REPRESENTATIONS OF INTEGERS BY AN INVARIANT POLYNOMIAL AND UNIPOTENT FLOWS

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Abstract. We study a refined version of Linnik problem on the asymptotic behavior of the number of representations of integers m by an integral polynomial as m tends to infinity. Assuming that the polynomials arise from invariant theory, we reduce the question to the study of limiting behavior of measures invariant under unipotent flows. Our main tool is then Ratner's theorem on the classification of measures invariant under unipotent flows, in a form developed by Dani and Margulis.

1. Introduction

Let f be an integral homogeneous polynomial of degree d in n variables. A basic problem in Diophantine analytic number theory is to understand the behavior of the integral representations of integers m by f as m tends to infinity.

For each $m \in \mathbb{N}$, consider the level variety $V_m := \{x \in \mathbb{R}^n : f(x) = m\}$. For instance, if $f(x_1, \cdots, x_n) = x_1^2 + \cdots + x_n^2$ $(n \geq 3)$, then V_m is the sphere of radius $\sqrt{m}$ centered at the origin and the set $V_m(\mathbb{Z}) = V_m \cap \mathbb{Z}^n$ consists of integral vectors the sum of whose squares is equal to m. In this case, the asymptotic of $\#V_m(\mathbb{Z})$ is well known. For $n \geq 5$ the classical Hardy-Littlewood circle method applies and for $n = 4$ the Kloosterman sum method works. For $n = 3$, Linnik gave a conditional answer and later Iwaniec and Duke gave a complete answer (see [Sa], [Iw], [Du]).

In the case when V_m is non-compact, the number $\#V_m(\mathbb{Z})$ may be infinite. In this case one asks if there exists an asymptotic density for $V_m(\mathbb{Z})$ as $m \to \infty$. To be more precise, for a compact subset Ω of V_1, set

$$N_m(f, \Omega) := \#V_m(\mathbb{Z}) \cap m^{1/d}\Omega. \tag{1.1}$$

Or equivalently,

$$N_m(f, \Omega) = \#V_m(\mathbb{Z}) \cap \mathbb{R}^+\Omega$$

where $\mathbb{R}^+\Omega$ is the radial cone $\{x \in V : tx \in \Omega \text{ for some } t \in \mathbb{R}^+\}$. The question we study in this paper is if there exists a sequence of numbers ω_m, independent of the compact subset Ω, such that

$$N_m(f, \Omega) \sim_{m\to\infty} \omega_m \cdot \mathrm{vol}(\Omega).$$

This formulation is basically due to Linnik ([Li2], see also [Sa]).

The only known general method is the Hardy-Littlewood circle method. However for this method to work, the number of variables needs to be much larger than the degree of the polynomial in general.

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In this paper, we focus on the polynomials that are invariant under an action of a semisimple real algebraic group. In such cases, the level varieties admit actions of a semisimple algebraic group and the dynamics of such groups plays a crucial role in understanding this question. In particular, when V_1 is a homogeneous space of a semisimple real algebraic group with the stabilizer being generated by unipotent flows, we are able to use a well developed theory of unipotent flows on a homogeneous space of a Lie group.

We formulate our main results for a family of varieties V_m which are more general than level sets of a given polynomial. Let V be a finite dimensional real vector space with a $\mathbb{Z}$-structure. Let G be a semisimple real algebraic group defined over $\mathbb{Q}$ and let $\rho : G \to \mathrm{GL}(V)$ be a $\mathbb{Q}$-rational representation of G. Fix a non-zero vector $v_0 \in V(\mathbb{Z})$ such that $v_0 G$ is Zariski closed and a sequence $\{\lambda_m \in \mathbb{R}^+ : m \in \mathbb{N}\}$ of strictly increasing numbers. We set

$$V_m := \lambda_m v_0 G \quad \text{ for each } m \in \mathbb{N}.$$

Denote by $\mathcal{A}_V$ be the collection of all arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$ of G such that $V(\mathbb{Z})\Gamma \subset V(\mathbb{Z})$. Since ρ is defined over $\mathbb{Q}$, $\mathcal{A}_V$ is non-empty. Consider a sequence $\{\mathcal{O}_m \subset V_m(\mathbb{Z}) : m \in \mathbb{N}\}$ of Γ-invariant subsets of $V_m(\mathbb{Z})$ for some $\Gamma \in \mathcal{A}_V$. For a compact subset Ω of V_1, we define

$$N(\mathcal{O}_m, \Omega) := \# \operatorname{pr}(\mathcal{O}_m) \cap \Omega$$

where $\operatorname{pr} : V_m \to V_1$ denotes the radial projection given by $\operatorname{pr}(x) = \lambda_1 \lambda_m^{-1} x$.

Let H denote the stabilizer of v_0 in G. The notation H^0 and G^0 denote the identity components of H and G respectively. In the rest of the introduction, we assume that H is semisimple without compact factors.

Theorem 1.2. *Suppose that H^0 is a maximal connected closed subgroup of G^0. Let $\{\mathcal{O}_m \subset V_m(\mathbb{Z})\}$ be a sequence of Γ-invariant subsets for some $\Gamma \in \mathcal{A}_V$, for example, $\mathcal{O}_m = V_m(\mathbb{Z})$. Suppose that for each $m_0 \in \mathbb{N}$,*

$$\#\{m \in \mathbb{N} : \operatorname{pr}(\mathcal{O}_{m_0}) = \operatorname{pr}(\mathcal{O}_m)\} < \infty. \tag{1.3}$$

Fix a G-invariant Borel measure μ on V_1. Then for any compact subset Ω of V_1 with boundary of measure 0,

$$N(\mathcal{O}_m, \Omega) \sim_{m\to\infty} \omega(\mathcal{O}_m) \cdot \mu(\Omega) \tag{1.4}$$

where $\omega(\mathcal{O}_m)$ is given below (1.6).

An immediate consequence of the above theorem is that the sequence $\{\operatorname{pr}(\mathcal{O}_m)\}$ of projections is equidistributed on V_1 as $m \to \infty$, since it follows from (1.4) that for any two compact subsets Ω_1 and Ω_2 of V_1 with boundary measure 0,

$$\frac{N_m(\mathcal{O}_m, \Omega_1)}{N_m(\mathcal{O}_m, \Omega_2)} \sim_{m\to\infty} \frac{\mu(\Omega_1)}{\mu(\Omega_2)}.$$

The condition (1.3) says that there is no infinite sequence of varieties V_{m_i} where all the integer points in $\mathcal{O}_{m_i}$ are simply the radial dilations of a fixed $\mathcal{O}_{m_1}$. Clearly this is a necessary condition for the conclusion of the above theorem. We remark that the condition (1.3) is also equivalent to saying $\lim_{m\to\infty} \omega(\mathcal{O}_m) = \infty$ (see Lemma 4.3).

In order to give the formula of the number $\omega(\mathcal{O}_m)$ in Theorem 1.2, let μ_G and μ_H denote Haar measures on G and H respectively so that the triple (μ, μ_G, μ_H) are topologically compatible in the sense of Weil [We], namely, for any continuous function f on G with compact support,

$$\int_G f\, d\mu_G(g) = \int_{H\backslash G} d\mu(\bar{g}) \int_H f(hg) d\mu_H(h) \tag{1.5}$$

where μ, by slight abuse of notation, denotes the G-invariant measure on $H\backslash G$ pulled back from μ on V_1 via the identification of $H\backslash G$ with V_1 by $Hg \mapsto v_0 g$.

The number $\omega(\mathcal{O}_m)$ is defined as follows: for any $\Gamma \in \mathcal{A}_V$ preserving each $\mathcal{O}_m$,

$$\omega(\mathcal{O}_m) := \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m} \mu_H\left((H \cap g_\xi{}^{-1}\Gamma g_\xi)\backslash H\right)}{\mu_G(\Gamma\backslash G)}, \tag{1.6}$$

where the sum is taken over the set of disjoint Γ-orbits in $\mathcal{O}_m$ and $g_\xi \in G$ is an element such that $v_0 = \mathrm{pr}(\xi) g_\xi$. Under our assumption, $\omega(\mathcal{O}_m) < \infty$, and moreover $\omega(\mathcal{O}_m)$ does not depend on the choice of Γ in $\mathcal{A}_V$ preserving $\mathcal{O}_m$, justifying our notation. This is shown in [Oh] for $\mathcal{O}_m = V_m(\mathbb{Z})$ but the same proof works for any $\mathcal{O}_m$. For the case of $\mathcal{O}_m = V_m(\mathbb{Z})$, we simply write

$$\omega_m = \omega(V_m(\mathbb{Z})). \tag{1.7}$$

Note also that the product $\omega(\mathcal{O}_m)\cdot\mu(\Omega)$ is independent of the choice of compatible measures (μ, μ_H, μ_G). For this reason, we sometimes write $\omega(\mathcal{O}_m)\cdot \mathrm{vol}(\Omega)$ in what follows.

We remark that Theorem 1.2 applies to the cases where V_1 is an affine symmetric space, i.e., H is the set of fixed points of an involution of G, with additional assumptions that G is $\mathbb{Q}$-simple and that H is semisimple without compact factors. It is so since H^0 is then a maximal connected closed subgroup of G^0 (cf. [Bo, Lemma 8.0]).

We present some examples which follow from Theorem 1.2 for the case $\mathcal{O}_m = V_m(\mathbb{Z})$. See (1.1) for the notation $N_m(f, \Omega)$. The following three theorems are proven in section 6 where we realize each f as an invariant polynomial of a certain representation of a semisimple algebraic group. Once we do that, the number ω_m is defined as in (1.6) and (1.7).

