ON A PROBLEM CONCERNING ARITHMETICITY OF DISCRETE GROUPS ACTING ON $\mathbb{H} \times \cdots \times \mathbb{H}$

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ABSTRACT. We discuss an open problem on the discreteness of subgroups of $(SL_2(\mathbb{R}))^n$ ($n \geq 2$) generated by n linearly independent upper triangular matrices and n linearly independent lower triangular matrices. According to a conjecture by Margulis, only Hilbert modular groups can arise this way. The purpose of this note is to explain how this open problem is related to another conjecture on the orbit behavior of diagonal subgroups in the homogeneous space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.

1. **Introduction**

Let $G := (SL_2(\mathbb{R}))^n$ be the product of n copies of $SL_2(\mathbb{R})$. A discrete subgroup Γ of G is said to be a *lattice* if $\Gamma\backslash G$ is a finite volume space. A lattice in G is called *non-uniform* if $\Gamma\backslash G$ is not compact, and *irreducible* if for any proper connected normal subgroup N of G, $\Gamma \cap N$ is not discrete.

An example of a non-uniform irreducible lattice in G is a Hilbert modular group acting on $\mathbb{H}^n$, $\mathbb{H}$ the hyperbolic plane. If k is a totally real number field of degree n over the rationals $\mathbb{Q}$, and $A^{(i)}$, $i = 1, \cdots, n$ denotes the n conjugates of $A \in SL_2(k)$ under the different embeddings of k into $\bar{\mathbb{Q}}$ over $\mathbb{Q}$, then the subgroup $\{(A^{(1)}, \cdots, A^{(n)}) \in G : A \in SL_2(\mathcal{O}_k)\}$, $\mathcal{O}_k$ the ring of integers of k, is called the *Hilbert modular group* related to the field k. We denote this group by $SL_2(\mathcal{O}_k)$.

Selberg proved in the late sixties [Se]:

Theorem 1.1. *If $n \geq 2$ and Γ is a non-uniform irreducible lattice in G, then Γ is a Hilbert modular group up to conjugation in $GL_2(\mathbb{R})^n$ and up to commensurability, that is, there exist a totally real number field k and an element $g \in (GL_2(\mathbb{R}))^n$ such that $g\Gamma g^{-1} \cap SL_2(\mathcal{O}_k)$ has finite index both in $g\Gamma g^{-1}$ and $SL_2(\mathcal{O}_k)$.*

This was the first instance where the arithmeticity of irreducible lattices in higher rank (meaning that the real rank is at least 2) semisimple real algebraic groups, which was conjectured by Selberg for non-uniform lattices and by Piateski Shapiro for uniform lattices, was settled (cf. [Ti]). Both conjectures were completely settled by Margulis in the mid seventies by his celebrated super-rigidity theorem [Ma2].

A main characteristic of a *non-uniform* lattice used in the proof of the above theorem given by Selberg [Se] was that Γ contains a non-trivial unipotent element and

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moreover, up to conjugation, Γ intersects U_1 and U_2 as lattices where U_1, U_2 denote the (strictly) upper and lower triangular subgroup of G respectively, or equivalently $\Gamma \cap U_i$ contains n linearly independent vectors over $\mathbb{R}$, with U_i considered as $\mathbb{R}^n$ in a natural way.

The following conjecture was made by Margulis in 1993:

Conjecture 1.2. *Let $n \geq 2$ and Γ be a discrete subgroup of G such that for each $i = 1, 2$, $\Gamma \cap U_i$ is a lattice in U_i and for any proper connected normal subgroup N of G, $\Gamma \cap U_i \cap N$ is not discrete. Then Γ is commensurable with a Hilbert modular group up to conjugation in $(GL_2(\mathbb{R}))^n$.*

Note that the above conjecture describes a sufficient (also necessary by the preceding discussion) condition for a discrete subgroup Γ of G to be an arithmetic subgroup in G with $\Gamma\backslash G$ non-compact.

One can also view the above as a statement about discreteness criterion on the subgroups generated by some unipotent elements in G.

