spaces of G admit compact Clifford-Klein forms. If H is compact, then G/H has a compact Clifford-Klein form by a result of Borel [Bor]. When H is non-compact, the situation is far from being well understood. Some examples of such homogeneous spaces admitting a compact Clifford-Klein forms have been constructed by Kulkarni [Kul], Goldman [Gol], and T. Kobayashi [Kb6]. Their constructions are quite special as they concern specific groups. More generally one suspects that most non-Riemannian homogeneous spaces do not admit a compact Clifford-Klein forms (see [Kb5] and [Lab] for a survey on the general problem).

If $\mathbb{R}\text{-rank}\, G = 0$, or equivalently if G is compact, it is obvious that there is no interesting example. The same is true for the case of $\mathbb{R}\text{-rank}\, G = 1$ even though it is not as obvious.

1.2. **Proposition** (see 2.7 and 2.3)**.** *If $\mathbb{R}\text{-rank}\, G = 1$, then G/H does not have a compact Clifford-Klein form, unless either H or G/H is compact.*

So the first interesting case is when $\mathbb{R}\text{-rank}\, G = 2$. For $G = \mathrm{SL}(3,\mathbb{R})$, there are again no interesting examples unless H is compact. This was proved by Y. Benoist when H is reductive. Generalizing the same method to other subgroups leads to the following:

1.3. **Theorem** (Benoist [Ben], Oh-Witte [OW2])**.** *If $G = \mathrm{SL}(3,\mathbb{R})$, then G/H does not have a compact Clifford-Klein form, unless either H or G/H is compact.*

We now consider $G = \mathrm{SO}(2,n)$. We determine exactly which homogeneous spaces of $\mathrm{SO}(2,n)$ have a compact Clifford-Klein form in the case where n is even (see 1.7), and we have almost complete results in the case where n is odd (see 1.9). The work leads to new examples of homogeneous spaces of $\mathrm{SO}(2,n)$, (n even), that have compact Clifford-Klein forms (see 1.5).

In the following we realize $\mathrm{SO}(2,n)$ as isometries of the indefinite form $\langle v \mid v\rangle = v_1 v_{n+2} + v_2 v_{n+1} + \sum_{i=3}^{n} v_i^2$ on $\mathbb{R}^{n+2}$ (for $v = (v_1, v_2, \ldots, v_{n+2}) \in \mathbb{R}^{n+2}$).

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1.4. **Notation.** Let A be the subgroup consisting of the diagonal matrices in $\mathrm{SO}(2,n)$ whose diagonal entries are all positive, and let N be the subgroup consisting of the upper-triangular matrices in $\mathrm{SO}(2,n)$ with only 1's on the diagonal. Thus the Lie algebra of AN is

$$\mathfrak{a}+\mathfrak{n} = \left\{ \begin{pmatrix} t_1 & \phi & x & \eta & 0 \\ & t_2 & y & 0 & -\eta \\ & & 0 & -y^T & -x^T \\ & & & -t_2 & -\phi \\ & & & & -t_1 \end{pmatrix} \middle| \; t_1, t_2, \phi, \eta \in \mathbb{R}, x, y \in \mathbb{R}^{n-2} \right\}.$$

Note that the first two rows of any element of $\mathfrak{a}+\mathfrak{n}$ are sufficient to determine the entire matrix.

Before we describe our new examples of homogeneous spaces of $\mathrm{SO}(2,2m)$ having compact Clifford-Klein forms, let us first recall the construction of compact Clifford-Klein forms found by Kulkarni [Kul, Thm. 6.1] (see also [Kb1, Prop. 4.9]). Consider the subgroup $\mathrm{SU}(1,m)$ as a subgroup of $\mathrm{SO}(2,2m)$ embedded in a standard way. Then $\mathrm{SU}(1,m)$ acts properly and transitively on the homogeneous space $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$. Therefore any co-compact lattice Γ in $\mathrm{SO}(1,2m)$ acts properly on $\mathrm{SO}(2,2m)/\mathrm{SU}(1,m)$, and the quotient $\Gamma\backslash\mathrm{SO}(2,2m)/\mathrm{SU}(1,m)$ is compact. Now let $H_{\mathrm{SU}} = \mathrm{SU}(1,m)\cap(AN)$. Since H_{SU} is a connected co-compact subgroup of $\mathrm{SU}(1,m)$, it is not difficult to see that $\Gamma\backslash\mathrm{SO}(2,2m)/H_{\mathrm{SU}}$ is a compact Clifford-Klein form as well. (Similarly, Kulkarni also constructed compact Clifford-Klein forms $\Lambda\backslash\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$, by letting Λ be a co-compact lattice in $\mathrm{SU}(1,m)$.)

The following theorem demonstrates how to construct new examples of compact Clifford-Klein forms $\Gamma\backslash\mathrm{SO}(2,2m)/H_B$. The subgroup H_B of $\mathrm{SO}(2,2m)$ is obtained by deforming H_{SU}, but H_B is almost never contained in any conjugate of $\mathrm{SU}(1,m)$.

