UNIPOTENT FLOWS ON HIGHER DIMENSIONAL HYPERBOLIC MANIFOLDS

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Abstract. For $d \geq 4$, we establish an analogue of Ratner's orbit closure theorem for the action of any connected subgroup generated by unipotent elements on the frame bundle of a convex cocompact hyperbolic d-manifold M with Fuchsian ends. For $d = 3$, this was proved earlier by McMullen, Mohammadi and Oh. As an application, we obtain that for any $k \geq 1$, the closure of any k-horosphere or of any geodesic $(k+1)$-plane in M is a properly immersed submanifold.

1. Introduction

Let G be a connected simple linear Lie group and $\Gamma < G$ be a discrete subgroup. Let U be a connected closed subgroup of G generated by unipotent elements in it. We are interested in the action of U on the homogeneous space $\Gamma\backslash G$ by right translations. If the volume of the homogeneous space $\Gamma\backslash G$ is finite, the celebrated Ratner's orbit closure theorem, which was a conjecture of Raghunathan, states that

$$\text{the closure of every } U\text{-orbit is homogeneous,} \tag{1.1}$$

that is, for any $x \in \Gamma\backslash G$, $\overline{xU} = xL$ for some connected closed subgroup $L < G$ containing U [27]. Ratner's proof is based on her classification of all U-invariant ergodic probability measures [26] and the work of Dani and Margulis [11] on the non-divergence of unipotent flow. Prior to her work, some important special cases of (1.1) were established by Margulis [17], Dani-Margulis ([9], [10]) and Shah ([31], [30]) by topological methods. This theorem is a fundamental result with numerous applications.

It is natural to ask if there exists a family of homogeneous spaces of infinite volume where an analogous orbit closure theorem holds. When the volume of $\Gamma\backslash G$ is infinite, the geometry of the associated locally symmetric space turns out to play an important role in this question. The first orbit closure theorem in the infinite volume case was established by McMullen, Mohammadi, and Oh ([19], [20]) for a class of homogeneous spaces $\Gamma\backslash \mathrm{SO}(3,1)$ which arise as the frame bundles of convex cocompact hyperbolic 3-manifolds with Fuchsian ends.

Our goal in this paper is to show that a similar type of orbit closure theorem holds in the higher dimensional analogues of these manifolds. We present a complete hyperbolic d-manifold $M = \Gamma\backslash\mathbb{H}^d$ as the quotient of the hyperbolic space by a discrete subgroup Γ of $G = \mathrm{SO}^\circ(d,1) \simeq \mathrm{Isom}^+(\mathbb{H}^d)$. The geometric boundary of $\mathbb{H}^d$ can be identified with the sphere $\mathbb{S}^{d-1}$. The limit set $\Lambda \subset \mathbb{S}^{d-1}$ of Γ is the set of all accumulation points of an orbit Γx in the compactification $\mathbb{H}^d \cup \mathbb{S}^{d-1}$ for $x \in \mathbb{H}^d$. The convex core of M is given by the quotient $\operatorname{core} M = \Gamma\backslash \operatorname{hull}(\Lambda)$

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where $\text{hull}(\Lambda) \subset \mathbb{H}^d$ is the smallest convex subset containing all geodesics in $\mathbb{H}^d$ connecting points in Λ. When $\text{core}\, M$ is compact, M is called *convex cocompact*.

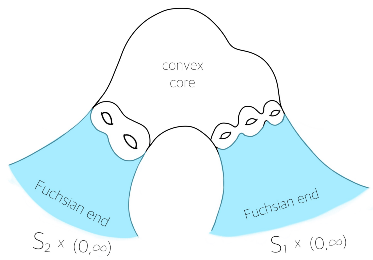

FIGURE 1. A convex cocompact hyperbolic manifold with non-empty Fuchsian ends

In the rest of the introduction, we assume that M is a convex cocompact hyperbolic d-manifold ($d \ge 2$) with Fuchsian ends, that is, $\text{core}\, M$ has non-empty interior and has totally geodesic boundary. The term *Fuchsian ends* reflects the fact that each component of the boundary of $\text{core}\, M$ is a $(d-1)$-dimensional closed hyperbolic manifold, and each component of the complement $M - \text{core}(M)$ is diffeomorphic to the product $S \times (0,\infty)$ for some closed hyperbolic $(d-1)$-manifold S. We remark that for $d \ge 4$, Kerckhoff and Storm showed that a convex cocompact hyperbolic manifold with Fuchsian ends does not allow any non-trivial deformation [13].