For simplicity, we write $1_n = (1, \cdots, 1) \in \mathbb{R}^n$ and $0_n = (0, \cdots, 0) \in \mathbb{R}^n$. We sometimes write

$$\begin{pmatrix} (a_1 \cdots, a_n) & (b_1, \cdots, b_n) \\ (c_1, \cdots, c_n) & (d_1, \cdots, d_n) \end{pmatrix} \text{ instead } \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right),$$

$$(v_1, \cdots, v_n) \in U_1 \text{ instead } \begin{pmatrix} 1_n & (v_1, \cdots, v_n) \\ 0_n & 1_n \end{pmatrix} \in U_1,$$

and similarly for U_2 as well.

For each $i = 1, 2$ and $v = (v_1, \cdots, v_n) \in U_i$, the k-th component of v means v_k. Conjecture 1.2 has the following equivalent form (see section 3):

Conjecture 1.3. *Let $n \geq 2$. For each $i = 1, 2$, let V_i be a set of n-linearly independent vectors in U_i such that no non-zero $\mathbb{Z}$-linear combination of V_i has 0 component in U_i. If V_1 and V_2 generate a discrete subgroup of G, then there exist a totally real number field k of degree n over $\mathbb{Q}$, an n-tuple α of non-zero real numbers and a non-zero integer p such that*

$$V_1 \subset \alpha \mathcal{O}_k \quad \text{and} \quad V_2 \subset \frac{1}{p}\alpha^{-1}\mathcal{O}_k.$$

Here $\alpha^{\pm 1}\mathcal{O}_k$ denotes the set $\{(\alpha_1^{\pm 1}x^{(1)}, \cdots, \alpha_n^{\pm 1}x^{(n)}) : x \in \mathcal{O}_k\}$, respectively, for $\alpha := (\alpha_1, \cdots, \alpha_n)$.

Even though we stated Conjecture 1.2 only for the case where G is a direct product of n copies of $SL_2(\mathbb{R})$ $(n \geq 2)$, it is the expectation of Margulis conjecture (see [Oh1] for a general statement) that the analogous statement should be true for any higher rank connected semisimple real algebraic group G with no compact factors and for any pair U_1, U_2 of the unipotent radicals of opposite parabolic subgroups of G; of course the conclusion would be that any such Γ is an arithmetic subgroup.

In [Oh1-2], this general version of Conjecture 1.2 has been settled in some cases, for instance, including the cases $G = SL_n(\mathbb{R})$ $(n \geq 4)$. One of main ingredients of the proof there is Ratner's theorem on Raghunathan's topological conjecture on the behavior of orbits of unipotent subgroups in homogeneous spaces [Ra]. To apply this method, one needs to have a unipotent one parameter subgroup contained in the common normalizer of U_1' and U_2' for some pair $U_1' \subset U_1$, $U_2' \subset U_2$ of the unipotent radicals of opposite parabolic subgroups of G.

This is certainly not available in the situation of $G = (SL_2(\mathbb{R}))^n$, since the common normalizer of any such U_1' and U_2' is a torus. Instead, it turns out the following conjecture on the orbits of diagonal subgroups of $SL_n(\mathbb{R})$ on the homogeneous space $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ is relevant at least in the case of $n \geq 3$.

Conjecture 1.4 (Ma3, Conjecture 9)**.** *Let $n \geq 3$ and D denote the diagonal subgroup of $SL_n(\mathbb{R})$. For any $x \in SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$, if the orbit xD is relatively compact, then xD is closed.*

We remark that the Littlewood conjecture follows from Conjecture 1.4 (see [Ma3]).

Letting $F_1 := \Gamma \cap U_1$, we may consider $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ as the space of lattices in $U_1 \simeq \mathbb{R}^n$ with the same determinant as the lattice F_1.

Theorem 1.5. *Let $n \geq 2$ and Γ be as in Conjecture 1.2.*

(1) *Then the orbit $F_1 D$ is relatively compact in $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.*
(2) *If the closure $\overline{F_1 D}$ contains a closed D-orbit in $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ then Conjecture 1.2 holds.*

If we assume that Conjecture 1.4 is true, the above theorem in particular implies Conjectures 1.2 and 1.3 for $n \geq 3$.

Under the assumption that Γ is discrete, the compactness of the orbit $F_1 D$ provides some non-trivial diagonal elements of G which normalize F_1 and F_2 simultaneously. Utilizing such elements one is then able to find a $\mathbb{Q}$-structure of G with respect to which Γ is contained in $G(\mathbb{Q})$. We give a detailed proof of the above theorem in the next section.