1.5. **Theorem** (cf. Thm. 4.1). *Let $G=\mathrm{SO}(2,2m)$ with $m\ge 2$. Let $B\colon \mathbb{R}^{2m-2}\to\mathbb{R}^{2m-2}$ be a linear transformation. Set*

$$\mathfrak{h}_B = \left\{ \begin{pmatrix} t & 0 & x & \eta & 0 \\ & t & B(x) & 0 & -\eta \\ & & \cdots & & \end{pmatrix} \middle| \; x\in\mathbb{R}^{2m-2}, t,\eta\in\mathbb{R} \right\} \tag{1.6}$$

and let H_B be the corresponding closed, connected subgroup of G. Suppose that B has no real eigenvalue. Then for any co-compact lattice Γ in $\mathrm{SO}(1,2m)$, the quotient $\Gamma\backslash\mathrm{SO}(2,2m)/H_B$ is a compact Clifford-Klein form.

Furthermore H_B is conjugate via $\mathrm{O}(2,2m)$ to a subgroup of $\mathrm{SU}(1,m)$ if and only if for some $a,b\in\mathbb{R}$ (with $b\ne 0$), the matrix of B with respect to some orthonormal basis of $\mathbb{R}^{2m-2}$ is a block diagonal matrix each of whose blocks is $\begin{pmatrix} a & b \\ -b & a\end{pmatrix}$.

In fact one can obtain uncountably many pairwise non-conjugate subgroups of the form H_B by varying B (cf.[OW2]) We also obtain similar new examples of compact Clifford Klein forms of homogeneous spaces of $\mathrm{SU}(2,2m)$ and $\mathrm{SO}(4,4m)$ (see section 4).

We recall that extending work of Goldman [Gol], Kobayashi [Kb6, Thm. B] showed that a co-compact lattice in $\mathrm{SU}(1,m)$ can be deformed to a discrete subgroup Λ such that Λ acts properly on $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$ and the quotient space $\Lambda\backslash\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$ is compact, but Λ is not contained in any conjugate of $\mathrm{SU}(1,m)$. Note that Kobayashi created new compact Clifford-Klein forms by deforming the discrete group while keeping the homogeneous space $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$ fixed. In contrast, we deform the homogeneous space $\mathrm{SO}(2,2m)/H_{\mathrm{SU}}$ to another homogeneous space $\mathrm{SO}(2,2m)/H_B$ while keeping the discrete group Γ in $\mathrm{SO}(1,2m)$ fixed.

For even n, we show that the Kulkarni examples and our deformations are essentially the only interesting homogeneous spaces of $\mathrm{SO}(2,n)$ that have compact Clifford-Klein forms when H is non-compact. We assume that $H\subset AN$ as the general case reduces to this (see 3.5).

1.7. **Theorem** (cf. Thm. 5.1). *Let $G=\mathrm{SO}(2,2m)$ with $m\ge 2$ and H a connected closed subgroup of AN such that neither H nor G/H is compact. The homogeneous space G/H has a compact Clifford-Klein form if and only if either*

1) *H is conjugate to a co-compact subgroup of $\mathrm{SO}(1,2m)$; or*
2) *H is conjugate to H_B for some B as described in Theorem 1.5.*

It is conjectured [Kb6, 1.4] that if H is reductive and G/H has a compact Clifford-Klein form, then there exists a reductive subgroup L of G such that L acts properly on G/H and the double-coset space $L\backslash G/H$ is compact. Because there is no such subgroup L in the case where $G = \mathrm{SO}(2, 2m+1)$ and $H = \mathrm{SU}(1, m)$, the following is a special case of the general conjecture.

1.8. **Conjecture.** For $m \geq 1$, the homogeneous space $\mathrm{SO}(2, 2m + 1)/\mathrm{SU}(1, m)$ does not have a compact Clifford-Klein form.

If this conjecture is true, then there is no interesting example of a homogeneous space of $\mathrm{SO}(2, 2m + 1)$ admitting a compact Clifford-Klein form unless H is compact.

1.9. **Theorem** (cf. Thm. 5.1)**.** *Let $G = \mathrm{SO}(2, 2m + 1)$ with $m \geq 1$. Assume that $G/\mathrm{SU}(1, m)$ does not have a compact Clifford-Klein form. If H is a connected closed subgroup of G such that neither H nor G/H is compact, then G/H does not have a compact Clifford-Klein form.*

Here is a summary of the paper. In section 2, we define the notion of "Cartan-decomposition subgroup" and note that if H is such, then no non-compact subgroup of G acts properly on G/H. We also discuss some of the main results of [OW1] which list all the subgroups of $\mathrm{SO}(2, n)$ and $\mathrm{SL}(3, \mathbb{R})$ that are not Cartan-decomposition subgroups, and hence all the homogeneous spaces admitting a proper action by a non-compact subgroup. Our proofs of Theorems 1.7 and 1.9 then reduce to determining whether each of these homogeneous spaces has a compact Clifford-Klein form. In section 3, we state some results of Kobayashi (Theorem 3.1(1)) and of Margulis (Theorem 3.2) which imply that certain of these homogeneous spaces do not have compact Clifford-Klein forms. Theorem 3.1(3) provides a method to determine whether a double-coset space $\Gamma\backslash G/H$ is compact or not. In section 4, we describe our new examples of compact Clifford-Klein forms of $\mathrm{SO}(2, 2m)$, $\mathrm{SU}(2, 2m)$ and $\mathrm{SO}(4, 4m)$, and sketch the proof of Theorem 1.6. In section 5, we outline the proof of our classification results (Theorems 1.7 and 1.9). Finally in section 6, we state similar results for finite volume-Clifford-Klein forms of $\mathrm{SO}(2, n)$.