The homogeneous space $\Gamma\backslash G$ can be regarded as the bundle $\mathrm{F}\, M$ of oriented frames over M. Let $A = \{a_t : t \in \mathbb{R}\} < G$ denote the one parameter subgroup of diagonalizable elements whose right translation actions on $\Gamma\backslash G$ correspond to the frame flow. Let $N \simeq \mathbb{R}^{d-1}$ denote the contracting horospherical subgroup: $N = \{g \in G : a_{-t} g a_t \to e \text{ as } t \to +\infty\}$. We denote by $\mathrm{RF}\, M$ the renormalized frame bundle of M:

$$\mathrm{RF}\, M := \{x \in \Gamma\backslash G : \ xA \text{ is bounded}\},$$

and also set

$$\mathrm{RF}_+ M := \{x \in \Gamma\backslash G : xA^+ \text{ is bounded}\}$$

where $A^+ = \{a_t : t \ge 0\}$. The sets $\mathrm{RF}\, M$ and $\mathrm{RF}_+ M$ are precisely non-wandering sets for the actions of A and N respectively [34].

For a connected closed subgroup $U < N$, we denote by $H(U)$ the smallest closed simple Lie subgroup of G which contains both U and A. If $U \simeq \mathbb{R}^k$, then $H(U) \simeq \mathrm{SO}^\circ(k+1,1)$. A connected closed subgroup of G generated by one-parameter unipotent subgroups is, up to conjugation, of the form $U < N$ or $H(U)$ for some $U < N$ (Cor. 2.7).

We set $F_{H(U)} := \mathrm{RF}_+ M \cdot H(U)$, which is a closed subset. It is easy to see that if $x \notin \mathrm{RF}_+ M$ (resp. $x \notin F_{H(U)}$), then xU (resp. $xH(U)$) is closed in $\Gamma\backslash G$.

Orbit closures are relatively homogeneous. We define the following collection of closed connected subgroups of G:

$$\mathcal{L}_U := \left\{ L = H(\widehat{U})C : \begin{array}{c} \text{for some } z \in \mathrm{RF}_+ M,\, zL \text{ is closed in } \Gamma\backslash G \\ \text{and } \mathrm{Stab}_L(z) \text{ is Zariski dense in } L \end{array} \right\}.$$

where $U < \widehat{U} < N$ and C is a closed subgroup of the centralizer of $H(\widehat{U})$. We also define: $\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N\}$. The following theorem gives a classification of orbit closures for all connected closed subgroups of G generated by unipotent one-parameter subgroups:

Theorem 1.1. *Let $U < N$ be a non-trivial connected closed subgroup.*

(1) ($H(U)$-orbit closures) *For any $x \in \mathrm{RF}\, M \cdot H(U)$,*
$$\overline{xH(U)} = xL \cap F_{H(U)}$$
where xL is a closed orbit of some $L \in \mathcal{L}_U$.

(2) (U-orbit closures) *For any $x \in \mathrm{RF}_+ M$,*
$$\overline{xU} = xL \cap \mathrm{RF}_+ M$$
where xL is a closed orbit of some $L \in \mathcal{Q}_U$.

(3) (Equidistributions) *Let $x_i L_i$ be a sequence of closed orbits intersecting $\mathrm{RF}\, M$, where $x_i \in \mathrm{RF}_+ M$ and $L_i \in \mathcal{Q}_U$. Assume that no infinite subsequence of $x_i L_i$ is contained in a subset of the form $y_0 L_0 D$ where $y_0 L_0$ is a closed orbit of $L_0 \in \mathcal{L}_U$ with $\dim L_0 < \dim G$ and D is a compact subset of the normalizer $\mathrm{N}(U)$ of U. Then*
$$\lim_{i\to\infty} x_i L_i \cap \mathrm{RF}_+ M = \mathrm{RF}_+ M.$$

Remark 1.2. If $x \in F_{H(U)} - \mathrm{RF}\, M \cdot H(U)$, then $\overline{xH(U)} = xLV^+H(U)$ for some $L \in \mathcal{L}_U$, and some one-parameter semigroup $V^+ < N$ (Thm. 10.5).

Theorem 1.1(1) and (2) can be presented as follows in a unified manner:

Corollary 1.3. *Let $H < G$ be a connected closed subgroup generated by unipotent elements in it. Assume that H is normalized by A. For any $x \in \mathrm{RF}\, M$, the closure of xH is homogeneous in $\mathrm{RF}\, M$, that is,*
$$\overline{xH} \cap \mathrm{RF}\, M = xL \cap \mathrm{RF}\, M \tag{1.2}$$
where xL is a closed orbit of some $L \in \mathcal{Q}_U$.