As well known, Conjecture 1.4 is not true for $n = 2$, in which case the structure of geodesic flows is far from being rigid. We are then in the following situation: Let $\alpha = (\alpha_1, \alpha_2)$ be a vector in $\mathbb{R}^2$ of non-zero reals and $v = (v_1, v_2)$ be a vector in $\mathbb{R}^2$ such that $v_i \notin \mathbb{Q}$ for each $i = 1, 2$. Let $\Gamma_{\alpha,v}$ be the subgroup of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ generated by

$$\begin{pmatrix} 1_2 & 1_2 \\ 0_2 & 1_2 \end{pmatrix}, \begin{pmatrix} 1_2 & (v_1, v_2) \\ 0_2 & 1_2 \end{pmatrix}, \begin{pmatrix} 1_2 & 0_2 \\ (\alpha_1, \alpha_2) & 1_2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1_2 & 0_2 \\ (\alpha_1 v_1, \alpha_2 v_2) & 1_2 \end{pmatrix}.$$

If $F := \mathbb{Z}(1,1) + \mathbb{Z}(v_1, v_2)$ then the set $FD \subset SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ is the collection of lattices $\{(aw_1, a^{-1}w_2) : (w_1, w_2) \in F\}$ in $\mathbb{R}^2$ where a ranges over non-zero real numbers. Theorem 1.5 then implies:

Theorem 1.6. *If $\Gamma_{\alpha,v}$ is discrete and the closure $\overline{FD}$ contains a closed D-orbit, in $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$, then there exists a real quadratic extension field k over $\mathbb{Q}$ such that v_1 is an algebraic integer in k, $\alpha_1 \in k$ and*

$$\alpha_2 = \sigma(\alpha_1) \quad \text{and} \quad v_2 = \sigma(v_1)$$

where σ is the non-trivial Galois automorphism of k.

In fact for $n = 2$, it suffices to prove Conjecture 1.2 for the subgroups of the form $\Gamma_{\alpha,v}$ as above (see Lemma 2.1). Conjecture 1.2 will hence be settled if the following is true:

Conjecture 1.7. *If $\Gamma_{\alpha,v}$ is discrete, $\overline{FD}$ contains a closed D-orbit in $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$.*

Lastly we mention that the analogous problems remain open in the groups of the form $SL_2(k_1)\times\cdots\times SL_2(k_n)$, $n \geq 2$ where $k_1, \cdots, k_n$ are the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the p-adic field $\mathbb{Q}_p$.

2. Proof of Theorem 1.5

In this section we give a proof of Theorem 1.5. We try to make the arguments self contained and as elementary as possible. The scheme of the proof essentially follows steps of [Oh2] which is in turn heavily influenced by [Se] and [Ma1]. We remark that in [Ma1], Margulis first gave the proof of the arithmeticity of *non-uniform* irreducible lattices in higher rank semisimple real algebraic groups before [Ma2] (see also [Ra1] for an independent approach in this direction).

Let Γ be as in Conjecture 1.2 and $n \geq 2$. Set $F_i := \Gamma \cap U_i$ for each $i = 1, 2$. Without loss of generality, we may assume that Γ is the subgroup generated by F_1 and F_2.

Lemma 2.1. *There exist an element $u \in U_1$ and a diagonal element $x \in G$ such that $u\Gamma u^{-1}$ contains the subgroup generated by F_1 and $xwF_1w^{-1}x^{-1}$ where $w = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$.*

Proof. Let $\gamma \in F_2$ be non-trivial element. Then $\gamma N(U_1)\gamma^{-1} \cap N(U_1)$ is conjugate to the diagonal subgroup in G, which coincides with $N(U_1) \cap N(U_2)$. Here $N(U_i)$ denotes the normalizer of U_i in G for both $i = 1, 2$. Hence for some $u \in U_1$, $u\gamma N(U_1)\gamma^{-1}u^{-1} = N(U_2)$. Since $wN(U_1)w^{-1} = N(U_2)$, it follows that $u\gamma w^{-1} \in N(U_2)$, which we can write xy for a diagonal element $x \in G$ and $y \in U_2$. Hence $u\gamma F_1\gamma^{-1}u^{-1} = xy(wF_1w^{-1})y^{-1}x^{-1}$. Since both wF_1w^{-1} and y belong to U_2, which is commutative, $y(wF_1w^{-1})y^{-1} = wF_1w^{-1}$. Since $F_1 = uF_1u^{-1}$ and $u\gamma F_1\gamma^{-1}u^{-1} = xwF_1w^{-1}x^{-1}$ and they are both contained in $u\Gamma u^{-1}$, the claim is proved. □