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2. Cartan-decomposition subgroups

Let G be a Zariski-connected semisimple linear Lie group as in the introduction. We fix an Iwasawa decomposition $G = KAN$ and a corresponding Cartan decomposition $G = KA^+K$, where A^+ is the (closed) positive Weyl chamber of A in which the roots occurring in the Lie algebra of N are positive. Thus K is a maximal compact subgroup, A is the identity component of a maximal split torus and N is a maximal unipotent subgroup.

The terminology introduced in the following definition is new, but the underlying concept is well known.

2.1. **Definition.** A connected closed subgroup H of G is said to be a *Cartan-decomposition subgroup* of G if $G = CHC$ for some compact subset C of G.

Note that C is only assumed to be a sub*set* of G; it need not be a sub*group*. Some examples of Cartan-decomposition subgroups are the maximal split torus A (due to the Cartan decomposition $G = KAK$) and the maximal unipotent subgroup N (by a result of Kostant asserting that $G = KNK$ [Kos, Thm. 5.1]). If G is compact, then all subgroups of G are Cartan-decomposition subgroups. On the other hand, if G is non-compact then not all subgroups are Cartan-decomposition subgroups, because it is obvious that every Cartan-decomposition subgroup of G must be non-compact. It is a somewhat less obvious fact that if H is a Cartan-decomposition subgroup of G, then $\dim H \geq \mathbb{R}\text{-rank}\, G$.

Our interest in Cartan-decomposition subgroups is largely motivated by the following basic observation that, to construct nicely behaved actions on homogeneous spaces, one must find subgroups that are *not* Cartan-decomposition subgroups. (See [Kb5, §3] for some historical background on this result.)

2.2. **Proposition** (Calabi-Markus phenomenon, cf. [Kul, pf. of Thm. A.1.2]). *If H is a Cartan-decomposition subgroup of G, then no closed non-compact subgroup of G acts properly on G/H.*

Proof. Suppose that $G = CHC$ for some compact subset C. We may assume $C = C^{-1}$ without loss of generality. If L acts properly on G/H, then the set $\{g \in L | gCH \cap CH \neq \emptyset\}$ must be compact by the definition of proper action. But this set is L itself since $L \subset G = CHC$. Therefore L must be compact. □

2.3. **Corollary.** *If H is a Cartan-decomposition subgroup of G, then G/H does not have a compact Clifford-Klein form, unless either H or G/H is compact.*

This leads to the following outline for our proof of Theorems 1.7 and 1.9:

1) Find all closed, connected subgroups H that are *not* Cartan-decomposition subgroups.
2) For each such H, determine whether there is a discrete subgroup Γ of G such that $\Gamma\backslash G/H$ is a compact Clifford-Klein form.

To classify the Cartan-decomposition subgroups of G, our main tool is the Cartan projection.

2.4. **Definition** (Cartan projection). For each element g of G, the Cartan decomposition $G = KA^+K$ implies that there is a unique element a of A^+ with $g \in KaK$. So there is a well-defined function $\mu\colon G \to A^+$ given by $g \in K\,\mu(g)\,K$.

It is easy to see that the function μ is continuous and proper. We now recall a fundamental result of Benoist and Kobayashi that enables one to use the Cartan projection to determine whether an action is proper or not.

2.5. **Theorem** (Benoist [Ben, Prop. 1.5], Kobayashi [Kb4, Cor. 3.5]). *Let H_1 and H_2 be closed subgroups of G. The subgroup H_1 acts properly on G/H_2 if and only if $\big(\mu(H_1)C\big) \cap \mu(H_2)$ is compact for any compact subset C of A.*

As an immediate corollary we obtain:

2.6. **Corollary** (Benoist [Ben, Prop. 5.1], Kobayashi [Kb4, Thm. 1.1]). *A closed connected subgroup H of G is a Cartan-decomposition subgroup if and only if $A^+ \subset \mu(H)C$ for some compact subset C of A, or equivalently $\mu(H)$ comes within a bounded distance of every point in A^+.*

We noted above that every subgroup is a Cartan-decomposition subgroup if $\mathbb{R}\text{-rank}\, G = 0$. The following simple proposition shows that the characterization is again very easy if $\mathbb{R}\text{-rank}\, G = 1$.

2.7. **Proposition** (cf. [Kb3, Lem. 3.2]). *Let $\mathbb{R}\text{-rank}\, G = 1$. A closed connected subgroup H of G is a Cartan-decomposition subgroup if and only if H is non-compact.*

Proof. ($\Leftarrow$) We have $\mu(e) = e$ and because μ is a proper map, we have $\mu(h) \to \infty$ as $h \to \infty$ in H. Because $\mathbb{R}\text{-rank}\, G = 1$, we know that A^+ is homeomorphic to the half-line $[0, \infty)$ (with the point e in A^+ corresponding to the endpoint 0 of the half-line). Hence, by continuity, it must be the case that $\mu(H) = A^+$. Therefore $KHK = G$, so H is a Cartan-decomposition subgroup. □

It seems to be much more difficult to characterize the Cartan-decomposition subgroups when $\mathbb{R}\text{-rank}\, G = 2$, so these are the first interesting cases. In [OW1], using Corollary 2.6, we study two examples in detail. When $G = \mathrm{SL}(3,\mathbb{R})$ or $\mathrm{SO}(2,n)$, we give an approximate calculation of the image of each subgroup of G under the Cartan projection. This yields an explicit description of all the Cartan-decomposition subgroups of G.

Obviously any connected closed subgroup that contains a Cartan-decomposition subgroup is itself a Cartan-decomposition subgroup. Therefore the minimal Cartan-decomposition subgroups are the most interesting ones. As a simple example of our results, we state the following theorem.