Theorem 1.8. *Let $n \geq 2$ and* Det *denote the determinant polynomial on the vector space* $\mathrm{M}_n(\mathbb{R})$ *of $n \times n$ matrices. For any compact subset Ω of $V_1 = \{X \in \mathrm{M}_n(\mathbb{R}) : \mathrm{Det}(X) = 1\}$ of boundary of measure* 0*, we have*

$$N_m(\mathrm{Det}, \Omega) \sim_{m\to\infty} \omega_m \cdot \mathrm{vol}\,(\Omega).$$

Theorem 1.8 was proven first by Linnik [Li1] (for $n = 2$) and by Linnik and Skubenko (for $n \geq 3$) [LS] in the early sixties using methods in analytic number theory. A different proof was also given using methods based on Hecke operators (see [Sa], [COU] and [GO]).

Let V be the subspace of $\mathrm{M}_{2n}(\mathbb{R})$ consisting of the skew-symmetric matrices, i.e., $V = \{X \in \mathrm{M}_{2\mathrm{n}}(\mathbb{R}) : X^t = -X\}$ $(n \geq 2)$ and consider the Pfaffian as a polynomial on V defined by $\mathrm{Pff}(X)^2 = \mathrm{Det}(X)$ and

$$\mathrm{Pff}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = 1$$

where I_n denotes the identity matrix of order n.

Theorem 1.9. *Let $n \geq 2$. For any compact subset Ω of $V_1 = \{X \in V : \mathrm{Pff}(X) = 1\}$ of boundary of measure* 0*, we have*

$$N_m(\mathrm{Pff}, \Omega) \sim_{m\to\infty} \omega_m \cdot \mathrm{vol}\,(\Omega).$$

An integer m is called a fundamental discriminant if and only if m is either a square free integer congruent to 1 mod 4, or 4 times of a square free integer which is congruent to 2 or 3 mod 4.

Theorem 1.10. *Let Q be an integral quadratic form of signature (r, s) where $r + s \geq 4$, $r \geq 2$ and $s \geq 1$. For any compact subset Ω of $V_1 = \{X \in \mathbb{R}^{r+s} : Q(X) = 1\}$ of boundary of measure 0,*

$$N_m(Q, \Omega) \sim \omega_m \cdot \mathrm{vol}\,(\Omega)$$

as $m \to \infty$ along the fundamental discriminants with $V_m(\mathbb{Z}) \neq \emptyset$.

In fact, the Hardy-Littlewood circle method together with the Kloosterman sum method (needed for $r+s = 4$) (cf. [Va], [Es]) also gives an asymptotic density in Theorem 1.10, in the form of product of local densities. By comparing the two different forms of the asymptotic for $N_m(Q, \Omega)$, one obtains a new proof of Siegel mass formula for quadratic forms (see [Oh]) in the same spirit of the work of Eskin, Rudnick and Sarnak [ERS]. The above theorem is still true for the case of $r+s = 3$ and $rs > 0$ though our method does not apply. This follows from a theorem of Duke [Du].

More generally, a natural question is whether the asymptotic $\omega_m \cdot \mathrm{vol}(\Omega)$ in Theorem 1.2 coincides with the heuristics predicted by the Hardy-Littlewood circle method. It is shown in [Oh] that this is true in many cases but not always.

To state our main result in a more general setting without the maximality assumption on H, we define the following: the notation $Z(H^0)$ denotes the centralizer of H^0 in G.

Definition 1.11. *A sequence $\{\mathcal{O}_m \subset V_m(\mathbb{Z})\}$ is called* **focused** *if, for every $\Gamma \in \mathcal{A}_V$ preserving $\{\mathcal{O}_m\}$, there exist a proper connected closed subgroup L of G^0 in which $L \cap \Gamma$ is a Zariski dense lattice, $g \in G$ with $g^{-1}H^0 g \subset L$ and a compact subset $\mathcal{C} \subset v_0 Z(H^0) gL$ such that for every compact subset $\Omega \subset V_1$, there exists $\gamma_\Omega \in \Gamma$ with*

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m} \{\omega(\xi\Gamma) : \mathrm{pr}(\xi\Gamma) \cap \Omega \subset \mathcal{C}(L \cap \Gamma)\gamma_\Omega\}}{\omega(\mathcal{O}_m)} > 0. \tag{1.12}$$

In the above and also in the rest of the paper, the notation $\sum_{\xi\Gamma\subset\mathcal{O}_m}$ means that the sum is taken over the set of disjoint Γ-orbits contained in $\mathcal{O}_m$.

Theorem 1.13. *Suppose that $\{\mathcal{O}_m : m \in \mathbb{N}\}$ is not focused. Then for any compact subset Ω of V_1 with boundary of measure 0,*

$$N(\mathcal{O}_m, \Omega) \sim_{m\to\infty} \omega(\mathcal{O}_m) \cdot \mathrm{vol}(\Omega).$$

Note that in the case when $\mathcal{O}_m$ consists of one Γ-orbit, the focusing of $\{\mathcal{O}_m\}$ implies that for every given compact subset Ω of V_1, there exists an infinite sequence m_i such that $\mathrm{pr}(\mathcal{O}_{m_i}) \cap \Omega$ lies completely inside a proper subvariety of V_1. In general, if $\{\mathcal{O}_m\}$ is focused, one expects in view of Theorem 1.13 that a positive proportion of points in $\mathrm{pr}(\mathcal{O}_{m_i}) \cap \Omega$ lies inside a proper subvariety of V_1 of the form $v_0 Z(H) gL\gamma$ for some $\gamma \in \Gamma$ and for some proper subgroup L of G^0.

Assuming H connected for simplicity, we explain some of schemes in the proofs. Our main tool is Ratner's work on the classification of measures invariant under unipotent flows [Ra]. For each $\xi\Gamma \subset \mathcal{O}_m$, consider the H-invariant measure $\nu_{\xi\Gamma}$ on $\Gamma\backslash G$ supported on the closed orbit $\Gamma\backslash\Gamma g_\xi H$ with the total measure $\omega(\xi\Gamma)$. Let σ_m denote the averaging measure $\frac{\sum_{\xi\Gamma\subset\mathcal{O}_m} \nu_{\xi\Gamma}}{\omega(\mathcal{O}_m)}$ on $\Gamma\backslash G$. We first observe that the equidistribution of $\mathrm{pr}(\mathcal{O}_m)$ (Theorem 1.13) follows if σ_m

converges to the G-invariant probability measure on $\Gamma\backslash G$ as $m \to \infty$ (Proposition 2.2). We then show that under the non-focusing assumption of $\{\mathcal{O}_m\}$, the H-orbits $\Gamma\backslash\Gamma g_\xi H$ which stay outside a given compact subset C as well as the H-orbits $\Gamma\backslash\Gamma g_\xi H$ which stay inside $\Gamma\backslash\Gamma L g_\xi$ for some proper subgroup L of G are ignorable, in considering the weak limits of σ_m. Then by applying Ratner's measure classification invariant under unipotent flows [Ra] and the results of Dani and Margulis in [DM1-2], we show that each H-orbit $\Gamma\backslash\Gamma g_\xi H$ is getting longer and longer and moreover uniformly distributed on $\Gamma\backslash G$ as $m \to \infty$; hence the average measure σ_m tends to the Haar measure on $\Gamma\backslash G$ as $m \to \infty$.

In particular in the case where the subgroup H is a maximal closed subgroup of G^0, the equidistribution of $\{\mathrm{pr}(\mathcal{O}_m)\}$ is a consequence of the phenomenon that for any sequence of non-repeated individual Γ-orbits $\{\mathrm{pr}(\xi)\Gamma \subset \mathrm{pr}(\mathcal{O}_m)\}$, the corresponding sequence $\{\Gamma\backslash\Gamma g_\xi H\}$ of H-orbits is uniformly distributed on $\Gamma\backslash G$ and hence $\mathrm{pr}(\xi)\Gamma$ is equidistributed on V_1 as $m \to \infty$.

Remark Gan and Oh [GO] showed that for any invariant polynomial f in the above setting but with a more general H (not necessarily semisimple), if $V_{d_0}(\mathbb{Z})$ contains at least one integer point for some $d_0 \in \mathbb{N}$, then there exist explicitly computable constants c and r depending only on G, ρ and deg (f) such that $\mathrm{pr}(V_{cd_0m^r}(\mathbb{Z}))$ becomes dense on V_1 in a strong sense as $m \to \infty$. In particular it follows that the limit supreme in (1.12) is strictly less than 1.

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2. Measure theoretic formulation of a counting problem

Let G be a real semisimple algebraic group defined over $\mathbb{Q}$. This means that there exists a connected semisimple algebraic group $\underline{G}$ defined over $\mathbb{Q}$ such that G is a closed subgroup of $\underline{G}(\mathbb{R})$ containing the identity component $\underline{G}(\mathbb{R})^\circ$. Let $\rho : G \to \mathrm{GL}(V)$ be a $\mathbb{Q}$-rational representation for a finite dimensional real vector space V defined over $\mathbb{Z}$. Fix a non-zero vector $v_0 \in V$ such that $v_0 G$ is Zariski closed and a sequence $\{\lambda_m : m \in \mathbb{N}\}$ of strictly increasing positive numbers. For each $m \in \mathbb{N}$, we set

$$V_m = \lambda_m v_0 G.$$

Let H denote the stabilizer of v_0 in G. Since ρ is rational, H is a real algebraic subgroup of G. In particular, the identity component H^0 is a finite index normal subgroup of H. Assume that H^0 has no non-trivial $\mathbb{R}$-character. Let μ_G denote a Haar measure on G. Since both G and H are unimodular, we may choose a G-invariant Borel measure μ on V_1 and an H-invariant measure μ_H on H so that the triple (μ, μ_G, μ_H) is compatible in the sense of (1.5).