Hence there is no loss of generality in assuming that Γ is generated by F_1 and F_2 where

$$F_2 = xwF_1w^{-1}x^{-1}$$

when $x \in G$ is a diagonal element. Let H denote the subgroup

$$\{\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \cdots, \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1} \end{pmatrix}\right) \in G : \prod_{i=1}^{n} a_i = 1\}.$$

Note that the elements of H belong to the common normalizer of U_1 and U_2, and preserve a Haar measure on U_1. Hence via conjugation, H acts on the space of lattices in U_1 with the same determinant as F_1, which can be identified with $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$. Furthermore under this action we have $F_1.H = F_1D^0$ where D^0 is the identity component of D, that is, the subgroup of $SL_n(\mathbb{R})$ consisting of positive diagonals.

Note that D^0 is a normal subgroup of G, and the orbit xD is a disjoint union of finitely many translates of xD^0. Hence F_1D is relatively compact (resp. closed) if and only if F_1D^0 is relatively compact (resp. closed).

Theorem 2.2. *If Γ is discrete, then both orbits F_1D and F_2D are relatively compact in $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.*

Proof. There exists an $\epsilon_0 > 0$ (depending only G) such that the intersection of any discrete subgroup of G and the ϵ_0 neighborhood (the so-called Zassenhause neighborhood) in G generates a nilpotent subgroup (cf. [Ma1]). By taking ϵ_0 small enough, we may also assume that for any g_1 and g_2 in ϵ_0-neighborhood of G, the commutator $g_1g_2g_1^{-1}g_2^{-1}$ is contained in the $\epsilon_0/2$-neighborhood of G (cf. [Se]).

By a theorem of Minkowski (cf. [Ca, Ch VIII]), there exists a constant $c > 0$ (depending only on n) such that any lattice in U_2 with the same determinant as F_2 contains a non-zero vector whose norm is at most c. Let $g = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \cdots, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \in G$ be a suitable diagonal element which contracts U_2 so that if v is an element in U_2 of norm less than c, then gvg^{-1} has norm less than ϵ_0. Note that $g\Gamma g^{-1}$ is contains $a^2F_1 := gF_1g^{-1}$ and $a^{-2}F_2 := gF_2g^{-1}$. Suppose that $F_1.H$ and hence $a^2F_1.H$ is unbounded, then by Mahler's compactness criterion (cf. [Ca, Ch V]), there exist an $h \in H$ and a non-zero vector $v = (v_1, \cdots v_n) \in a^2hF_1h^{-1}$ with norm less than ϵ_0, considered as an Euclidean space. From the irreducibility assumption on Γ, we have $v_i \neq 0$ for each $1 \leq i \leq n$. Consider the subgroup $hg\Gamma g^{-1}h^{-1}$ which is generated by $a^2hF_1h^{-1}$ and $a^{-2}hF_2h^{-1}$. Let $w \in a^{-2}hF_2h^{-1}$ be a non-trivial element with norm less than ϵ_0, provided by Minkowski theorem mentioned in the beginning. Then the subgroup generated by v and $vwv^{-1}w^{-1}$ has to be nilpotent; in fact unipotent, since any nilpotent subgroup generated by unipotent elements is unipotent. However from the matrix multiplication it is easy to see that $vwv^{-1}w^{-1}$ cannot be an unipotent element for any $v \in U_1$ and $w \in U_2$ with non-zero components. This contradiction shows that $F_1.H$ hence F_1D are relatively compact. Since $F_2H = (xw)F_1H(xw)^{-1}$, F_2H is relatively compact as well. □

Theorem 2.3. *If $F_1.H = F_1D^0$ is closed and Γ is discrete, then Γ is commensurable with a Hilbert modular subgroup up to conjugation in $(GL_2(\mathbb{R}))^n$.*

Remark A subgroup of G (locally compact and second countable) is called L-subgroup if for any $x_n \in G$, $\pi(x_n)$ has no convergent subsequence for the natural projection $\pi : G \to \Gamma\backslash G$ if and only of there exists $\gamma_n \neq e \in \Gamma$ such that $x_n\gamma_n x_n^{-1}$ converges to e as $n \to \infty$ [Ra2, Definition 1.21].