2.8. **Theorem.** *Let $G = \mathrm{SL}(3,\mathbb{R})$. Up to conjugation by automorphisms of G, the only minimal Cartan-decomposition subgroups of G are:*

$$A, \qquad \left\{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r, s \in \mathbb{R} \right\}, \qquad \left\{ \begin{pmatrix} e^t & te^t & s \\ 0 & e^t & r \\ 0 & 0 & e^{-2t} \end{pmatrix} \,\middle|\, r, s, t \in \mathbb{R} \right\},$$

and subgroups of the form

$$\left\{ \begin{pmatrix} e^{pt} & r & 0 \\ 0 & e^{qt} & 0 \\ 0 & 0 & e^{-(p+q)t} \end{pmatrix} \,\middle|\, r,t \in \mathbb{R} \right\},$$

where p and q are fixed real numbers with $\max\{p,q\} = 1$ *and* $\min\{p,q\} \geq -1/2$, *or of the form*

$$\left\{ \begin{pmatrix} e^t \cos pt & e^t \sin pt & s \\ -e^t \sin pt & e^t \cos pt & r \\ 0 & 0 & e^{-2t} \end{pmatrix} \,\middle|\, r,s,t \in \mathbb{R} \right\},$$

where p is a fixed nonzero real number.

Note that AN contains uncountably many non-conjugate minimal Cartan-decomposition subgroups of G since the minimum of the two parameters p and q in the above theorem can be varied continuously. However, up to conjugacy under $\operatorname{Aut} G$, there is only one minimal Cartan-decomposition subgroup contained in A (namely, A itself) and only one contained in N.

Theorem 2.9 is a sample of our results on Cartan-decomposition subgroups of SO$(2,n)$.

2.9. **Theorem.** *Let G = SO$(2,5)$. Then there are exactly 6 non-conjugate minimal Cartan-decomposition subgroups of G contained in N. The Lie algebra of each such subgroup is conjugate to one of the following:*

1) $\left\{ \begin{pmatrix} 0 & \phi & 0 & 0 & 0 & \eta & 0 \\ & 0 & \epsilon_1\phi & 0 & 0 & 0 & -\eta \\ & & & \cdots & & & \end{pmatrix} \,\middle|\, \phi, \eta \in \mathbb{R} \right\}$ *where* $\epsilon_1 \in \{0,1\}$

2) $\left\{ \begin{pmatrix} 0 & \phi & x & 0 & 0 & 0 & 0 \\ & 0 & 0 & \epsilon_2\phi & 0 & 0 & 0 \\ & & & \cdots & & & \end{pmatrix} \,\middle|\, \phi, x \in \mathbb{R} \right\}$ *where* $\epsilon_2 \in \{0,1\}$

3) $\left\{ \begin{pmatrix} 0 & 0 & x & 0 & \epsilon_3 y & 0 & 0 \\ & 0 & 0 & y & 0 & 0 & 0 \\ & & & \cdots & & & \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ *where* $\epsilon_3 \in \{0,1\}$.

In [OW1], we describe all the Cartan-decomposition subgroups of SO$(2,n)$.

3. General results on compact Clifford-Klein forms

In this section we state some general results on compact Clifford-Klein forms. Recall that G is a Zariski-connected semisimple linear Lie group.

For a connected Lie group H, we use the notation $d(H) = \dim H - \dim K_H$, where K_H is a maximal compact subgroup of H (cf. [Kb1, (2.5), §5]). Since all the maximal compact subgroups of H are conjugate [Hoc, Thm. XV.3.1, p. 180–181], this is well defined. Note that if $H \subset AN$ then $d(H) = \dim H$ because AN has no non-trivial compact subgroups.

The following theorem is a very useful generalization of Corollary 2.3.

3.1. **Theorem** (Kobayashi, cf. [Kb1, Cor. 5.5] and [Kb2, Thm. 1.5])**.** *Let H be a closed connected subgroup of G. Assume that there exists a closed connected subgroup L such that $L \subset CHC$ for some compact subset C of G.*

1) *If $d(L) > d(H)$, then G/H does not have a compact Clifford-Klein form.*
2) *If $d(L) = d(H)$ and G/H has a compact Clifford-Klein form, then G/L also has a compact Clifford-Klein form.*
3) *If there is a closed subgroup L' of G such that L' acts properly on G/H, $d(H)+d(L') = d(G)$ and there is a co-compact lattice Γ in L', then the quotient $\Gamma\backslash G/H$ is compact.*

Kobayashi assumed that H is reductive, but the same proof works with only minor changes. Let us give an elementary proof of 3.1(1) under the simplifying assumption that $H \subset L$. Let Γ be a discrete group that acts properly on G/H. Because $L \subset CHC$, we know that Γ also acts properly on G/L, so $\Gamma\backslash\Gamma L/H$ is closed in $\Gamma\backslash G/H$. By replacing Γ with a finite-index subgroup, we may assume that $\Gamma \cap L = e$. Then $\Gamma\backslash\Gamma L/H$ is homeomorphic to L/H, which is non-compact (because $d(L) > d(H)$). Thus $\Gamma\backslash G/H$ has a closed non-compact subset and hence $\Gamma\backslash G/H$ is not compact.

Recall the following notion introduced by Margulis (cf. [Mar, Defn. 2.2]): A closed subgroup H of G is said to be (G,K)-*tempered* if there exists a function $q \in L^1(H)$ such that for every non-trivial irreducible unitary representation π of G with a K-fixed unit vector v, we have $|\langle \pi(h)v, v\rangle| \le q(h)$ for all $h \in H$.