Let $\mathcal{A}_V$ be defined as in the introduction, and let $\Gamma \in \mathcal{A}_V$. Since G acts transitively on V_1, for any $\xi \in V_m$, there exists $g_\xi \in G$ such that $v_0 = \mathrm{pr}(\xi) g_\xi$. The choice of g_ξ is unique only up to modulo H. If H_ξ denotes the stabilizer of ξ in G for $\xi \in V_m(\mathbb{Z})$, then H_ξ is a $\mathbb{Q}$-subgroup of G and $H_\xi = g_\xi H g_\xi^{-1}$. Therefore H_ξ^0 has no non-trivial $\mathbb{Q}$-character and hence by a theorem of Borel and Harish-Chandra, $H_\xi \cap \Gamma$ is a lattice in H_ξ. Hence

$$\omega(\xi\Gamma) := \frac{\mu_H(H \cap g_\xi^{-1}\Gamma g_\xi\backslash H)}{\mu_G(\Gamma\backslash G)} < \infty.$$

Observe that the definition of $\omega(\xi\Gamma)$ depends only on the Γ-orbit not on its representative.

Let $\{\mathcal{O}_m \subset V_m(\mathbb{Z})\}$ be a sequence of non-empty Γ-invariant subsets of $V_m(\mathbb{Z})$ for some $\Gamma \in \mathcal{A}_V$. Since each V_m is Zariski closed, by a theorem of Borel and Harish-Chandra, the number of Γ-orbits in $V_m(\mathbb{Z})$ and hence in $\mathcal{O}_m$ is finite. Hence

$$\omega(\mathcal{O}_m) := \sum_{\xi\Gamma \subset \mathcal{O}_m} \omega(\xi\Gamma) < \infty.$$

Lemma 2.1 (Oh)**.** *The number $\omega(\mathcal{O}_m)$ is independent of the choice of $\Gamma \in \mathcal{A}_V$ preserving $\mathcal{O}_m$.*

The space $\mathcal{P}(\Gamma\backslash G)$ of the probability measures on $\Gamma\backslash G$ is equipped with the weak*-topology. Now fix any $\Gamma \in \mathcal{A}_V$ which preserves each $\mathcal{O}_m$. For each Γ-orbit $\xi\Gamma \subset \mathcal{O}_m$, let $\nu_{\xi\Gamma}$ denote the unique H-invariant measure on $\Gamma\backslash G$ supported on the closed orbit $\Gamma\backslash\Gamma g_\xi H$ and with the total measure given by $\omega(\xi\Gamma)$. Hence

$$\frac{1}{\omega(\mathcal{O}_m)} \sum_{\xi\Gamma \subset \mathcal{O}_m} \nu_{\xi\Gamma} \in \mathcal{P}(\Gamma\backslash G).$$

Here is a main proposition suggested by Sarnak which translates the counting problem to the question of whether the weak-limits of the above measures are G-invariant.

Proposition 2.2. *If*

$$\lim_{m\to\infty} \frac{1}{\omega(\mathcal{O}_m)} \sum_{\xi\Gamma \subset \mathcal{O}_m} \nu_{\xi\Gamma} = \frac{1}{\mu_G(\Gamma\backslash G)} \mu_G \quad \text{in } \mathcal{P}(\Gamma\backslash G)$$

then for any compact subset Ω of V_1 with boundary of measure 0*,*

$$N(\mathcal{O}_m, \Omega) \sim_{m\to\infty} \omega(\mathcal{O}_m) \cdot \mu(\Omega).$$

Proof. Without loss of generality, we assume $\mu_G(\Gamma\backslash G) = 1$. Let ϕ be any continuous function with compact support on $H\backslash G = V_1$. Define a function F_ϕ^m as follows: for each $g \in G$

$$F_\phi^m(g) := \frac{1}{\omega(\mathcal{O}_m)} \sum_{\xi\Gamma \subset \mathcal{O}_m} \sum_{\gamma \in (H_\xi \cap \Gamma)\backslash\Gamma} \phi(\mathrm{pr}(\xi)\gamma g).$$

Since F_ϕ^m is left Γ-invariant, it may be considered as a function on $\Gamma\backslash G$. Let ψ be a continuous function on $\Gamma\backslash G$ with compact support. Note that

$$\begin{aligned}
\omega(\mathcal{O}_m)\cdot\langle F_\phi^m,\psi\rangle &= \sum_{\xi\Gamma\subset\mathcal{O}_m}\int_{\Gamma\backslash G}\left(\sum_{\gamma\in(H_\xi\cap\Gamma)\backslash\Gamma}\phi(\mathrm{pr}(\xi)\gamma g)\psi(g)\right)d\mu_G(g)\\
&= \sum_{\xi\Gamma\subset\mathcal{O}_m}\int_{g\in(H_\xi\cap\Gamma)\backslash G}\phi(\mathrm{pr}(\xi)g)\psi(g)\,d\mu_G(g)\\
&= \sum_{\xi\Gamma\subset\mathcal{O}_m}\int_{t\in(H\cap g_\xi^{-1}\Gamma g_\xi)\backslash G}\phi(\mathrm{pr}(\xi)g_\xi t)\,\psi(g_\xi t)\,d\mu_G(t)\\
&= \sum_{\xi\Gamma\subset\mathcal{O}_m}\int_{g\in H\backslash G}\phi(v_0 g)\left(\int_{h\in(H\cap g_\xi^{-1}\Gamma g_\xi)\backslash H}\psi(g_\xi hg)d\mu_H(h)\right)d\mu(g)\\
&= \sum_{\xi\Gamma\subset\mathcal{O}_m}\int_{g\in H\backslash G}\phi(v_0 g)\left(\int_{s\in\Gamma\backslash G}\psi(sg)\,d\nu_\xi(s)\right)d\mu(g)
\end{aligned}$$

Consider a function ψ_g on $\Gamma\backslash G$ defined by $\psi_g(s):=\psi(sg)$. Then

$$\int_{\Gamma\backslash G}\psi_g\,d\mu_G=\int_{\Gamma\backslash G}\psi\,d\mu_G.$$

Hence by the assumption,

$$\lim_{m\to\infty}\frac{1}{\omega(\mathcal{O}_m)}\sum_{\xi\Gamma\subset\mathcal{O}_m}\int_{s\in\Gamma\backslash G}\psi(sg)\,d\nu_\xi(s)=\int_{\Gamma\backslash G}\psi\,d\mu_G.$$

Now by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty}\langle F_\phi^m,\psi\rangle=\int_{H\backslash G}\phi(v_0 g)d\mu(g)\cdot\int_{\Gamma\backslash G}\psi\,d\mu_G.$$

If follows that

$$\lim_{m\to\infty}\langle F_{\chi_\Omega}^m,\psi\rangle=\mu(\Omega)\cdot\int_{\Gamma\backslash G}\psi\,d\mu_G \tag{2.3}$$

where χ_Ω denotes the characteristic function of Ω.

Fix $\epsilon>0$. Let U_ϵ be a symmetric neighborhood of e in G such that

$$\mu(\Omega_\epsilon^+-\Omega_\epsilon^-)\le\epsilon$$

where $\Omega_\epsilon^+=\cup_{u\in U_\epsilon}\Omega u$ and $\Omega_\epsilon^-=\cap_{u\in U_\epsilon}\Omega u$. Then for all $g\in U_\epsilon$,

$$F^m_{\chi_{\Omega_{\epsilon-}}}(g)\le F^m_{\chi_\Omega}(e)\le F^m_{\chi_{\Omega_{\epsilon+}}}(g). \tag{2.4}$$

Let ψ_ϵ be a non-negative continuous function on $\Gamma\backslash G$ with support in U_ϵ and $\int_{\Gamma\backslash G}\psi_\epsilon\,d\mu_G=1$. Integrating (2.4) against ψ_ϵ now gives

$$\langle F^m_{\chi_{\Omega_{\epsilon-}}},\psi_\epsilon\rangle\le F^m_{\chi_\Omega}(e)\le\langle F^m_{\chi_{\Omega_{\epsilon+}}},\psi_\epsilon\rangle.$$

Since both sides tend to $\mu(\Omega_{\epsilon\pm})$ respectively as $m\to\infty$ by (2.3) and $\epsilon>0$ is arbitrary, we have

$$F^m_{\chi_\Omega}(e)\to\mu(\Omega)\quad\text{ as } m\to\infty.$$

Since

$$F^m_{\chi_\Omega}(e) = \frac{N(\mathcal{O}_m, \Omega)}{\omega(\mathcal{O}_m)},$$

this proves the claim. □

3. **Asymptotic behavior of unipotent flows**

We recall the following fundamental result of Dani and Margulis.

Theorem 3.1 (DM2, Theorem 6.1). *Let G be a connected Lie group and Γ a lattice in G. Given a compact subset $C \subset \Gamma\backslash G$ and an $\epsilon > 0$, there exists a compact set $K \subset \Gamma\backslash G$ such that the following holds: for any $x \in C$, any unipotent one-parameter subgroup $\{u(t)\}$ of G, and any $T > 0$,*

$$|\{t \in [0, T] : xu(t) \in K\}| > (1 - \epsilon)T$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

Let G be a connected semisimple real algebraic group defined over $\mathbb{Q}$, and H a connected semisimple real algebraic subgroup of G. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup of G. The results in this section have meanings only when $\Gamma\backslash G$ is non-compact, which we assume. Consider the one point compactification $\Gamma\backslash G \cup \{\infty\}$ of $\Gamma\backslash G$. The space $\mathcal{P}(\Gamma\backslash G \cup \{\infty\})$ of the probability measures on $\Gamma\backslash G \cup \{\infty\}$ equipped with the weak*-topology is weak* compact.