Note that any lattice in G is known to be an L-subgroup [Ra2, Theorem 1.21]. We remark that if we know that our discrete subgroup Γ in Conjecture 1.2 is an L-subgroup, we can directly show that F_1H is compact. In fact, for any discrete subgroup Γ, $\pi(H)$ is closed in $\Gamma\backslash G$. It suffices to show that $\pi(H)$ is relatively compact. If not, there exists $\gamma_n \neq e \in \Gamma$ and $a_n \in H$ such that $a_n\gamma_n a_n^{-1}$ converges to e as $n \to \infty$ [Ra2, Definition 1.21]. It is easy to see that the elements γ_n are unipotent elements and moreover cotained in $F_1 \cup F_2$. Hence this would contradict that F_1H and F_2H are relativey compact. Therefore $\pi(H)$ is compact, which clearly implies that F_1H is compact as well.

The property of being an L-subgroup which describes the cusp structure of a fundamental domain of Γ has proved to be very critical in the study of non-uniform lattices. For instance, many properties of lattices in a connected semisimple real algebraic group with no compact factors are known to be shared by discrete subgroups with this property (cf. Ra2).

We now begin a proof of Theorem 2.3. Letting H^0 denote the identity component of H, set

$$\Delta := \{g \in H^0 : g^{-1}F_1 g = F_1\}.$$

Then

$$a := \left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1}\end{pmatrix}, \cdots, \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1}\end{pmatrix}\right) \in \Delta$$

if and only if $(a_1^2x_1, \cdots, a_n^2x_n) \in F_1$ for any $(x_1, \cdots, x_n) \in F_1$.

For each $1 \leq i \leq n$, define a map $\phi_i : \Delta \to \mathbb{R}^+$ by

$$\phi_i\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1}\end{pmatrix}, \cdots, \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1}\end{pmatrix}\right) = a_i^2.$$

Note that the assumption that $\Gamma \cap U_1 \cap N$ is not discrete for any proper connected normal subgroup N of G implies that for any non-zero $(v_1, \cdots, v_n) \in F_1$, $v_i \neq 0$ for all $1 \leq i \leq n$. It follows that ϕ_i is an injective homomorphism. Also if $\phi_i = \phi_j$ for some $i \neq j$, this would imply that Δ is contained in a subgroup of H of dimension strictly less than n, which is a contradiction since $\Delta\backslash H$ is compact. Hence all ϕ_i, $1 \leq i \leq n$ are distinct from each other. Denote the image of ϕ_i by Δ_i. If we now consider ϕ_i as a map from Δ to Δ_i, ϕ_i is an isomorphism. Furthermore, since Δ is a co-compact lattice in H^0, Δ_i has rank $n-1$ as a free abelian group.

Let $\{Y_i = (y_{i1}, \cdots, y_{in}) : i = 1, \cdots, n\}$ be a $\mathbb{Z}$-basis of F_1. Then for any

$$a := \left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1}\end{pmatrix}, \cdots, \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1}\end{pmatrix}\right) \in \Delta,$$

$aY_ia^{-1} = (a_1^2 y_{i1}, \cdots, a_n^2 y_{in})$ has to be a $\mathbb{Z}$-linear combination of $Y_1, \cdots, Y_n$ for each $i = 1, \cdots, n$. So there exists an $n \times n$ matrix of integer coefficients whose characteristic polynomial has $a_1^2, \cdots, a_n^2$ as its zeros.

It follows that every element in Δ_i is an algebraic number with degree at most n. Hence if k_i denotes the field generated by Δ_i, then $[k_i : \mathbb{Q}] \leq n$. Since $x \in \Delta_i$ implies $x_i^{-1} \in \Delta_i$ as well, $\Delta_i \subset \mathcal{O}_{k_i}^*$. On the other hand, by Dirichlet unit theorem (cf. [La]), the rank of $\mathcal{O}_{k_i}^*$ is equal to $[k_i : \mathbb{Q}] - 1$. Hence this forces that the degree of k_i over $\mathbb{Q}$ be precisely n.

Define $\sigma_i : \Delta_1 \to \Delta_i$ by $\sigma_i = \phi_i \circ \phi_1^{-1}$ for each $1 \leq i \leq n$. Then σ_i is a group isomorphism and extends to a field isomorphism from k_1 to k_i, which we denote by σ_i as well, by slight abuse of notation. Since ϕ_i, $1 \leq i \leq n$, are distinct, so are σ_i, $1 \leq i \leq n$.