3.2. **Theorem** (Margulis [Mar, Thm. 3.1]). *If H is a closed non-compact (G,K)-tempered subgroup of G, then G/H does not have a compact Clifford-Klein form.*

We refer to [Oh] for a method to determine when H is (G,K)-tempered as well as for examples of (G,K)-tempered subgroups.

Using Theorems 3.1 and 3.2, we prove the following:

3.3. **Proposition.** *Let G be a connected simple linear Lie group. If H is a closed connected non-compact one-parameter subgroup of G, then G/H does not have a compact Clifford-Klein form.*

Sketch of proof. By Proposition 1.2, we may assume that $\mathbb{R}$-rank G is at least 2. If H is unipotent, there exists a connected closed subgroup L locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ which contains H. Then H is a Cartan-decomposition subgroup of L by Proposition 2.7 and $d(L) = 2 > 1 = d(H)$. So Theorem 3.1(1) applies. If H is diagonalizable, then H is (G,K)-tempered subgroup (cf. [Mar]). Hence Theorem 3.2 applies. The remaining case is when $H = \{a_t u_t | t \in \mathbb{R}\}$ where a_t is semisimple and u_t is unipotent such that a_t commutes with u_t. It can be seen that H is a (G,K)-tempered subgroup in this case as well. □

3.4. *Remark.* Benoist and Labourie [BL] proved that if H is unimodular and the center of H contains a non-trivial connected subgroup of A, then G/H does not have a compact Clifford-Klein form. This provides an alternate proof of Proposition 3.3 in the special case when H is conjugate to a subgroup of A.

We use the following well-known lemma to reduce the study of compact Clifford-Klein forms of G/H to the case where $H \subset AN$. We remark that the proof is constructive.

3.5. **Lemma.** *Let H be a closed connected subgroup of G. Then there is a closed connected subgroup H' of G such that*

1) *H' is conjugate to a subgroup of AN;*
2) $\dim H' = d(H)$ *; and*
3) *$CH = CH'$ for some compact connected subgroup C of G.*

Moreover G/H has a compact Clifford-Klein form if and only if G/H' has a compact Clifford-Klein form.

Sketch of proof. Replace H by a conjugate so that $\overline{H} \cap AN$ is co-compact in $\overline{H}$ where $\overline{H}$ is the Zariski closure of $\overline{H}$, and choose a maximal connected compact subgroup $\overline{C}$ of $\overline{H}$. Then write $\overline{C} = C_1 C_2$, where C_1 is a maximal compact subgroup of H and C_2 is contained in the Zariski closure of $\operatorname{Rad} H$. Finally, let $H' = (HC_2) \cap (AN)$. □

4. New examples of compact Clifford-Klein forms

The first half of this section is devoted to a proof of the following theorem, and in the other half, we describe analogous examples of homogeneous spaces $\mathrm{SU}(2,2m)$ and $\mathrm{SO}(4,4m)$ admitting compact Clifford-Klein forms.

4.1. **Theorem.** *Let $G = \mathrm{SO}(2,2m)$ with $m \ge 2$ and H_B be as in Theorem 1.5. Suppose that B has no real eigen-value. Then $\Gamma \backslash \mathrm{SO}(2,2m)/H_B$ is a compact Clifford-Klein for any co-compact lattice Γ in $\mathrm{SO}(1,2m)$.*

To establish the above theorem, we will first show that $\mathrm{SO}(1,2m)$ and hence Γ acts properly on $\mathrm{SO}(2,2m)/H_B$. By Theorem 2.5, it suffices to show that $\mu(\mathrm{SO}(1,2m))$ and $\mu(H_B)$ diverge from each other.

To calculate the approximate image of any subgroup of $\mathrm{SO}(2,2m)$ under the Cartan projection, we use a method of Benoist [Ben] that approximates $\mu(h)$ by using the norms of the image of h under the (two) fundamental representations of $\mathrm{SO}(2,2m)$. Consider the representation

$$\rho\colon \mathrm{SO}(2,2m) \to \mathrm{SL}\big(\mathbb{R}^{2m+2} \wedge \mathbb{R}^{2m+2}\big) \text{ given by } \rho(g) = g \wedge g.$$

Then the two representations $g \mapsto g$ and $g \mapsto \rho(g)$ form a set of fundamental representations of G.

4.2. **Notation.** For subsets $X, Y \subset A^+$, we write $X \approx Y$ if there is a compact subset C of A with $X \subset YC$ and $Y \subset XC$.

4.3. **Notation.** For a subgroup H of G and for $i, j \in \{1, 2\}$, we write $\mu(H) \approx \left[\|h\|^i, \|h\|^j\right]$ if, for every sufficiently large $C > 1$, we have

$$\mu(H) \approx \left\{ a \in A^+ \;\middle|\; C^{-1}\|a\|^i \leq \|\rho(a)\| \leq C\|a\|^j \right\},$$

where $\|h\|$ denotes the norm of the linear transformation h.

If we denote the two walls of A^+ by W_1 and W_2, then $\|h\|^i = \|\rho(h)\|$ for $h \in W_i$ (possibly after interchanging W_1 and W_2). Because $\mathrm{SO}(1, 2m)$ and $\mathrm{SU}(1, m)$ are reductive, their Cartan projections are easy to calculate; we have $\mu(\mathrm{SO}(1, 2m)) = W_1$ and $\mu(\mathrm{SU}(1, m)) = W_2$. In the following proposition, we show that $\mu(H_B)$ stays within a bounded distance of W_2 as long as B has no real eigen-value.

