ON DISCRETE SUBGROUPS CONTAINING A LATTICE IN A HOROSPHERICAL SUBGROUP

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ABSTRACT. We showed in [Oh] that for a simple real Lie group G with real rank at least 2, if a discrete subgroup Γ of G contains lattices in two opposite horospherical subgroups, then Γ must be a non-uniform arithmetic lattice in G, under some additional assumptions on the horospherical subgroups. Somewhat surprisingly, a similar result is true even if we only assume that Γ contains a lattice in one horospherical subgroup, provided Γ is Zariski dense in G.

1. INTRODUCTION

Let G be a connected semisimple algebraic $\mathbb{R}$-group. A unipotent subgroup U of G is called a *horospherical* $\mathbb{R}$-subgroup if U is the unipotent radical of some proper parabolic $\mathbb{R}$-subgroup of G.

The main theorem in this paper is as follows:

1.1. Theorem. *Let G be a connected absolutely simple $\mathbb{R}$-split algebraic group with rank at least 2. Suppose that G is not of type A_2. Let Γ be a discrete Zariski dense subgroup of $G(\mathbb{R})$. Then Γ is a non-uniform arithmetic lattice in $G(\mathbb{R})$ if and only if there exists a horospherical subgroup U of G such that $\Gamma \cap U$ is Zariski dense in U.*

1.2. Corollary. *Let G be as in Theorem 1.1 and U a horospherical $\mathbb{R}$-subgroup of G.*

(1) *Let X be any subset of $U(\mathbb{R})$ such that the subgroup generated by the elements of X is Zariski dense in U.*
(2) *Let Y be any subset of $G(\mathbb{R})$ such that the subgroup $\Gamma_{X\cup Y}$ generated by the elements of $X \cup Y$ is Zariski dense.*

Then either $\Gamma_{X\cup Y}$ is a non-uniform arithmetic lattice in $G(\mathbb{R})$ or $\Gamma_{X\cup Y}$ is not discrete.

The proof of Theorem 1.1 is based on some results obtained in [Oh], which we will recall in the following.

Two parabolic subgroups are called *opposite* if their intersection is a common Levi subgroup in both of them. Two horospherical subgroups are called *opposite* if they are the unipotent radicals of opposite parabolic subgroups.

1.3. Theorem. *(cf.* [Oh, Theorem 0.2]*) Let G be a connected absolutely simple $\mathbb{R}$-algebraic group with real rank at least 2 and U_1, U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups of G. Let F_1 and F_2 be lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively. Suppose that G is split over $\mathbb{R}$ and that if G is of type A_2, then U_1 is not the unipotent radical of a Borel subgroup. Then F_1 and F_2 generate a discrete subgroup if and only if there exists a $\mathbb{Q}$-form of G with respect to which U_1 and U_2 are defined over $\mathbb{Q}$ and $F_i \subset U_i(\mathbb{Z})$ for each $i = 1, 2$. Furthermore the discrete subgroup generated by F_1 and F_2 is a subgroup of finite index in $G(\mathbb{Z})$.*

In [Oh, Theorem 0.2], we proved that if the subgroups F_1 and F_2 as in the statement of Theorem 1.3 generate a discrete subgroup, then there exists a $\mathbb{Q}$-form of G such that the subgroup Γ_{F_1,F_2} generated by F_1 and F_2 is commensurable to the subgroup $G(\mathbb{Z})$. We deduce Theorem 1.3 from this theorem; since G is absolutely simple; hence the center of G is trivial, we can assume that $G \subset SL_N(\mathbb{C})$ by considering the adjoint representation of G. Since Γ_{F_1,F_2} is an arithmetic subgroup, it is not difficult to see that there exists a Γ_{F_1,F_2}-invariant lattice L in $\mathbb{Q}^N$ (cf. [P-R, Proposition 4.2]); hence $\Gamma_{F_1,F_2} \subset G^L = \{g \in G \mid g(L) \subset L\}$. Now by applying the automorphism of $SL_N(\mathbb{C})$ that changes the standard basis to a basis of L, we can change the $\mathbb{Z}$-form of G so that $G(\mathbb{Z}) = G^L$, proving Theorem 1.3.