For simplicity, we set $k = k_1$ and write $a^{(i)}$ for $\sigma_i(a)$. Hence the subgroup

$$\{\begin{pmatrix} a^{(1)} & 0 \\ 0 & 1/a^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} a^{(n)} & 0 \\ 0 & 1/a^{(n)} \end{pmatrix} : a \in \Delta_1 \subset \mathcal{O}_k^*\}$$

is a subgroup of finite index in Δ.

By conjugating Γ using a diagonal element in $(GL_2(\mathbb{R}))^n$, we may assume that F_1 contains the element $1_n = (1, \cdots, 1)$. Since Δ normalizes F_1, F_1 contains the $\mathbb{Z}$-linear combinations of $(x^{(1)}, \cdots, x^{(n)})$ where $x \in \Delta_1$, which we denote by $\mathfrak{a}$. Since the rank of Δ_1 is $n-1$ and hence $\{(\log|x^{(1)}|, \cdots, \log|x^{(n)}|) : x \in \Delta_1$ is a co-compact lattice in the subspace $\{(v_1, \cdots, v_n) \in \mathbb{R}^n : \sum_{i=1}^n v_i = 0\}$, we can deduce that $\mathfrak{a}$ is a subgroup of finite index in

$$\{(x^{(1)}, \cdots, x^{(n)}) : x \in \mathcal{O}_k\},$$

which we denote simply by $\mathcal{O}_k$.

Since $F_2 = xwF_1w^{-1}x^{-1}$ for a diagonal element $x \in G$ and $w = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$, we have for an n-tuple of some positive real numbers $\alpha = (\alpha_1, \cdots, \alpha_n)$, $F_2 = \alpha F_1$ where αF_1 denotes the set

$$\{(\alpha_1 x_1, \cdots, \alpha_n x_n) : (x_1, \cdots, x_n) \in F_1\}.$$

Since Δ is contained in the diagonal subgroup normalizing F_1, Δ normalizes F_2 as well. Hence by the same argument as above, we see that $F_2 \supset \alpha\mathfrak{a}$. To distinguish the vector notations for F_1 and F_2, we now write $F_1 \supset U_1(\mathfrak{a})$ and $F_2 \supset \alpha U_2(\mathfrak{a})$. We may assume that $\mathfrak{a}$ is an ideal of $\mathcal{O}_k$ by making it smaller if necessary.

Without loss of generality we may now assume that Γ is generated by $U_1(\mathfrak{a})$ and $\alpha U_2(\mathfrak{a})$.

Proposition 2.4. *Let $n \geq 2$. If $U_1(\mathfrak{a})$ and $\alpha U_2(\mathfrak{a})$ generate a discrete subgroup, then*

$$(\alpha_1, \cdots, \alpha_n) = (b^{(1)}, \cdots, b^{(n)})$$

for some $b \in k^*$.

Proof. Denote by Γ_0 the normalizer of Γ in G. We claim that Γ_0 is discrete. Suppose not; then there exists a sequence of distinct elements $g_j \in \Gamma_0$ which converges to the identity as $j \to \infty$. For each $\gamma \in \Gamma$, $g_j\gamma g_j^{-1} \in \Gamma$ and it converges to γ as $j \to \infty$. Since Γ is discrete, it follows that there exists a positive integer $j(\gamma)$ such that for all $j > j(\gamma)$, g_j is in the centralizer of γ. Since γ is an arbitrary element of Γ and Γ is finitely generated, we can find j_0 such that for any $j > j_0$, g_j is in the centralizer of Γ and hence in the center of G, since Γ is Zariski dense. Since the center of G is finite, this contradiction yields the claim.

Note that $\Delta \subset \Gamma_0$. Set $A_1 := H \ltimes U_1$. Then since $\Delta \ltimes F_1 \subset A_1 \cap \Gamma_0$ and hence $A_1 \cap \Gamma_0$ is co-compact in A_1, $\gamma A_1 \gamma^{-1} \cap \Gamma_0$ is co-compact in $\gamma A_1 \gamma^{-1}$ for any $\gamma \in \Gamma_0$. Since Γ_0 is discrete, the intersection $\gamma A_1 \gamma^{-1} \cap A_1 \cap \Gamma_0$ is co-compact in $\gamma A_1 \gamma^{-1} \cap A_1$ for any $\gamma \in \Gamma_0$.