Let $\{g_m \in G\}$ be a sequence such that $g_m H g_m^{-1}$ is a $\mathbb{Q}$-subgroup of G for each m. By a theorem of Borel and Harish-Chandra [BH], it follows that $g_m^{-1}\Gamma g_m \cap H$ is a lattice in H. Hence each $\Gamma\backslash\Gamma g_m H$ is closed in $\Gamma\backslash G$ (cf. [Rag]) and there exists the unique H-invariant probability measure μ_m in $\Gamma\backslash G$ supported on $\Gamma\backslash\Gamma g_m H$.

Proposition 3.2. *Assume either that H has no compact factors or that $g_m H g_m^{-1}$ is $\mathbb{Q}$-simple for each m. Then the following are equivalent:*

(1) *There exists a compact subset C of $\Gamma\backslash G$ such that*

$$\Gamma\backslash\Gamma g_m H \cap C \neq \emptyset \quad \text{for all sufficiently large } m \in \mathbb{N}.$$

(2) *Every weak limit of $\{\mu_m\}$ in $\mathcal{P}(\Gamma\backslash G \cup \{\infty\})$ is supported on $\Gamma\backslash G$.*

Proof. Assume that (1) is true. Without loss of generality, we may assume that $\Gamma\backslash\Gamma g_m \in C$ for all $m \in \mathbb{N}$. Let H_N denote the unique maximal connected normal closed subgroup of H without compact factors. Let $U = \{u(t)\}$ be a unipotent one parameter subgroup in H_N not contained in any proper normal subgroup of H_N. Such a subgroup exists (see for example, [MS, Lemma 2.3]). Under our assumption, either $H = H_N$ or $g_m^{-1}\Gamma g_m \cap H$ is an irreducible lattice in H. Hence it follows from Moore's ergodicity theorem (cf. Theorem 2.1 in [BM]) that U acts ergodically with respect to each μ_m. Moreover by the Birkhoff ergodic theorem (cf. [BM]), for almost all $h \in H$, $\Gamma\backslash\Gamma g_m h u(t)$ is uniformly distributed on $\Gamma\backslash G$ with respect to μ_m, i.e., for any $f \in C_c(\Gamma\backslash G)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\Gamma\backslash\Gamma g_m h u(t))\, dt = \int f\, d\mu_m.$$

Therefore we may assume that for each $m \in \mathbb{N}$, there exists $h_m \in H$ such that $\Gamma\backslash\Gamma g_m h_m \in C$ and $\Gamma g_m h_m u(t)$ is uniformly distributed on $\Gamma\backslash G$ with respect to μ_m. For any given $\epsilon > 0$, let

K be a compact subset of $\Gamma\backslash G$ as in Theorem 3.1 with respect to C. Then for each $m \in \mathbb{N}$,

$$\mu_m(K) \geq 1 - \epsilon.$$

Therefore $\mu(K) \geq 1 - \epsilon$ for any weak limit μ of $\{\mu_m\}$. Since $\epsilon > 0$ is arbitrary, we have $\mu(\Gamma\backslash G) = 1$, proving that (2) holds.

Now suppose that (1) fails. First write $\Gamma\backslash G$ as $\cup_{i=1}^{\infty} C_i$ where C_i are compact subsets such that $C_i \subset C_{i+1}$ for all i. Then for each i, there exists m_i such that

$$\Gamma\backslash\Gamma g_{m_i} H \cap C_i = \emptyset.$$

Since C_i is increasing, we have

$$\Gamma\backslash\Gamma g_{m_j} H \cap C_i = \emptyset \quad \text{for all } j \geq i. \tag{3.3}$$

This implies that any weak limit of $\{\mu_{m_i}\}$ cannot be supported on $\Gamma\backslash G$, for if so, then for some i_0, $\mu_{m_i}(C_{i_0}) > 1/2$ for infinitely many i. This is contradiction to (3.3). Hence (2) implies (1). □

Let $\{\mathfrak{O}_m \subset H\backslash G\}$ be a sequence of a finitely many union of Γ-orbits. For each $H\backslash Hg \in \mathfrak{O}_m$, we assume that gHg^{-1} is a $\mathbb{Q}$-subgroup of G. For each Γ-orbit $\eta\Gamma \subset \mathfrak{O}_m$, set

$$\omega(\eta\Gamma) := \frac{\mu_H((H \cap g_\eta^{-1}\Gamma g_\eta)\backslash H)}{\mu_G(\Gamma\backslash G)} \quad \text{and} \quad \omega(\mathfrak{O}_m) := \sum_{\eta\Gamma \subset \mathfrak{O}_m} \omega(\eta\Gamma).$$

Here $g_\eta \in G$ is such that $\eta = H\backslash Hg_\eta$.

Let $\nu_{\eta\Gamma}$ denote the H-invariant measure on $\Gamma\backslash G$ supported on $\Gamma\backslash\Gamma g_\eta H$ with the total measure given by $\omega(\eta\Gamma)$. Define an H-invariant probability measure σ_m on $\Gamma\backslash G$:

$$\sigma_m = \frac{1}{\omega(\mathfrak{O}_m)} \sum_{\eta\Gamma \subset \mathfrak{O}_m} \nu_{\eta\Gamma}.$$

The notation H_N denotes the unique maximal connected normal closed subgroup of H without compact factors.

Proposition 3.4. *Assume either that H has no compact factors or that $g_\eta H g_\eta^{-1}$ is $\mathbb{Q}$-simple for any $\eta \in \mathfrak{O}_m$. Suppose that $g_\eta H_N g_\eta^{-1}$ is not contained in any proper $\mathbb{Q}$-parabolic subgroup of G for any $\eta \in \cup_m \mathfrak{O}_m$. Then any weak limit of $\{\sigma_m : m \in \mathbb{N}\}$ in $\mathcal{P}(\Gamma\backslash G \cup \{\infty\})$ is supported in $\Gamma\backslash G$.*

Proof. Without loss of generality we may assume that $\{\sigma_m\}$ converges in $\mathcal{P}(\Gamma\backslash G \cup \{\infty\})$. It suffices to show that, for any $\epsilon > 0$, there exists a compact subset $K \subset \Gamma\backslash G$ such that

$$\sigma_m(K) > 1 - \epsilon \quad \text{for all sufficiently large } m.$$

Assume not; then for any compact subset $K \subset \Gamma\backslash G$, (after going to a subsequence) there exists $\eta_m \in \mathfrak{O}_m$ such that

$$\nu_{\eta_m\Gamma}(K) < (1 - \epsilon)\omega(\eta_m\Gamma) \quad \text{for each } m \in \mathbb{N}.$$

Let $U = \{u(t)\}$ be a unipotent one-parameter subgroup of H_N as in the proof of previous proposition. Let R be the set of $h \in H$ such that $\Gamma\backslash\Gamma g_{\eta_m} h u(t)$ is uniformly distributed in $\Gamma\backslash\Gamma g_{\eta_m} H$ with respect to the probability measure $\frac{1}{\omega(\eta_m\Gamma)}\nu_{\eta_m\Gamma}$. Then R has the full measure in H (see the proof of the previous proposition). Fix any $h \in R$. Then for each $m \in \mathbb{N}$, there exists $T_m > 0$ (depending on h) such that

$$\frac{1}{T}|\{t \in [0,T] : \Gamma\backslash\Gamma g_{\eta_m} hu(t) \in K\}| < 1 - \epsilon/2$$

for all $T > T_m$ where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. Applying a theorem of Dani and Margulis [DM1, Theorem 2] (see also [EMS1]), we obtain that for any given $\alpha_m > 0$ with $\lim_{m\to\infty} \alpha_m = 0$, after passing to a subsequence, there exist a proper parabolic $\mathbb{Q}$-subgroup P of G, a non-zero vector $q \in \wedge^k \operatorname{Lie}(W)(\mathbb{Q})$ (W being the unipotent radical of P and $k = \dim(W)$) and a sequence $\{\gamma_m(h) \in \Gamma\}$ such that for all $m \in \mathbb{N}$ and $t > 0$,

$$\|q.\gamma_m(h) g_{\eta_m} hu(t)\| < \alpha_m$$

where the action is through the k-th exterior of the adjoint representation of G on $\wedge^k \operatorname{Lie}(G)$. Since $u(t)$ acts as a unipotent one-parameter subgroup on $\wedge^n \operatorname{Lie}(G)$ and any orbit of a unipotent one-parameter subgroup is unbounded except for a fixed point, it follows that for all $0 \le t < \infty$ and for all $m \in \mathbb{N}$,

$$q.\gamma_m^h g_{\eta_m} hu(t) = q.\gamma_m^h g_{\eta_m} h.$$

Hence

$$U \subset (\gamma_m^h g_{\eta_m} h)^{-1} P(\gamma_m^h g_{\eta_m} h),$$

since the latter group contains the stabilizer of the vector $q.\gamma_m^h g_{\eta_m} h$. Hence we have shown that for almost all $h \in H$ and for any $m \in \mathbb{N}$,

$$hUh^{-1} \subset (\gamma_m^h g_{\eta_m})^{-1} P(\gamma_m^h g_{\eta_m})$$

for some $\gamma_m^h \in \Gamma$. Since Γ is countable, it follows that for each $m \in \mathbb{N}$, there exist an element $\gamma_m \in \Gamma$ and a subset $S_m \subset H$ of positive measure such that

$$hUh^{-1} \subset (\gamma_m g_{\eta_m})^{-1} P(\gamma_m g_{\eta_m}) \quad \text{for all } h \in S_m.$$