4.4. **Proposition.** *If B has no real eigen-value, then $\mu(H_B) \approx [\|h\|^2, \|h\|^2] \approx \mu(\mathrm{SU}(1, m))$ (hence $SO(1, 2m)$ acts properly on G/H_B).*

Proof. Given $h \in H_B$, write $h = au$ with $a \in A \cap H_B$ and $u \in N \cap H_B$. There is some $t \in \mathbb{R}^+$ with

$$a = \mathrm{diag}(t, t, 1, 1, \ldots, 1, 1, t^{-1}, t^{-1}).$$

In the following we use the notation $f_1 \asymp f_2$ if $f_1 = O(f_2)$ and $f_2 = O(f_1)$. Because $h \asymp h^{-1}$ and $\rho(h^{-1}) \asymp \rho(h)$, we may assume that $t \geq 1$.

There is some $Z \in \mathfrak{h}_B \cap \mathfrak{n}$ with $u = \exp Z$. From (1.6), we know that there exist $x \in \mathbb{R}^{2m-2}$ and $\eta \in \mathbb{R}$ with

$$Z = \begin{pmatrix} 0 & 0 & x & \eta & 0 \\ & 0 & B(x) & 0 & -\eta \\ & & \cdots & & \end{pmatrix}.$$

Calculating $\exp Z$, we obtain

$$u = \begin{pmatrix} 1 & 0 & x & \eta - \frac{1}{2}(x \cdot Bx) & -\frac{1}{2}\|x\|^2 \\ & 1 & Bx & -\frac{1}{2}\|Bx\|^2 & -\eta - \frac{1}{2}(x \cdot Bx) \\ & & \mathrm{Id} & -(Bx)^T & -x^T \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}.$$

Then it is clear that $\|h\| = \|au\| \asymp \max\{t, t|\eta|, t\|x\|^2\}$. To calculate $\|\rho(h) = h \wedge h\|$ approximately, we use the norm $\|\rho(h)\|$ that is the maximum absolute value among the determinants of all the 2×2 submatrices of h.

We have $\det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} = t^2$ and

$$\begin{aligned} 4\left|\det \begin{pmatrix} h_{1,2m+1} & h_{1,2m+2} \\ h_{2,2m+1} & h_{2,2m+2} \end{pmatrix}\right| &= 4t^2 \left|\det \begin{pmatrix} u_{1,2m+1} & u_{1,2m+2} \\ u_{2,2m+1} & u_{2,2m+2} \end{pmatrix}\right| \\ &= t^2\Big(4\eta^2 + \big(\|x\|^2\,\|Bx\|^2 - (x \cdot Bx)^2\big)\Big). \end{aligned}$$

Since B has no real eigen-value, x and Bx are linearly independent for all non-trivial $x \in \mathbb{R}^{2m-2}$. Hence we have

$$\|x\|^2\,\|Bx\|^2 - (x \cdot Bx)^2 \asymp \|x\|^2\,\|Bx\|^2 \asymp \|x\|^4.$$

Therefore $\|\rho(h)\| \asymp \max(t^2, t^2\eta^2, t^2\|x\|^4) \asymp \|h\|^2$, that is, $\mu(H_B) \approx [\|h\|^2, \|h\|^2]$. It now follows from Theorem 2.5 that $\mathrm{SO}(1, 2m)$ acts properly on G/H_B. □

Now note that $d\big(\mathrm{SO}(2, 2m)\big) = 4m$, $d\big(\mathrm{SO}(1, 2m)\big) = 2m$ and $d(H_B) = \dim(H_B) = 2m$; hence $d\big(\mathrm{SO}(2, 2m)\big) = d\big(\mathrm{SO}(1, 2m)\big) + d(H_B)$. Therefore the quotient space $\Gamma\backslash \mathrm{SO}(2, 2m)/H_B$ is compact for any co-compact lattice Γ in $\mathrm{SO}(1, 2m)$, by Theorem 3.1(3). This finish the proof of Theorem 4.1.

4.5. *Remark.* The subgroups H_B of Theorem 1.5 are not all isomorphic (unless $m = 2$). For example, let $m = 3$ and let

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of B is $\det(\lambda - B) = \lambda^4 - \lambda^2 + 1$, which has no real zeros, so B has no real eigenvalues. Let $v = (0,0,0,1)$. We have $B^T v = Bv$, so, for every $x \in \mathbb{R}^4$, we have $x \cdot Bv - v \cdot Bx = 0$. Thus if h is any element of $\mathfrak{h}_B \cap \mathfrak{n}$ with $x_h = v$, then h is in the center of $\mathfrak{h}_B \cap \mathfrak{n}$. Therefore the center of $\mathfrak{h}_B \cap \mathfrak{n}$ contains $\langle h, \mathfrak{u}_{\alpha+2\beta} \rangle$, so the dimension of the center is at least 2. (In fact, the center is 3-dimensional.) Because the center of $\mathfrak{h}_{\mathrm{SU}} \cap \mathfrak{n}$ is $\mathfrak{u}_{\alpha+2\beta}$, which is one-dimensional, we conclude that $\mathfrak{h}_B$ is not isomorphic to $\mathfrak{h}_{\mathrm{SU}}$.