The following corollary follows from Theorem 1.3 (see [Oh, Corollary 0.5]).

1.4. Corollary. *Let G be a connected absolutely simple $\mathbb{R}$-split algebraic group with rank at least 2 and Γ a discrete subgroup of $G(\mathbb{R})$. Suppose that G is not of type A_2. Then Γ is a non-uniform lattice in $G(\mathbb{R})$ if and only if there exists a pair U_1, U_2 of opposite horospherical $\mathbb{R}$-subgroups of G such that $\Gamma \cap U_i$ is Zariski dense in U_i for each $i = 1, 2$.*

Using Corollary 1.4, Thoerem 1.1 is then a consequence of the following proposition.

1.5. Proposition. *Let G be a connected semisimple $\mathbb{R}$-algebraic group and Γ a discrete Zariski dense subgroup of $G(\mathbb{R})$. Suppose that there exists a horospherical $\mathbb{R}$-subgroup U of G such that $\Gamma \cap U$ is Zariski dense in U. Then there exists a pair U_1, U_2 of opposite horospherical $\mathbb{R}$-subgroups of G such that $U \subset U_1$ and $\Gamma \cap U_i$ is Zariski dense in U_i for each $i = 1, 2$.*

For simplicity, Theorem 1.3 is stated here only for $\mathbb{R}$-split groups but we proved it in much greater generality (see [Oh, Theorem 0.3]). All the above theorems are valid for all situations in which Theorem 0.3 in [Oh] holds.

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2. Unipotent subgroups of Γ

2.1. Definition. Let H be a locally compact topological group, and D and M closed subgroups of H. We say that D is M-proper if the natural map $D/(D\cap M)\to H/M$ is proper, and M-compact if the factor space $D/(D\cap M)$ is compact.

2.2. Lemma [Ma, Lemma 2.1.4]**.** *If M is a discrete subgroup of H, D_1 is M-proper and D_2 is M-compact, then $D_1\cap D_2$ is M-compact.*

2.3. Let G be a connected semisimple algebraic $\mathbb{R}$-group. For a subgroup F of $G(\mathbb{R})$, we denote by $N(F)$ the normalizer of F in G, by $[F,F]$ the commutator subgroup of F, by $S(F)$ the subgroup $[N(F),N(F)]\cap G(\mathbb{R})$. The notation $\overline{F}$ denotes the Zariski closure of F.

The following theorem in [Ma], which was originally stated only for the case where Γ is a lattice, holds for arbitrary discrete subgroups, as we can see from the proof given in [Ma].

Theorem [Ma, Lemma 5.2.2]**.** *Let Γ be a discrete subgroup of $G(\mathbb{R})$ and F a unipotent subgroup of Γ such that $\overline{F}\cap\Gamma=F$. Then $S(\overline{F})$ is Γ-proper.*

2.4. Corollary. *Let G be a connected semisimple algebraic $\mathbb{R}$-group and Γ a discrete subgroup of $G(\mathbb{R})$. Let F_1 and F_2 be unipotent subgroups of Γ such that $\overline{F}_i\cap\Gamma=F_i$ for each $i=1,2$. Then $S(\overline{F}_1)\cap\overline{F}_2$ is Γ-compact. In particular $S(\overline{F}_1)\cap\overline{F}_2\cap\Gamma$ is a co-compact lattice in $S(\overline{F}_1)\cap\overline{F}_2$.*

Proof. By Theorem 2.3, $S(\overline{F_1})$ is Γ-proper. Since a Zariski dense and discrete subgroup of a unipotent algebraic group is a co-compact lattice, $\overline{F_2}\cap G(\mathbb{R})$ is Γ-compact. Therefore, by Lemma 2.2, $S(F_1)\cap F_2$ is Γ-compact. □

Note that the above corollary gives a way of obtaining a subgroup of $\overline{F}_2$ that intersects Γ as a lattice when $S(\overline{F_1})\cap\overline{F_2}$ is non-trivial.