Since the set

$$\{h \in H : hUh^{-1} \subset (\gamma_m g_{\eta_m})^{-1} P(\gamma_m g_{\eta_m})\}$$

is a real analytic submanifold of H with a positive measure, it is indeed equal to H. Hence

$$hUh^{-1} \subset (\gamma_m g_{\eta_m})^{-1} P(\gamma_m g_{\eta_m}) \quad \text{for all } h \in H.$$

Since U is not contained in any proper normal subgroup of H_N, it follows that

$$g_{\eta_m} H_N g_{\eta_m}^{-1} \subset {\gamma_m}^{-1} P \gamma_m.$$

This contradicts the assumption since ${\gamma_m}^{-1} P \gamma_m$ is a proper parabolic $\mathbb{Q}$-subgroup of G. □

4. Projections of $\mathcal{O}_m$ and stabilizer subgroups

We recall the following theorem of Dani and Margulis: let G be any connected Lie group and Γ a discrete subgroup of G. We fix a left invariant Riemannian metric on G. Let M be any closed subgroup of G such that $M \cap \Gamma$ is a lattice in M. Then $\Gamma\backslash\Gamma M$ is a closed Riemannian submanifold of $\Gamma\backslash G$ and hence it has a right M-invariant Riemannian volume form, denoted by $\mathcal{V}$, induced by the Riemannian metric.

Theorem 4.1 (DM2, Theorem 5.1)**.** *For any $c > 0$, let $\mathcal{W}_c$ be the collection of all closed connected subgroups of G such that $\Gamma\backslash\Gamma M$ is closed in $\Gamma\backslash G$ and $\mathcal{V}((M \cap \Gamma)\backslash M) \le c$. Then there are only finitely many subgroups of the form $M \cap \Gamma$ with $M \in \mathcal{W}_c$.*

We also need the following simple consequence of a theorem of Kazhdan and Margulis ([KM], [Ra, Theorem 11.8]):

Lemma 4.2. *Let G be a connected linear semisimple Lie group without compact factors. There exists a constant $c > 0$ such that for any discrete subgroup Γ of G, the co-volume of Γ in G (with respect to a fixed Haar measure on G) is at least c.*

We keep the same notation from section 2 for G, $\rho : G \to \mathrm{GL}(V)$, Γ, H, H_ξ, v_0, etc. Let $\mathcal{O}_m$ be a Γ-invariant subset of $V_m(\mathbb{Z})$ for each m. We assume that H is semisimple without compact factors. Denote by $N(H)$ the normalizer of H in G.

Lemma 4.3. *Assume that $[N(H) : H] < \infty$ and that for each $\xi \in \mathcal{O}_m$, H_ξ^0 is not contained in any proper $\mathbb{Q}$-parabolic subgroup of G. Then the following are equivalent:*

(1) *Suppose that for each $m_0 \in \mathbb{N}$,*

$$\#\{m \in \mathbb{N} : \mathrm{pr}(\mathcal{O}_{m_0}) = \mathrm{pr}(\mathcal{O}_m)\} < \infty.$$

(2) $\lim_{m\to\infty} \omega(\mathcal{O}_m) = \infty$.

Proof. It is easy to see that $\mathrm{pr}(\mathcal{O}_m) = \mathrm{pr}(\mathcal{O}_k)$ implies $\omega(\mathcal{O}_m) = \omega(\mathcal{O}_k)$. Hence (2) implies (1). Assume now that (2) fails. Then by passing to a subsequence we may assume that $\omega(\mathcal{O}_m)$ is uniformly bounded. By Lemma 4.2 which we may apply since H^0 has finite index in H, there exists some $c > 0$ such that

$$\omega(\xi\Gamma) = \frac{\mu_H((H \cap {g_\xi}^{-1}\Gamma g_\xi)\backslash H)}{\mu_G(\Gamma\backslash G)} > c \quad \text{ for all } \xi \in \cup_m \mathcal{O}_m.$$

Since $\omega(\mathcal{O}_m) \ge h_m \cdot c$ where h_m is the number of disjoint Γ-orbits in $\mathcal{O}_m$, we may also assume that h_m is constant, say r, for all m, by passing to a subsequence. Now write $\mathcal{O}_m = \cup_{i=1}^r \xi_{m_i}\Gamma$. It suffices to show that for each $1 \le i \le r$, $\mathrm{pr}(\xi_{m_i}\Gamma)$ is the same set for infinitely many m. For, this implies that $\mathrm{pr}(\mathcal{O}_m)$ is the same set for infinitely many m, which contradicts (1). Fix $1 \le i \le r$ and set $\xi_{m_i} = \xi_m$ for simplicity.

It follows from Propositions 3.2 and 3.4 that there exists a compact subset C of $\Gamma\backslash G$ such that $\Gamma\backslash\Gamma g_{\xi_m} H \cap C \ne \emptyset$ for all $m \in \mathbb{N}$. Hence we may choose g_{ξ_m} so that $\{g_{\xi_m} : m \in \mathbb{N}\}$ is relatively compact.

On the other hand, if δ_g denotes the factor by which the volumes of subsets gets multiplied under the transformation $h \to ghg^{-1}$, $h \in H$, then

$$\mathcal{V}((H_{\xi_m} \cap \Gamma)\backslash H_{\xi_m}) = \delta_{g_{\xi_m}} \cdot \omega(\xi_m\Gamma)$$

up a uniform constant multiple depending only on the choice of Haar measure μ_H. Since $\{g_{\xi_m} : m \in \mathbb{N}\}$ is relatively compact, $\sup_m \delta_{g_{\xi_m}} < \infty$.

Therefore

$$\sup_m \mathcal{V}((H_{\xi_m} \cap \Gamma)\backslash H_{\xi_m}) \le \sup_m \delta_{g_{\xi_m}} \cdot \sup_m \omega(\mathcal{O}_m) < \infty.$$

By Theorem 4.1, this implies that $H_{\xi_m} \cap \Gamma$ are all equal to each other by passing to a subsequence. Since $H_{\xi_m} \cap \Gamma$ is Zariski dense in H_{ξ_m} by Borel density theorem, it follows that

H_{ξ_m} are all equal to each other, that is, $g_{\xi_m}^{-1} g_{\xi_k} \in N(H)$ for all m, k. Since $[N(H) : H] < \infty$, by passing to a subsequence, we have

$$g_{\xi_m}^{-1} g_{\xi_k} \in H, \quad \text{and hence} \quad \mathrm{pr}(\xi_m \Gamma) = \mathrm{pr}(\xi_k \Gamma)$$

for all m, k. This finishes the proof. $\square$

Observe that for compact subsets $\Omega \subset V_1$, $C_0 \subset G$ such that $H \backslash H C_0 = \Omega$ and $\xi \in V_m(\mathbb{Z})$,

$$N(\xi\Gamma, \Omega) = 0 \quad \text{if and only if} \quad \Gamma \backslash \Gamma g_\xi H \cap \Gamma \backslash \Gamma C_0^{-1} = \emptyset. \tag{4.4}$$

Proposition 4.5. *Consider a sequence $\{\xi_m \in V_m(\mathbb{Z})\}$ such that $\{\mathrm{pr}(\xi_m)\}$ is relatively compact in V_1. Suppose that L is a closed subgroup of G containing $H_{\xi_m}^0$ for all m. such that $L \cap \Gamma$ is a lattice in L. Then for any compact subset Ω of V_1, there exists a finite subset $\Lambda_\Omega \subset \Gamma$ such that for all m,*

$$\mathrm{pr}(\xi_m)\Gamma \cap \Omega \subset \mathrm{pr}(\xi_m)(L \cap \Gamma)\Lambda_\Omega.$$

Proof. Let Ω_0 be a compact subset of G such that $H \backslash H \Omega_0 = \Omega$. Write H as a disjoint union $\cup_{i=1}^k h_i H^0$. By the assumption, there exists a choice of $\{g_{\xi_m}\}$ so that $\{g_{\xi_m} \in G\}$ is relatively compact. Let $\Omega_1 \subset G$ be a compact subset which contains $\{g_{\xi_m} h_i : 1 \le i \le k,\, m \in \mathbb{N}\}\Omega_0$.

Since $L \cap \Gamma$ is a lattice in L, $\Gamma \backslash \Gamma L$ is closed in $\Gamma \backslash G$ [Rag], and this implies easily that $L \backslash L\Gamma$ is closed in $L \backslash G$. Since $L \backslash L\Gamma$ is a closed countable subset of $L \backslash G$, it follows from Baire category theorem that there exists at least one isolated point. Since Γ acts transitively on $L \backslash L\Gamma$, every point of $L \backslash L\Gamma$ is an isolated point. Therefore $L \backslash L\Gamma$ is discrete in $L \backslash G$. Hence there exists a finite subset Λ_Ω of Γ such that

$$L\Omega_1 \cap L\Gamma \subset L\Lambda_\Omega.$$

Note that

$$g_{\xi_m} H \Omega_0 \cap g_{\xi_m} H^0 g_{\xi_m}^{-1} \Gamma \subset L g_{\xi_m} (\cup_{i=1}^k h_i) \Omega_0 \cap L\Gamma \subset L\Omega_1 \cap L\Gamma \subset L\Lambda_\Omega.$$

Hence

$$H\Omega_0 \cap H^0 g_{\xi_m}^{-1} \Gamma \subset g_{\xi_m}^{-1} L \Lambda_\Omega. \tag{4.6}$$

If $v_0 x \in \mathrm{pr}(\xi_m \Gamma) \cap \Omega$ for $x \in G$, then

$$x = h g_{\xi_m}^{-1} \gamma = h' w$$

for some $h, h' \in H$, $\gamma \in \Gamma$ and $w \in \Omega_0$.