It can be easily checked that for any non-trivial $g \in U_2$, $gA_1g^{-1} \cap A_1$ is conjugate to H. Hence if $\gamma \in F_2$ is non-trivial, we have that $\gamma A_1 \gamma^{-1} \cap A_1 \cap \Gamma_0$ is infinite. Since

$$\gamma A_1 \gamma^{-1} \cap A_1 \cap \Gamma_0 = \gamma(A_1 \cap \Gamma_0)\gamma^{-1} \cap (A_1 \cap \Gamma_0)$$

there exist elements $\delta_1, \delta_2 \in \Delta \ltimes F_1 \subset A_1 \cap \Gamma_0$ such that δ_1, $\delta_2 \notin F_1$ and

$$\gamma\delta_1 = \delta_2\gamma. \tag{2.5}$$

Let

$$\gamma := \begin{pmatrix} 1_n & 0_n \\ (\alpha_1 x^{(1)}, \cdots, \alpha_n x^{(n)}) & 1_n \end{pmatrix} \in F_2;$$

and for each $j = 1, 2$, let

$$\delta_j = \left(\begin{pmatrix} {z_j}^{(1)} & {y_j}^{(1)} \\ 0 & 1/{z_j}^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} {z_j}^{(n)} & {y_j}^{(n)} \\ 0 & 1/{z_j}^{(n)} \end{pmatrix} \right) \in \Delta \ltimes F_1$$

for a non-zero $x \in \mathfrak{a}$, for a unit $z_j \in \mathcal{O}_k$ and for a non-zero $y_j \in \mathcal{O}_k$.

Then the equation 2.5 yields that $y_2 \neq 0$, $z_1 \neq \pm 1$ and

$$\alpha_i = \frac{(xy_2)^{(i)}}{(z_1 - z_1^{-1})^{(i)}}$$

for each $1 \leq i \leq n$. Hence it suffices to set $b = \frac{(xy_2)}{(z_1 - z_1^{-1})}$. ☐

Therefore we can find a non-trivial ideal $\mathfrak{b}$ of $\mathcal{O}_k$ contained in $\mathfrak{a}$ such that

$$F_1 \supset U_1(\mathfrak{b}) \quad \textit{and} \quad F_2 \supset U_2(\mathfrak{b}).$$

Applying the following, which was first proven by Vasserstein [Va], we now conclude that Γ is commensurable with a Hilbert modular group, completing the proof of Theorem 2.3.

Theorem 2.6. *Let $n \geq 2$. For any non-trivial ideal $\mathfrak{b}$ of $\mathcal{O}_k$, the subgroup of G generated by $U_1(\mathfrak{b})$ and $U_2(\mathfrak{b})$ is of finite index in the Hilbert modular group $SL_2(\mathcal{O}_k)$.*

Theorem 1.5 (2) now follows from Theorem 2.3 toghether with the following:

Theorem 2.7. *If* Γ *is discrete and the closure* $\overline{F_1D^0}$ *contains a closed* D^0*-orbit, then* F_1D^0 *is closed.*

Proof. Let $E_1 \in SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ be such that E_1D^0 is a closed orbit contained in $\overline{F_1D^0}$. There exists a sequence $h_n \in H$ such that $h_nF_1h_n^{-1}$ converges to E_1. Since $H.F_2$ is relatively compact, $h_nF_2h_n^{-1}$ has a convergeant subsequence. By passing to a subsequence, we may assume that $h_nF_2h_n^{-1}$ converges to a lattice, say, E_2 in $\mathbb{R}^n$. Since $H.E_1$ and $H.E_2$ are relatively compact, it follows from Mahler's compactness criterion that no non-zero vector in E_1 or E_2 has a zero component. We can furtheremor show that there exists a neighborhood W of e such that $h_n\Gamma h_n^{-1} \cap W = \{e\}$. If we denote by Γ_{E_1,E_2} the subgroup generated by E_1 and E_2, then it follows that Γ_{E_1,E_2} is a limit of $h_n\Gamma h_n^{-1}$ and that it is discrete. Hence by Theorem 2.3, Γ_{E_1,E_2} is a Hilbert modular subgroup up to commensurability and up to conjugation. In particular, it is locally rigid and finitely presentable. Using the property that Γ_{E_1,E_2} is finitely presentable and $h_n\Gamma h_n^{-1} \cap W = \{e\}$, we can define a sequence of homomorphisms $\phi_n : \Gamma_{E_1,E_2} \to h_n\Gamma h_n^{-1}$ such that ϕ_n converges to the identity map. By the local rigidity, it follows that ϕ_n is a conjugation. Since $\phi_n(U_i) \subset U_i$, ϕ_n is a conjugation by a diagonal element in G. Since the determinant of E_i must be equal to that of F_i, it follows that ϕ_n is in fact conjugation by an element of H. Hence $E_i \subset H.F_i$. Since $H.E_1$ is closed, so is $H.F_1$; which in trun implies the same for $H.F_2$. This finishes the proof. □