We now describe how to construct examples analogous to those of the above theorem for $\mathrm{SU}(2,2m)$ and $\mathrm{SO}(4,4m)$. Let us recall that $\Gamma_1 \backslash \mathrm{SU}(2,2m)/\mathrm{Sp}(1,m)$ and $\Gamma_2 \backslash \mathrm{SO}(4,4m)/\mathrm{Sp}(1,m)$ are compact Clifford-Klein forms where Γ_1 and Γ_2 are co-compact lattices of $\mathrm{SU}(1,2m)$ and $\mathrm{SO}(3,4m)$ respectively (see [Kb5]). We shall define a subgroup $\hat{H}_B$ of $\mathrm{SU}(2,2m)$ for every linear transformation $B : \mathbb{R}^{2m-2} \to \mathbb{R}^{2m-2}$, which can be considered as a deformation of a co-compact subgroup of $\mathrm{Sp}(1,m)$. More precisely, let us realize $\mathrm{SU}(2,2m)$ as isometries of the following form $\langle v \mid v \rangle = v_1 \bar{v}_{2m+2} + v_2 \bar{v}_{2m+1} + \sum_{i=3}^{2m} |v_i|^2$ on $\mathbb{C}^{2m+2}$ (for $v = (v_1, v_2, \ldots, v_{2m+2}) \in \mathbb{C}^{2m+2}$). Thus the Lie algebra of AN is

$$\left\{ \left. \begin{pmatrix} t_1 & \phi & x & \eta & ui \\ & t_2 & y & wi & -\bar{\eta} \\ & & 0 & -\bar{y}^T & -\bar{x}^T \\ & & & -t_2 & -\bar{\phi} \\ & & & & -t_1 \end{pmatrix} \right| t_1, t_2, u, w \in \mathbb{R}, \phi, \eta \in \mathbb{C}, x, y \in \mathbb{C}^{2m-2} \right\}.$$

For any linear transformation $B \in \mathrm{Mat}(2m-2, \mathbb{R}) \subset \mathrm{Mat}(2m-2, \mathbb{C})$, set $\hat{H}_B$ to be the connected closed subgroup of G whose Lie algebra is

$$\left\{ \left. \begin{pmatrix} t & 0 & x & \eta & wi \\ & t & B\bar{x} & wi & -\bar{\eta} \\ & & 0 & -(B\bar{x})^T & -\bar{x}^T \\ & & & -t & 0 \\ & & & & -t \end{pmatrix} \right| t, w \in \mathbb{R}, \eta \in \mathbb{C}, x \in \mathbb{C}^{2m-2} \right\}.$$

Note that if B_0 is the block diagonal matrix each of whose blocks is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $\hat{H}_{B_0}$ is a co-compact subgroup of $\mathrm{Sp}(1,m)$, more precisely $\mathrm{Sp}(1,m) \cap AN$.

4.6. **Theorem.** *Suppose that $B \in \mathrm{SO}(2m-2)$ and that B has no real eigenvalue. Then*

1) $\Gamma \backslash \mathrm{SU}(2,2m)/\hat{H}_B$ *is a compact Clifford-Klein for any co-compact lattice* Γ *in* $\mathrm{SU}(1,2m)$.
2) $\Gamma \backslash \mathrm{SO}(4,4m)/\hat{H}_B$ *is a compact Clifford-Klein for any co-compact lattice* Γ *in* $\mathrm{SO}(3,4m)$ *(since* $\mathrm{SU}(2,2m) \subset \mathrm{SO}(4,4m)$*, we may consider* $\hat{H}_B \subset \mathrm{SO}(4,4m)$*).*

The proof of the above theorem is similar to that of Theorem 4.1.

5. Non-existence results on compact Clifford-Klein forms of $\mathrm{SO}(2,n)/H$

The "if" direction of Theorem 1.7 is obtained by Theorem 4.1 and Kulkarni's construction [Kul, Thm. 6.1] of compact Clifford-Klein forms of $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$. In this section we outline the proof of the following theorem which contains Theorem 1.9 and the "only if" direction of Theorem 1.7.

5.1. **Theorem.** *Let $G = \mathrm{SO}(2,n)$ with $n \geq 3$. Let H be a closed connected subgroup of AN such that neither H nor G/H is compact. Suppose that G/H has a compact Clifford-Klein form.*

1) *If n is even, then H is conjugate either to $\mathrm{SO}(1,n) \cap AN$ or to H_B (see Theorem 1.5 for notation).*

2) *If n is odd, then* $\dim H = n-1$ *and* $\mathrm{SU}(1,(n-1)/2) \subset CHC$ *for some compact subset C of G.*

Let $L_5 \subset \mathrm{SL}_5(\mathbb{R})$ be the image of $\mathrm{SL}_2(\mathbb{R})$ under an irreducible 5-dimensional representation of $\mathrm{SL}_2(\mathbb{R})$. More concretely, we may take the Lie algebra of L_5 to be the image of the homomorphism $\pi\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(2,3)$ given by

$$\pi\begin{pmatrix} t & u \\ v & -t \end{pmatrix} = \left\{ \begin{pmatrix} 4t & 2u & 0 & 0 & 0 \\ 2v & 2t & \sqrt{6}u & 0 & 0 \\ 0 & \sqrt{6}v & 0 & -\sqrt{6}u & 0 \\ 0 & 0 & -\sqrt{6}v & -2t & -2u \\ 0 & 0 & 0 & -2v & -4t \end{pmatrix} \,\middle|\, t,u,v \in \mathbb{R} \right\}.$$

Via the embedding $\mathbb{R}^5 \hookrightarrow \mathbb{R}^{n+2}$ given by

$$(x_1,x_2,x_3,x_4,x_5) \mapsto (x_1,x_2,x_3,0,0,\ldots,0,0,x_4,x_5),$$

we may realize $\mathrm{SO}(2,3)$ as a subgroup of $\mathrm{SO}(2,n)$, so we may view L_5 as a subgroup of $\mathrm{SO}(2,n)$ for any $n \geq 3$.