Then

$$g_{\xi_m}^{-1} \gamma \in H\Omega_0 \cap H^0 g_{\xi_m}^{-1} \Gamma,$$

and hence by (4.6),

$$g_{\xi_m}^{-1} \gamma = g_{\xi_m}^{-1} g \gamma_1 \quad \text{for some } g \in L \text{ and } \gamma_1 \in \Lambda_\Omega.$$

In particular, $g = \gamma \gamma_1^{-1} \in L \cap \Gamma$. Therefore

$$x = h g_{\xi_m}^{-1} g \gamma_1 \in H g_{\xi_m}^{-1} (L \cap \Gamma) \Lambda_\Omega$$

proving

$$\mathrm{pr}(\xi_m \Gamma) \cap \Omega \subset \mathrm{pr}(\xi_m)(L \cap \Gamma)\Lambda_\Omega.$$

$\square$

5. Asymptotic behavior of $\mathcal{O}_m$

We start by recalling the following theorem of Dani and Margulis, which was built up on the measure classification theorem of Ratner [Ra]. For any two closed subgroups U and L of a connected Lie group G, set

$$X(U,L) := \{g \in G : gU \subset Lg\}.$$

Theorem 5.1 (DM2, Theorem 3). *Let G be a connected Lie group and Γ a lattice in G. Let $U = \{u(t)\}$ be a unipotent one-parameter subgroup of G and let ψ be a bounded continuous function of $\Gamma\backslash G$. Let K be a compact subset of $\Gamma\backslash G$ and let $\epsilon > 0$ be given. Then there exist finitely many proper closed subgroups $L_1, \cdots, L_k$ such that $L_i \cap \Gamma$ is a lattice in L_i for each $1 \le i \le k$, and compact subsets $C_1, \cdots, C_k$ of $X(U, L_1), \cdots, X(U, L_k)$ respectively, for which the following holds: for any compact subset $F \subset K - \cup_{i=1}^{k} \Gamma\backslash\Gamma C_i$, there exists $T_0 \ge 0$ such that for all $x \in F$ and $T > T_0$,*

$$\left| \frac{1}{T}\int_0^T \psi(xu(t))\, dt - \int_{\Gamma\backslash G} \psi \, d\mu_G \right| \le \epsilon.$$

In fact, it is shown in the proof of the above theorem [DM2] that the subgroups L_i can be taken so that $\mathrm{Ad}(L_i \cap \Gamma)$ is Zariski dense in $\mathrm{Ad}(L_i)$ as well where Ad denotes the adjoint representation of G.

We keep the same notation from section 2 for G, $\rho : G \to \mathrm{GL}(V)$, Γ, v_0, etc. Let $\mathcal{O}_m$ be a Γ-invariant subset of $V_m(\mathbb{Z})$. As before, we assume that the subgroup H, which is the stabilizer of v_0, is a semisimple real algebraic subgroup of G without compact factors. Set $X = \Gamma\backslash G$. We assume without loss of generality $\mu_G(X) = 1$. For each $\xi \in V_m(\mathbb{Z})$, we denote by $\nu_{\xi\Gamma}$ the unique H-invariant measure on X supported on $\Gamma\backslash\Gamma g_\xi H$ with the total measure given by $\omega(\xi\Gamma)$.

Denote by π the canonical projection from $H^0\backslash G$ to $H\backslash G$. By the identification of $V_1 = v_0 G$ with $H\backslash G$, we consider $\mathrm{pr}(\mathcal{O}_m)$ as a subset of $H\backslash G$. Set $\mathfrak{O}_m = \pi^{-1}(\mathrm{pr}(\mathcal{O}_m))$. Note that $\mathfrak{O}_m$ is Γ-invariant and has finitely many Γ-orbits. For each $\eta\Gamma \subset \mathfrak{O}_m$, the notation $\nu_{\eta\Gamma}$ denotes the H^0-invariant measure on $\Gamma\backslash G$ supported on $\Gamma\backslash\Gamma g_\eta H^0$ with the total measure given by

$$\omega(\eta\Gamma) := \frac{\mu_{H^0}((H^0 \cap g_\eta^{-1}\Gamma g_\eta)\backslash H^0)}{\mu_G(\Gamma\backslash G)}$$

where g_η is any element in G such that $H^0\backslash H^0 g_\eta = \eta$. Here μ_{H^0} is simply the restriction of μ_H to H^0. Note that if $\pi(\eta_1\Gamma) = \pi(\eta_2\Gamma)$, then $\omega(\eta_1\Gamma) = \omega(\eta_2\Gamma)$ and $\omega(\eta\Gamma) \le \omega(\xi\Gamma)$ if $\pi(\eta) = \mathrm{pr}(\xi)$.

For $\xi\Gamma \subset \mathcal{O}_m$, it is not hard to check that

$$\nu_{\xi\Gamma} = \sum_{\eta\Gamma \subset \pi^{-1}(\mathrm{pr}(\xi\Gamma))} \nu_{\eta\Gamma}$$

where the sum is taken over the disjoint Γ-orbits $\eta\Gamma$ in $\mathfrak{O}_m$ such that $\pi(\eta\Gamma) = \mathrm{pr}(\xi\Gamma)$. The number such Γ-orbits is clearly bounded by $[H : H^0]$.

Therefore

$$\sum_{\xi\Gamma \subset \mathcal{O}_m} \nu_{\xi\Gamma} = \sum_{\eta\Gamma \subset \mathfrak{O}_m} \nu_{\eta\Gamma} \text{ and } \omega(\mathcal{O}_m) = \sum_{\eta\Gamma \subset \mathfrak{O}_m} \omega(\eta\Gamma). \tag{5.2}$$

Let $U = \{u(t)\}$ be a unipotent one-parameter subgroup of H^0 not contained in any proper closed normal subgroup of H^0. For given compact subset $K \subset X$, $\epsilon > 0$ and a bounded continuous function ψ on X, let L_i and C_i, $1 \le i \le k$ be as in the above theorem, with respect to the given triples (K, ϵ, ψ). Set

$$\mathcal{S}(K, \psi, \epsilon) := K \cap (\cup_{i=1}^{k} \Gamma\backslash\Gamma C_i)$$

and let $\mathcal{G}(K, \psi, \epsilon)$ denote the complement of $\mathcal{S}(K, \psi, \epsilon)$ inside K.

Proposition 5.3. *Fix a compact subset K of X with a non-empty interior. Suppose that for any $\epsilon > 0$ and for any continuous function ψ on X with compact support,*

$$\lim_{m\to\infty} \frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0 \cap \mathcal{G}(K, \psi, \epsilon) \neq \emptyset\}}{\omega(\mathcal{O}_m)} = 1.$$

Then

$$\frac{1}{\omega(\mathcal{O}_m)} \sum_{\xi\Gamma\subset\mathcal{O}_m} \nu_{\xi\Gamma} \to \mu_G \qquad \text{as } m \to \infty.$$

Proof. We set

$$A_m(\epsilon) := \{\Gamma\backslash\Gamma g_\eta H^0 : \eta\Gamma \subset \mathfrak{O}_m, \Gamma\backslash\Gamma g_\eta H^0 \cap K \subset \mathcal{S}(K, \psi, \epsilon)\}$$

and

$$B_m(\epsilon) := \{\Gamma\backslash\Gamma g_\eta H^0 : \Gamma\backslash\Gamma g_\eta H^0 \cap \mathcal{G}(K, \psi, \epsilon) \neq \emptyset\}.$$

The assumption implies that

(5.4)

$$\lim_{m\to\infty} \frac{\sum\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0 \in A_m(\epsilon)\}}{\omega(\mathcal{O}_m)} = 0; \quad \lim_{m\to\infty} \frac{\sum\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0 \in B_m(\epsilon)\}}{\omega(\mathcal{O}_m)} = 1.$$

By the Birkhoff ergodic theorem and Moore's ergodicity theorem, U acts ergodically with respect to each $\frac{1}{\omega(\eta\Gamma)}\nu_{\eta\Gamma}$ and the following subset R has the zero co-measure in H^0:

$$R = \{h \in H^0 : \Gamma\backslash\Gamma g_\eta h u(t) \text{ is uniformly distributed in } \Gamma\backslash\Gamma g_\eta H^0 \text{ w. r. t. } \frac{1}{\omega(\eta\Gamma)}\nu_{\eta\Gamma}\}.$$

Let $\Gamma\backslash\Gamma g_\eta H^0 \in B_m(\epsilon)$. Since $\mathcal{G}(K, \psi, \epsilon)$ is open in K and R has co-measure 0 in H^0, we may assume by a suitable choice for g_η that $\Gamma\backslash\Gamma g_\eta \in \mathcal{G}(K, \psi, \epsilon)$ and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(\Gamma\backslash\Gamma g_\eta u(t))dt = \frac{1}{\omega(\eta\Gamma)} \int_X \psi \, d\nu_{\eta\Gamma}.$$