It follows from the proof of Theorem 2.7 that in order to prove Conjecture 1.2, we only need to know that any discrete subgroup Γ satisfying the assumptions in the conjecture is finitely presentable and locally rigid. Again, the latter property has known to be true, as shown by Selberg [Se]. However, a discrete subgroup of G being finitely presetable is a strong hypothesis about a fundamental domain.

3. Equivalence of Conjecture 1.2 and Conjecture 1.3

To show that Conjecture 1.2 implies Conjecture 1.3, let Γ be the subgroup generated by V_1 and V_2. Then for each $1 \le k \le n$, the k-th components of the elements in $\Gamma \cap U_i$ contains n numbers which are linearly independent over $\mathbb{Z}$. Since any proper connected normal subgroup of G is a product of less than n copies of $SL_2(\mathbb{R})$, Γ satisfies the assumptions in Conjecture 1.2. Hence Γ is commensurable with $gSL_2(\mathcal{O}_k)g^{-1}$ for some $g \in (GL_2(\mathbb{R}))^n$ and a totally real number field k of degree n over $\mathbb{Q}$. Set $\Lambda = \Gamma \cap gSL_2(\mathcal{O}_k)g^{-1}$. Let $p \in \mathbb{N}$ be an upper bound for the indices $[\Gamma \cap U_i : \Lambda \cap U_i]$ for $i = 1, 2$.

Fix a non-zero element $\alpha := (\alpha_1, \cdots, \alpha_n) \subset \Lambda \cap U_1$. Then for any $y := (y_1, \cdots, y_n) \subset \Lambda \cap U_2$, the trace of the element αy is equal to

$$(2 + \alpha_1 y_1, \cdots, 2 + \alpha_n y_n),$$

which must belong to $\{(x^{(1)}, \cdots, x^{(n)}) : x \in \mathcal{O}_k\}$. Hence $(y_1, \cdots, y_n) \in \alpha^{-1}\mathcal{O}_k$. So we have $V_2 \subset \Gamma \cap U_1 \subset \frac{1}{p}\alpha^{-1}\mathcal{O}_k$.

In the same way, if $x = (x_1, \cdots, x_n) \in \Lambda \cap U_1$, then $(x_1, \cdots, x_n) \in y^{-1}\mathcal{O}_k$ for each $y \in \Lambda \cap U_2$. So $\Lambda \cap U_1 \subset p\alpha\mathcal{O}_k$ and hence $V_1 \subset \alpha\mathcal{O}_k$.

To see the other direction, take V_i to be a basis of U_i contained in $\Gamma \cap U_i$ which exists since $\Gamma \cap U_i$ is a lattice in U_i. Clearly the assumption on V_i required in Conjecture 1.3 is satisfied; hence $V_1 \subset \alpha\mathcal{O}_k$ and $V_2 \subset \frac{1}{p}\alpha^{-1}\mathcal{O}_k$ for some n tuple α of non-zero real numbers and some non-zero integer p. By conjugating Γ using a suitable diagonal element, say g, in $(GL_2(\mathbb{R}))^n$, we have $g\Gamma g^{-1} \cap U_1$ and $g\Gamma g^{-1} \cap U_2$ are subgroups of finite indices in $U_1(\mathcal{O}_k)$ and $U_2(\frac{1}{p}\mathcal{O}_k)$ respectively. We can hence find a non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_k$ such that $g\Gamma g^{-1}$ contains $U_1(\mathfrak{a})$ and $U_2(\mathfrak{a})$. Now applying Theorem 2.6, since Γ is discrete, $g\Gamma g^{-1}$ is commensurable with $SL_2(\mathcal{O}_k)$.

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