3.1. Let G be a connected semisimple algebraic $\mathbb{R}$-group, S a maximal $\mathbb{R}$-split torus of G and T a maximal $\mathbb{R}$-torus containing S. Denote by ${}_{\mathbb{R}}\Phi = \Phi(S, G)$ (resp. $\Phi = \Phi(T, G)$) the set of roots of G with respect to S (resp. T). We choose compatible orderings on Φ and ${}_{\mathbb{R}}\Phi$, and let ${}_{\mathbb{R}}\Delta$ be the simple roots of ${}_{\mathbb{R}}\Phi$ with respect to this ordering. Let j be the map $\Phi \to {}_{\mathbb{R}}\Phi \cup \{0\}$ induced by the restriction onto S.

For each $b \in \Phi(T, G)$, we denote by U_b the unique one parameter root subgroup associated with b. For $\theta \subset {}_{\mathbb{R}}\Delta$, $[\theta]$ denotes the $\mathbb{Z}$-linear combinations of θ which are roots in ${}_{\mathbb{R}}\Phi$. A subset $\Psi \subset {}_{\mathbb{R}}\Phi$ is called *closed* if $a, b \in \Psi$ and $a + b \in {}_{\mathbb{R}}\Phi$ imply $a + b \in \Psi$. If Ψ is closed, then we denote by G_Ψ the subgroup generated by T and the subgroups U_a, $a \in j^{-1}(\Psi \cup \{0\})$, and by G^*_Ψ the subgroup generated by the subgroups U_a, $a \in j^{-1}(\Psi)$. If G^*_Ψ is unipotent, then it will also be denoted by U_Ψ and the set Ψ in this case will be called *unipotent*.

For $\theta \subset {}_{\mathbb{R}}\Delta$, we define the following closed subsets of ${}_{\mathbb{R}}\Phi$:

$$\pi_\theta = [\theta] \cup {}_{\mathbb{R}}\Phi^+, {\pi_\theta}^- = [\theta] \cup {}_{\mathbb{R}}\Phi^-, \beta_\theta = {}_{\mathbb{R}}\Phi^+ - [\theta], {\beta_\theta}^- = {}_{\mathbb{R}}\Phi^- - [\theta].$$

For the sake of simplicity, we shall denote by P_θ, ${P_\theta}^-$, V_θ and ${V_\theta}^-$, the subgroups G_{π_θ}, $G_{{\pi_\theta}^-}$, U_{β_θ} and $U_{{\beta_\theta}^-}$ respectively.

We recall some well-known facts about parabolic subgroups and horospherical subgroups (cf. [B-T]). Any parabolic (resp. horospherical) $\mathbb{R}$-subgroup of G is conjugate to P_θ (resp. V_θ) for some $\theta \subset {}_{\mathbb{R}}\Delta$. Any pair of opposite parabolic (resp. horospherical) $\mathbb{R}$-subgroups is conjugate to the pair P_θ, ${P_\theta}^-$ (resp. V_θ, ${V_\theta}^-$) for some $\theta \subset {}_{\mathbb{R}}\Delta$. Note that $P_\emptyset$ is a minimal parabolic $\mathbb{R}$-subgroup and $V_\emptyset$ is a maximal horospherical $\mathbb{R}$-subgroup. Any parabolic $\mathbb{R}$-subgroup containing $P_\emptyset$ is of the form P_θ for some $\theta \subset {}_{\mathbb{R}}\Delta$ and any horospherical $\mathbb{R}$-subgroup contained in $V_\emptyset$ is of the form V_θ for some $\theta \subset {}_{\mathbb{R}}\Delta$.