Therefore by applying Theorem 5.1 to each singleton $F = \{\Gamma\backslash\Gamma g_\eta\}$, we obtain that for any $\Gamma\backslash\Gamma g_\eta H^0 \in B_m(\epsilon)$,

$$\left| \frac{1}{\omega(\eta\Gamma)} \int_X \psi \, d\nu_{\eta\Gamma} - \int_X \psi \, d\mu_G \right| \le \epsilon. \tag{5.5}$$

Now

$$\left|\sum_{\eta\Gamma\subset\mathfrak{O}_m}\int_X \psi\, d\nu_{\eta\Gamma} - \int_X \psi\, d\mu_G\right| \leq \sum_{\Gamma\backslash\Gamma g_\eta H^0\in A_m(\epsilon)}\left|\int_X \psi\, d\nu_{\eta\Gamma} - \omega(\eta\Gamma)\int_X \psi\, d\mu_G\right|$$
$$+ \sum_{\Gamma\backslash\Gamma g_\eta H^0\in B_m(\epsilon)}\left|\int_X \psi\, d\nu_{\eta\Gamma} - \omega(\eta\Gamma)\int_X \psi\, d\mu_G\right|$$

By (5.5), the above is again less than or equal to

$$(\|\psi\|_\infty + \|\psi\|_1)(\textstyle\sum\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0 \in A_m(\epsilon)\}) + \epsilon(\textstyle\sum\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0 \in B_m(\epsilon)\}).$$

By applying Lebesgue dominated convergence theorem, we deduce from (5.4)

$$\limsup_{m\to\infty}\left|\frac{1}{\omega(\mathcal{O}_m)}\sum_{\eta\Gamma\subset\mathfrak{O}_m}\left(\int_X \psi\, d\nu_{\eta\Gamma}\right) - \int \psi d\mu_G\right| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have for any bounded continuous function ψ on X,

$$\lim_{m\to\infty}\frac{1}{\omega(\mathcal{O}_m)}\sum_{\eta\Gamma\subset\mathfrak{O}_m}\left(\int_X \psi\, d\nu_{\eta\Gamma}\right) = \int \psi\, d\mu_G.$$

This proves our claim by (5.2). □

Lemma 5.6 (EMS2, Lemma 5.1). *Let G, H and L be connected real algebraic groups such that $H \subset L \subset G$. If at least one of G, H, and L is reductive, then $X(H,L)$ is a union of finitely many closed double cosets of the form $L \cdot g \cdot Z(H)$ where $g \in X(H,L)$.*

Proof of Theorem 1.13 Since G^0 has a finite index in G and V_1 consists of finitely many open G^0 orbits, it suffices to prove the theorem for each G^0-orbit. Hence we may assume that G is connected without loss of generality. Since $\{\mathcal{O}_m\}$ is not focused, there exists an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ which preserves each $\mathcal{O}_m$ and $\{\mathcal{O}_m\}$ is not focused with respect to Γ.

First, for some compact subset C of V_1,

$$\limsup_{m\to\infty}\frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \mathrm{pr}(\xi)\Gamma\cap C = \emptyset\}}{\omega(\mathcal{O}_m)} = 0.$$

Note that the same holds for any compact subset of V_1 containing C.

Hence it follows from the observation (4.4) that for some relatively compact open subset C_0 of G, we have

$$\limsup_{m\to\infty}\frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0\cap\Gamma\backslash\Gamma C_0 = \emptyset\}}{\omega(\mathcal{O}_m)} = 0. \tag{5.7}$$

Set $K = \Gamma\backslash(\Gamma\overline{C_0})$ and $K' = \Gamma\backslash(\Gamma C_0)$. By Propositions 2.2, and 5.3, it suffices to show that for any $\epsilon > 0$ and for any bounded continuous function ψ on X,

$$\lim_{m\to\infty}\frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0\cap\mathcal{G}(K,\psi,\epsilon) \neq \emptyset\}}{\omega(\mathcal{O}_m)} = 1.$$

Suppose not. Since the orbits $\Gamma\backslash\Gamma g_\eta H^0$ disjoint from K' can be ignored by (5.7), it follows that there exist a bounded continuous function ψ on X and an $\epsilon > 0$ such that

$$\limsup_{m\to\infty} \frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \emptyset \neq K' \cap \Gamma\backslash\Gamma g_\eta H^0 \subset \mathcal{S}(K,\psi,\epsilon)\}}{\omega(\mathcal{O}_m)} > 0 \tag{5.8}$$

Let L_i and C_i, $1 \le i \le k$ be the subgroups and compact subsets in $X(U, L_i)$ respectively, used in the definition of $\mathcal{S}(K,\psi,\epsilon)$. Since $\mathcal{S}(K,\psi,\epsilon)$ is contained in the finite union $\cup_{i=1}^k \Gamma\backslash\Gamma C_i$, there exists $1 \le i \le k$ such that (5.8) holds with $\Gamma\backslash\Gamma C_i$ in place of $\mathcal{S}(K,\psi,\epsilon)$. Without loss of generality, we assume $i = 1$. By Lemma 5.6, there exists $g \in G$ such that $gUg^{-1} \subset L_1$ and

$$\limsup_{m\to\infty} \frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \emptyset \neq \Gamma g_\eta H^0 \cap K' \subset \Gamma(C_1 \cap L_1 g Z(U))\}}{\omega(\mathcal{O}_m)} > 0. \tag{5.9}$$

Whenever

$$\emptyset \neq \Gamma g_\eta H^0 \cap K' \subset \Gamma(C_1 \cap L_1 g Z(U))$$

we may assume that $g_\eta \in C_1 \cap L_1 g Z(U)$ by replacing η and g_η by suitable elements in $\pi^{-1}(\mathrm{pr}(\xi\Gamma))$ and $g_\eta H^0$ respectively. We may also assume that the set $\{h \in H^0 : g_\eta h \in C_1 \cap L_1 g Z(U)\}$ has a positive measure, since K' is open. Note that $g_\eta h \in C_1 \cap L_1 g Z(U)$ implies that $hUh^{-1} \subset {g_\eta}^{-1} L_1 g_\eta$. By a similar argument as in the proof of Proposition 3.4, it follows that $H^0_\xi \subset L_1$, i.e., $g_\xi \in X(H^0, L_1)$, whenever $\emptyset \neq \Gamma g_\eta H^0 \cap K' \subset \Gamma(C_1 \cap L_1 g Z(U))$ and $\pi(\eta) \in \mathrm{pr}(\xi\Gamma)$. Applying Lemma 5.6 again, we deduce from (5.9) that for some $g_0 \in G$ such that $g_0 H^0 g_0^{-1} \subset L_1$,

$$\limsup_{m\to\infty} \frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : g_\eta \in C_1 \cap L_1 g_0 Z(H^0)\}}{\omega(\mathcal{O}_m)} > 0.$$

Since $g_\eta \in C_1 \cap L_1 g_0 Z(H)$ and $\pi(\eta) = \mathrm{pr}(\xi)$ implies that $g_\eta H^0 g_\eta^{-1} = H^0_\xi \subset L_1$, it follows from Proposition 4.5 that for any compact subset Ω of V_1, there exists a finite subset $\Lambda_\Omega \subset \Gamma$ such that for all $g_\eta \in C_1 \cap L_1 g_0 Z(H)$,

$$\pi(\eta)\Gamma \cap \Omega \subset (H{g_\eta}^{-1})(L_1 \cap \Gamma)\Lambda_\Omega \subset H(L_1 g_0 Z(H) \cap C_0)^{-1}(L_1 \cap \Gamma)\Lambda_\Omega.$$

Hence we have shown that for any compact subset $\Omega \subset V_1$,

$$\limsup_{m\to\infty} \frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \pi(\eta)\Gamma \cap \Omega \subset H(Z(H)g_0^{-1}L_1 \cap C_0^{-1})(L_1 \cap \Gamma)\Lambda_\Omega\}}{\omega(\mathcal{O}_m)} > 0,$$

for some finite subset $\Lambda_\Omega \subset \Gamma$. Hence for some $\gamma_\Omega \in \Gamma$,

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \mathrm{pr}(\xi)\Gamma \cap \Omega \subset v_0(Z(H)g_0^{-1}L_1 \cap C_0^{-1})(L_1 \cap \Gamma)\gamma_\Omega\}}{\omega(\mathcal{O}_m)} > 0,$$

By the remark following Theorem 5.1, $L_1 \cap \Gamma$ is a Zariski dense lattice in L_i.

Hence the sequence $\{\mathcal{O}_m\}$ is focused, yielding contradiction. This finishes the proof.

Proof of Theorem 1.2 Without loss of generality we may assume that G is connected. Fix any $\Gamma \in \mathcal{A}_V$ preserving each $\mathcal{O}_m$. If (1.4) does not hold, then $\{\mathcal{O}_m\}$ is focused by Theorem 1.13. Since H^0 is a maximal connected closed subgroup of G, it follows that there exists $g \in G$ such that for any compact subset $\Omega \subset V_1$

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \mathrm{pr}(\xi)\Gamma \cap \Omega \subset v_0 Z(H^0)H^0 g\Gamma\}}{\omega(\mathcal{O}_m)} > 0.$$

We claim that

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \mathrm{pr}(\xi)\Gamma \cap v_0 Z(H^0)H^0 g\Gamma \neq \emptyset\}}{\omega(\mathcal{O}_m)} > 0 \tag{5.10}$$

Suppose not. Then it follows that for any compact subset Ω of V_1,

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \mathrm{pr}(\xi)\Gamma \cap \Omega = \emptyset\}}{\omega(\mathcal{O}_m)} > 0.$$

This is equivalent to saying that for any compact subset C of $\Gamma\backslash G$,

$$\limsup_{m\to\infty} \frac{\sum_{\eta\Gamma\subset\mathfrak{O}_m}\{\omega(\eta\Gamma) : \Gamma\backslash\Gamma g_\eta H^0 \cap C = \emptyset\}}{\omega(\mathcal{O}_m)} > 0.$$

Hence there exists a sequence $\eta_m \in \mathfrak{O}_m$ such that for any compact subset $C \subset X$, there exists m such that $\Gamma\backslash\Gamma g_{\eta_m} H^0 \cap C = \emptyset$. On the other hand, since H^0 is a proper maximal closed subgroup of G^0, by Proposition 3.4, any weak limit of $\{\nu_{\eta_m\Gamma}\}$ in $\mathcal{P}(X \cup \{\infty\})$ is supported on X. This is a contradiction by Proposition 3.2. Hence (5.10) is proved.