In view of Theorem 3.1(1), Proposition 3.3 and Lemma 3.5, the following theorem reduces the proof of Theorem 5.1 to the case when H is conjugate to a co-compact subgroup of $\mathrm{SO}(1,n)$ or L_5.

5.2. **Theorem.** *Let $G = \mathrm{SO}(2,n)$ with $n \geq 3$. Let H be a closed connected subgroup of AN such that neither H nor G/H is compact. Suppose that* $\dim H \geq 2$ *and that there is no non-trivial connected subgroup L of AN such that* $\dim L > \dim H$ *and $L \subset CHC$ for some compact subset C of G. Then one of the following holds:*

1) *H is conjugate to* $\mathrm{SO}(1,n) \cap AN$*;*
2) *H is conjugate to $L_5 \cap AN$;*
3) *n is even and H is conjugate to H_B;*
4) *n is odd,* $\dim H = n-1$ *and* $\mathrm{SU}\big(1,(n-1)/2\big) \subset CHC$ *for some compact subset C of G.*

Sketch of proof. For each closed connected subgroup H of $\mathrm{SO}(2,n)$, [OW1] gives explicit functions f_1 and f_2, such that $\mu(H) \approx \big[f_1(\|h\|), f_2(\|h\|)\big]$. This provides the means to check whether there is a compact set C such that $L \subset CHC$ for a given subgroup L. For example, we have $\mu(\mathrm{SU}(1,\lfloor n/2 \rfloor)) \approx \big[\|h\|^2, \|h\|^2\big]$. Thus, if $\mu(H)$ is of the form $\mu(H) \approx \big[\cdot, \|h\|^2\big]$, then there is a compact subset C of G such that $\mathrm{SU}(1,\lfloor n/2 \rfloor) \subset CHC$.

The result is obtained by inspection of the list of subgroups that are not Cartan-decomposition subgroups, and comparing their Cartan projections (see [OW2] for details). □

While the homogeneous space $\mathrm{SO}(2,n)/\mathrm{SO}(1,n)$ for even n does have a compact Clifford-Klein form, the following result of Kulkarni says that the situation is different for odd n.

5.3. **Proposition** (Kulkarni [Kul, Cor. 2.10])**.** *The homogeneous space* $\mathrm{SO}(2,2m+1)/\mathrm{SO}(1,2m+1)$ *does not have a compact Clifford-Klein form.*

To finish the proof of Theorem 5.1, we now only need to exclude the case when H is conjugate to $L_5 \cap AN$.

5.4. **Proposition** (Oh [Oh, Ex. 5.6])**.** *The subgroup L_5 is a (G,K)-tempered subgroup for $G = \mathrm{SO}(2,3)$.*

It easily follows that L_5 is a (G,K)-tempered subgroup of $G = \mathrm{SO}(2,n)$ for any $n \geq 3$. Hence by combining this proposition with Theorem 3.2, we obtain the following corollary, which concludes the proof of Theorem 5.1.

5.5. **Corollary.** *Let $G = \mathrm{SO}(2,n)$ with $n \geq 3$. Then G/L_5 does not have a compact Clifford-Klein form.*

6. Finite-volume Clifford-Klein forms

Let H be a closed connected subgroup of G such that G/H has a G-invariant regular Borel measure. (Because G is unimodular, this means that H is unimodular [Rag, Lem. 1.4, p. 18].)

6.1. **Definition** (cf. [Kb1, Def. 2.2]). We say that G/H has a *finite-volume Clifford-Klein form* if there is a discrete subgroup Γ of G such that

- Γ acts properly on G/H; and
- there is a Borel subset $\mathcal{F}$ of G/H such that $\mathcal{F}$ has finite measure and $\Gamma\mathcal{F} = G/H$.

Unfortunately, the study of finite-volume Clifford-Klein forms does not usually reduce to the case where $H \subset AN$, because the subgroup H' of Proposition 3.5 is usually not unimodular.

6.2. **Theorem.** [OW2] *Let $G = \mathrm{SO}(2,n)$ with $n \geq 3$. Let H be a closed connected subgroup of G. If G/H has a finite-volume Clifford-Klein form, then one of the following holds:*

1) *H has a co-compact normal subgroup of G that is conjugate under $\mathrm{O}(2,n)$ to the identity component of $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,\lfloor n/2 \rfloor)$ or L_5 (see section 5 for notation);*
2) *$d(H) \leq 1$;*
3) *$H = G$.*

It seems natural to conjecture that $\mathrm{SO}(2,2m+1)/\mathrm{SU}(1,m)$ and $\mathrm{SO}(2,n)/L_5$ do not have finite-volume Clifford-Klein forms, and that G/H does not have a finite-volume Clifford-Klein form when $d(H) = 1$, either.

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Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
Current address: Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
E-mail address: heeoh@math.huji.ac.il

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
E-mail address: dwitte@math.okstate.edu, http://www.math.okstate.edu/~dwitte