Let ${}_{\mathbb{R}}W = N(S)/C(S)$ be the relative Weyl group where $C(S)$ denotes the centralizer of S in G. For any $w \in {}_{\mathbb{R}}W$ and any subgroup H of G normalized by $C(S)$, we denote by wHw^{-1} the subgroup $n_w H n_w^{-1}$ where n_w is a representative of w in $N(S)$. We fix $w_0 \in {}_{\mathbb{R}}W$ such that $w_0 V_\emptyset w_0^{-1} = V_\emptyset^-$. Then w_0 takes Φ^+ into Φ^- and there exists an involution i of Φ such that $w_0(i(a)) = -a$ for all $a \in \Phi$. We call i the *opposition involution* of Φ (cf. [B-T, 2.1]). We have that $w_0 P_\theta w_0^{-1} = P^-_{i(\theta)}$ and $w_0 V_\theta w_0^{-1} = V^-_{i(\theta)}$. If $i(\theta) = \theta$, then P_θ (resp. V_θ) is called *reflexive* and so are all the elements in the conjugacy class of P_θ (resp. V_θ). Note that a parabolic subgroup P is reflexive if and only if its conjugacy class contains a parabolic subgroup opposite to P.

It is not difficult to see that the following lemma holds.

3.2. Lemma. *The followings are all equivalent.*

(1) V_θ *is reflexive;*
(2) $i(\theta) = \theta$*;*
(3) V_θ *and* $w_0 V_\theta w_0^{-1}$ *are opposite;*
(4) $N(V_\theta) \cap w_0 V_\theta w_0^{-1}$ *is trivial.*

3.3. Lemma [B-T, Proposition 3.22]**.** *If* ψ *and* ϕ *are two closed subsets of* ${}_{\mathbb{R}}\Phi$ *and* ψ *is unipotent, then* $U_\psi \cap G_\phi = U_\psi \cap G^*_\phi = U_{\psi\cap\phi}$.

3.4. Lemma [B-T, Lemma 4.12]**.** *Let* G *be a connected semisimple algebraic* $\mathbb{R}$*-group and* P_1, P_2 *two parabolic subgroups of* G*. Then the set* M *of all elements* $g \in G$ *such that* P_1 *and* gP_2g^{-1} *contain opposite minimal parabolic subgroups is Zariski dense and open.*

3.5. Corollary. *Let* G *be a connected semisimple* $\mathbb{R}$*-algebraic group and* P *a reflexive parabolic* $\mathbb{R}$*-subgroup of* G *. Then the set* M *of the elements* $g \in G$ *such that* P *and* gPg^{-1} *are opposite is Zariski dense and open.*

Proof. This follows from Lemma 3.4 and Proposition 4.10 in [B-T], which says that two conjugate parabolic subgroups are opposite if they contain opposite minimal parabolic subgroups. $\square$

3.6. Proposition. *Let* G *be a connected semisimple algebraic* $\mathbb{R}$*-group,* Γ *a Zariski dense subgroup of* $G(\mathbb{R})$ *and* U *a horospherical* $\mathbb{R}$*-subgroup of* G *such that* $\Gamma \cap U$ *is Zariski dense in* U*. Then there exists an element* $h \in G$ *such that*

(1) $hUh^{-1} = V_\theta$ *for some* $\theta \subset {}_{\mathbb{R}}\Delta$
(2) $w_0 hUh^{-1} w_0^{-1} \cap h\Gamma h^{-1}$ *is Zariski dense in* $w_0 hUh^{-1} w_0^{-1}$*.*