It is easy to check that $\mathrm{pr}(\xi)\Gamma \cap v_0 Z(H^0)H^0 g\Gamma \neq \emptyset$ implies $\gamma H_\xi \gamma^{-1} = gHg^{-1}$ for some $\gamma \in \Gamma$. Hence (5.10) implies:

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \gamma H_\xi \gamma^{-1} = gHg^{-1} \text{ for some } \gamma \in \Gamma\}}{\omega(\mathcal{O}_m)} > 0$$

It follows that there exists $\xi_0 \in \mathcal{O}_{m_0}$ for some m_0 such that

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \gamma H_\xi \gamma^{-1} = H_{\xi_0} \text{ for some } \gamma \in \Gamma\}}{\omega(\mathcal{O}_m)} > 0$$

Observe that the condition $\gamma H_\xi \gamma^{-1} = H_{\xi_0}$ implies that $g_{\xi_0}^{-1}\gamma g_\xi \in N(H)$, and the condition $g_{\xi_0}^{-1}\gamma g_\xi \in H$ implies that $\mathrm{pr}(\xi)\Gamma = \mathrm{pr}(\xi_0)\Gamma$.

Since H has a finite index in the normalizer $N(H)$, it follows that

$$\limsup_{m\to\infty} \frac{\sum_{\xi\Gamma\subset\mathcal{O}_m}\{\omega(\xi\Gamma) : \mathrm{pr}(\xi)\Gamma = \mathrm{pr}(\xi_0)\Gamma)\}}{\omega(\mathcal{O}_m)} > 0. \tag{5.11}$$

Note that if $\mathrm{pr}(\xi)\Gamma = \mathrm{pr}(\xi_0)\Gamma$, then $\omega(\xi\Gamma) = \omega(\xi_0\Gamma)$. Since there can be at most one Γ-orbit $\xi\Gamma$ in $\mathcal{O}_m$ such that $\mathrm{pr}(\xi)\Gamma = \mathrm{pr}(\xi_0)\Gamma$, (5.11) implies that

$$\omega(\xi_0\Gamma) \cdot \limsup_{m\to\infty} \frac{1}{\omega(\mathcal{O}_m)} > 0 \quad \text{or equivalently} \quad \liminf_{m\to\infty} \omega(\mathcal{O}_m) < \infty.$$

By Lemma 4.3, this contradicts the assumption on $\{\mathcal{O}_m\}$. Hence the proof is now complete.

6. Examples

Theorem 6.1. *Let Q be an integral quadratic form of signature (r, s) where $r + s \geq 4$, $r \geq 2$ and $s \geq 1$. For any compact subset Ω of V_1 with boundary of measure 0,*

$$N_m(Q, \Omega) \sim \omega_m \cdot \mathrm{vol}\,(\Omega)$$

as $m \to \infty$ along the fundamental discriminants.

Proof. Consider the standard representation of the orthogonal group $\mathrm{O}(Q)$ on $V := \mathbb{R}^{r+s}$. Let V_m be the level set $\{x \in V : Q(x) = m\}$. By Witt's theorem the orthogonal group $\mathrm{O}(Q)$ acts transitively on each V_1. The stabilizer of a vector v_0 in V_1 is isomorphic to $\mathrm{O}(r-1,s)$. Note that the assumptions on the size of the parameters r and s guarantee that $\mathrm{O}(r-1,s)$ is non-compact and simple. It is well known that $\mathrm{O}(r-1,s)^0$ is a maximal connected subgroup of $\mathrm{O}(r,s)^0$. If we set $\Gamma := \mathrm{O}(Q) \cap \mathrm{SL}_{r+s}(\mathbb{Z})$, then $\Gamma \in \mathcal{A}_V$. Let m be a positive integer such that $V_m(\mathbb{Z}) \neq \emptyset$ and m be a fundamental discriminant modulo $\mathbb{Z}^2$. Then we may take $\mathcal{O}_m = V_m(\mathbb{Z})$ to apply theorem 1.2. To check the condition (1.3), note that if $m \neq k$ are fundamental discriminants modulo $\mathbb{Z}^2$, then $\sqrt{m}^{-1} V_m(\mathbb{Z}) \cap \sqrt{k}^{-1} V_k(\mathbb{Z}) = \emptyset$; otherwise this would imply that $\sqrt{(m/k)} \in \mathbb{Q}$, which can be seen to be false by an easy computation. Therefore Theorem 1.2 implies the claim. □

Let $V := \{X \in \mathrm{M}_{2n}(\mathbb{R}) : X^t = -X\}$ be the space of skew-symmetric matrices, so that the Pfaffian on V is defined by

$$\mathrm{Pff}^2(X) = \mathrm{Det}(X) \quad \text{and} \quad \mathrm{Pff}(v_0) = 1$$

where

$$v_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

E.g., for $n = 2$, we have $\mathrm{Pff}(x_1, \cdots, x_6) = x_1x_2 - x_3x_4 + x_5x_6$.

Theorem 6.2. *Let $n \geq 2$. For any compact subset Ω of V_1 with boundary of measure* 0,

$$N_m(\mathrm{Pff}, \Omega) \sim_{m\to\infty} \omega_m \cdot \mathrm{vol}\,(\Omega).$$

Proof. Consider the representation $\rho : \mathrm{SL}_{2n}(\mathbb{R}) \to \mathrm{GL}(V)$ defined by

$$\rho(A)(X) = A^t X A$$

where $A \in \mathrm{SL}_{2n}(\mathbb{R})$ and $X \in V$. It is well known that $\mathrm{SL}_{2n}(\mathbb{R})$ acts transitively on $V_1 = \{X \in V : \mathrm{Pff}(X) = 1\}$, so that $V_m = m^{1/n} v_0 \,\mathrm{SL}_{2n}(\mathbb{R})$. The stabilizer H of v_0 is the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ corresponding to v_0, which is a maximal connected closed subgroup of $\mathrm{SL}_{2n}(\mathbb{R})$. Clearly $V_m(\mathbb{Z}) \neq \emptyset$ and $\mathrm{SL}_{2n}(\mathbb{Z})$ preserves $V_m(\mathbb{Z})$ for each m.

To check the condition 1.3 of Theorem 1.2, suppose for a given $m_0 \in \mathbb{N}$ that $\mathrm{pr}(V_{m_0}(\mathbb{Z})) = \mathrm{pr}(V_m(\mathbb{Z}))$. Then

$$m^{-1/n} \begin{pmatrix} & & m & 0 \\ & & 0 & I_{n-1} \\ -m & 0 & & \\ 0 & -I_{n-1} & & \end{pmatrix} \in m_0^{-1/n} V(\mathbb{Z}).$$

Hence $m^{-1/n} = m_0^{-1/n} k$ for some $k \in \mathbb{N}$. This leads to $m = m_0 k^{-n} \leq m_0$. Hence

$$|\{m \in \mathbb{N} : \mathrm{pr}(V_{m_0}(\mathbb{Z})) = \mathrm{pr}(V_m(\mathbb{Z}))\}| \leq m_0 < \infty.$$

Hence the claim follows from Theorem 1.2. □

Since the explicit representatives of $\mathrm{SL}_{2n}(\mathbb{Z})$-orbits on $V_m(\mathbb{Z})$ can easily be written down in the Pff case, we can compute ω_m, using the local density formula given in [GY]. For instance, for a prime p,

$$\omega_p = C \cdot \sum_{i=0}^{2n-2} p^i$$

for some constant C independent of p.

Theorem 6.3. *Let $n \geq 2$. For any compact subset Ω of $V_1 = \mathrm{SL}_n(\mathbb{R})$ with boundary of measure 0,*

$$N_m(\mathrm{Det}, \Omega) \sim_{m\to\infty} \omega_m \cdot \mathrm{vol}\,(\Omega).$$

Proof. Consider the representation $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ on the space $V = M_n(\mathbb{R})$ given by $X(A, B) = AXB^{-1}$ where $X \in V$ and $A, B \in \mathrm{SL}_n(\mathbb{R})$. The stabilizer of I_n is given by the diagonal embedding of $\mathrm{SL}_n(\mathbb{R})$ in the product $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$, which is a maximal connected closed subgroup. Clearly $V_m(\mathbb{Z}) \neq \emptyset$ and $\mathrm{SL}_n(\mathbb{Z}) \times \mathrm{SL}_n(\mathbb{Z})$ preserves $V(\mathbb{Z})$. The condition 1.3 can be checked similarly as in the case of Pfaffian. Hence our claim Theorem 1.2. □

For the determinant case, the constant ω_m is well known from the theory of Hecke operators. (cf. [Sa], or [GO]). For instance, for any fixed $k \in \mathbb{N}$ and a prime p,

$$\omega_{p^k} \sim_{p\to\infty} C \cdot p^{k(n-1)}.$$

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