Proof. By Lemma 3.4, there exists an element $\gamma \in \Gamma$ such that $B_1 \subset N(U)$ and $B_2 \subset \gamma N(U)\gamma^{-1}$ for some pair B_1, B_2 of opposite minimal parabolic subgroups. Therefore $U \subset R_u(B_1)$ and $\gamma U \gamma^{-1} \subset R_u(B_2)$. Since $R_u(B_1)$ and $R_u(B_2)$ are opposite maximal horospherical subgroups, there exists $h \in G(\mathbb{R})$ such that $hR_u(B_1)h^{-1} = V_\emptyset$ and $hR_u(B_2)h^{-1} = V_\emptyset^-$. Therefore $hUh^{-1} \subset V_\emptyset$; hence $hUh^{-1} \subset V_\theta$ for some $\theta \subset {}_{\mathbb{R}}\Delta$, and $h\gamma U\gamma^{-1}h^{-1} \subset V_\emptyset^-$. Since $h\gamma U\gamma^{-1}h^{-1} \cap h\Gamma h^{-1} = h\gamma(U \cap \Gamma)\gamma^{-1}h^{-1}$ and $U \cap \Gamma$ is Zariski dense in U by the assumption, we have that $h\gamma U\gamma^{-1}h^{-1} \cap h\Gamma h^{-1}$ is Zariski dense in $h\gamma U\gamma^{-1}h^{-1}$. It only remains to show that $h\gamma U\gamma^{-1}h^{-1} = w_0 hUh^{-1} w_0^{-1}$. Since $w_0 V_\emptyset w_0^{-1} = V_\emptyset^-$, $w_0^{-1} h\gamma U\gamma^{-1}h^{-1} w_0 \subset V_\emptyset$. Since $hN(U)h^{-1}$ and $w_0^{-1} h\gamma N(U)\gamma^{-1}h^{-1} w_0$ are two conjugate parabolic subgroups which contain the same minimal parabolic subgroup $P_\emptyset$, it follows that $hN(U)h^{-1} = w_0^{-1} h\gamma N(U)\gamma^{-1}h^{-1} w_0$ (cf. [B-T, 4.3]). Since the normalizer of a parabolic subgroup is the parabolic subgroup itself, there exists an element $n \in N(U)$ such that $hn = w_0^{-1} h\gamma$. Therefore $h\gamma U \gamma^{-1} h^{-1} = w_0 h U h^{-1} w_0^{-1}$, proving the proposition. □

3.7. Corollary. *Let G, Γ, U be as above and U be conjugate to V_θ for some $\theta \subset {}_{\mathbb{R}}\Delta$. Then for some $h \in G(\mathbb{R})$, $h\Gamma h^{-1}$ intersects both V_θ and $V_{i(\theta)}^{-1}$ Zariski densely.*

4. Proof of Proposition 1.5

4.1. Proposition. *Let G be a connected semisimple algebraic $\mathbb{R}$-group and Γ a discrete Zariski dense subgroup of $G(\mathbb{R})$. Suppose that there exists a horospherical $\mathbb{R}$-subgroup U of G such that $\Gamma \cap U$ is Zariski dense in U. Then there exists a reflexive horospherical $\mathbb{R}$-subgroup V of G such that $U \subset V$ and $\Gamma \cap V$ is Zariski dense in V.*

Proof. Note that a maximal horospherical $\mathbb{R}$-subgroup of G is always reflexive and the dimensions of the maximal horospherical $\mathbb{R}$-subgroups of G are all equal. Therefore by induction on the dimension of U, it is enough to prove that if U is not reflexive, then there exists a horospherical $\mathbb{R}$-subgroup V such that $V \cap \Gamma$ is Zariski dense in V and $U \subsetneq V$.

We use the notation ${}_{\mathbb{R}}\Phi$, ${}_{\mathbb{R}}\Delta$ etc. from section 3.1. By Corollary 3.7, there is no loss of generality in assuming that $U = V_\theta$ for some $\theta \subset {}_{\mathbb{R}}\Delta$ and $V_{i(\theta)}^{-} \cap \Gamma$ is Zariski dense in $V_{i(\theta)}^{-}$. Let $U' = V_{i(\theta)}^{-}$. Note that $w_0 U w_0^{-1} = U'$. Suppose that U is not reflexive. Then by Lemma 3.2, $N(U) \cap U'$ is not trivial. By Lemma 3.3, we have that $[N(U), N(U)] \cap U' = N(U) \cap U'$. Therefore it follows from Corollary 2.4 that $N(U)(\mathbb{R}) \cap U'(\mathbb{R})$ is Γ-compact; hence $N(U) \cap U' \cap \Gamma$ is Zariski dense in $N(U) \cap U'$. Hence $V \cap \Gamma$ is Zariski dense in V where $V = (N(U) \cap U') \ltimes U$. To finish the proof of the proposition, it remains to show that V is a horospherical $\mathbb{R}$-subgroup of G, which we will do in Proposition 4.2. □

Note that for $U = V_\theta$ for some $\theta \subset {}_{\mathbb{R}}\Delta$, we have

$$(N(U) \cap w_0 U w_0^{-1}) \ltimes U = (P_\theta \cap V_{i(\theta)}^{-}) \ltimes V_\theta.$$

4.2. Proposition. *Let $U = V_\theta$. Then $(N(U) \cap w_0 U w_0^{-1}) \ltimes U$ is a horospherical $\mathbb{R}$-subgroup of G.*

Proof. Note that $N(V_\theta) = P_\theta$. Let $L = P_\theta \cap P_\theta^{-}$ and $M = [L, L]$. Then M is a connected semisimple algebraic $\mathbb{R}$-group and $M = G_{[\theta]}^{*}$, i.e., the subgroup generated by U_a, $a \in j^{-1}[\theta]$. It is well known that $S \cap M$ is a maximal $\mathbb{R}$-split torus of M and

$\Phi(S\cap M, M) = [\theta]$, which will be denoted by Φ_M. The root system Φ_M has the induced ordering from Φ.

Since $w_0 U w_0^{-1} = V_{i(\theta)}^- = U_{{}_{\mathbb{R}}\Phi^- - [i(\theta)]}$ and the set ${}_{\mathbb{R}}\Phi^- - [i(\theta)]$ is unipotent, we have that $N(U)\cap w_0 U w_0^{-1} = M \cap w_0 U w_0^{-1} = U_{\Phi_M^- - [\theta\cap i(\theta)]}$ by Lemma 3.3. It follows that $M\cap w_0 U w_0^{-1}$ is a horospherical $\mathbb{R}$-subgroup of M. Note that $V_\emptyset \cap M$ is a maximal horospherical $\mathbb{R}$-subgroup of M. There exists an element $w \in N(S\cap M)\cap M$ such that $w(V_\emptyset \cap M)w^{-1} = V_\emptyset^- \cap M$ (cf. section 3.1). Let j be the opposition involution of Φ_M such that $w(j(a)) = -a$ for all $a\in\Phi_M$.

Then

$$wU_{\Phi_M^- - [\theta\cap i(\theta)]} \ltimes U_{{}_{\mathbb{R}}\Phi^+ - [\theta]} w^{-1} = wU_{\Phi_M^- - [\theta\cap i(\theta)]} w^{-1} \ltimes U_{{}_{\mathbb{R}}\Phi^+ - [\theta]}$$

since w normalizes U.

Since w sends $\Phi_M^- - [\theta\cap i(\theta)]$ to $\Phi_M^+ - [j(\theta\cap i(\theta))]$,

$$wU_{\Phi_M^- - [\theta\cap i(\theta)]} w^{-1} \ltimes U_{\Phi^+ - [\theta]} = U_{\Phi_M^+ - [j(\theta\cap i(\theta))]} \ltimes U_{{}_{\mathbb{R}}\Phi^+ - [\theta]} = U_{{}_{\mathbb{R}}\Phi^+ - [j(\theta\cap i(\theta))]}.$$

Therefore $wU_{\Phi_M^- - [\theta\cap i(\theta)]} \ltimes U_{{}_{\mathbb{R}}\Phi^+ - [\theta]} w^{-1}$ is the horospherical $\mathbb{R}$-subgroup $V_{j(\theta\cap i(\theta))}$. This proves that $(M\cap w_0 U w_0^{-1}) \ltimes U$ is a horospherical $\mathbb{R}$-subgroup of G. $\square$

4.3. Proof of Proposition 1.5. By Proposition 4.1, we may assume that U is reflexive. By Corollary 3.5, there exists $\gamma\in\Gamma$ such that U and $\gamma U\gamma^{-1}$ are opposite. Since $\gamma U \gamma^{-1}\cap\Gamma = \gamma(U\cap\Gamma)\gamma^{-1}$, it suffices to set $U_1 = U$ and $U_2 = \gamma U\gamma^{-1}$ to prove the proposition.

References

[B-T] A. Borel and J. Tits, *Groupes réductifs*, Publ. Math. I.H.E.S.27 (1965), 55–151.

[Ma] G. A. Margulis, *Arithmetic properties of discrete groups*, Russian Math. Surveys 29 (1974), 107–156.

[Oh] H. Oh, *Discrete subgroups generated by lattices in opposite horospherical subgroups*, to appear in J. Algebra.

[P-R] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, 1994.

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