A geodesic plane in M of dimension k is the image of a totally geodesic immersion $f : \mathbb{H}^k \to M$. A horosphere in $\mathbb{H}^d$ of dimension k is a Euclidean sphere of dimension k which is tangent to a point in $\mathbb{S}^{d-1}$. A horosphere in M is simply the image of a horosphere in $\mathbb{H}^d$ under the covering map $\mathbb{H}^d \to M = \Gamma\backslash\mathbb{H}^d$. Theorem 1.1 implies:

Corollary 1.4. *The closure of a geodesic plane of dimension at least 2 or of a horosphere is a properly immersed submanifold of M (possibly with boundary).*

For $d = 3$, the main result of this paper was proved earlier in [19] and [20]. In a higher dimensional case, the possibility of accumulation on closed orbits of intermediate subgroups causes serious issues. Calling the collection of all such closed orbits as the singular set and its complement as the generic set, the main achievement of this paper lies in establishing *avoidance of the singular set* along the k-thick recurrence of unipotent flows to $\mathrm{RF}\, M$ for a sequence of $\mathrm{RF}\, M$-points limiting at a generic point (sec. 5-6). Roughly speaking,[1] Theorem 1.1 is proved by induction on the co-dimension of U in N. For each $i = 1, 2, 3$, let us say that $(i)_m$

[1]To be precise, we need to carry out induction on the co-dimension of U in $\widehat{L} \cap N$ whenever xU is contained in a closed orbit $x_0\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_U$ as formulated in Theorem 13.1.

holds, if Theorem 1.1(i) is true for all U satisfying co-$\dim_N(U) \le m$. We show that the validity of $(2)_m$ and $(3)_m$ implies that of $(1)_{m+1}$ (sec. 15) the validity of $(1)_{m+1}$, $(2)_m$ and $(3)_m$ implies that of $(2)_{m+1}$ (sec. 16), and the validity of $(1)_{m+1}$, $(2)_{m+1}$, and $(3)_m$ implies that of $(3)_{m+1}$ (sec. 17).

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2. Lie subgroups and geodesic planes

Let G denote the connected simple Lie group $\mathrm{SO}^\circ(d,1) \simeq \mathrm{Isom}^+(\mathbb{H}^d)$ for $d \ge 2$. In order to present a family of subgroups of G explicitly, we fix a quadratic form $Q(x_1, \cdots, x_{d+1}) = 2x_1x_{d+1} + x_2^2 + x_3^2 + \cdots + x_d^2$, and identify $G = \mathrm{SO}^\circ(Q)$.

Subgroups of G. Inside G, we have the following subgroups:

$$K = \{g \in G : g^t g = \mathrm{Id}_{d+1}\} \simeq \mathrm{SO}(d),$$
$$A = \left\{a_s = \mathrm{diag}(e^s, \mathrm{Id}_{d-1}, e^{-s}) : s \in \mathbb{R}\right\},$$

$$M = \text{ the centralizer of } A \text{ in } K \simeq \mathrm{SO}(d-1),$$
$$N^\pm = \{\exp u^\pm(x) : x \in \mathbb{R}^{d-1}\},$$

where $u^-(x) = \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix}$ and $u^+(x) = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^t & 0 \end{pmatrix}$.

The Lie algebra of M consists of matrices of the form $m(C) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where $C \in \mathrm{M}_{d-1}(\mathbb{R})$ satisfies $C^t = -C$.

As we will be using the subgroup N^- frequently, we simply write $N = N^-$. We often identify the subgroup $N^\pm$ with $\mathbb{R}^{d-1}$ via the map $\exp u^\pm(x) \mapsto x$. For a connected closed subgroup $U < N$, we use the notation $U^\perp$ for the orthogonal complement of U in N as a vector subgroup of N, and $U^t = U^+$ for the transpose of U. We use the notation $B_U(r)$ to denote the ball of radius r centered at 0 in U for a Euclidean metric on $N = \mathbb{R}^{d-1}$.

We consider the upper-half space model of $\mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}$, so that its boundary is given by $\mathbb{S}^{d-1} = \{\infty\} \cup (\{0\} \times \mathbb{R}^{d-1})$. Set $o = (1, 0, \cdots, 0)$, and fix a standard basis $e_0, e_1, \cdots, e_{d-1}$ at $\mathrm{T}_o(\mathbb{H}^d)$. The map

$$g \mapsto (ge_0, \cdots, ge_{d-1})_{g(o)} \tag{2.1}$$

gives an identification of G with the oriented frame bundle $\mathrm{F}\,\mathbb{H}^d$. The stabilizer of o and e_0 in G are equal to K and M respectively, and hence the map (2.1) induces the identifications of the hyperbolic space $\mathbb{H}^d$ and the unit tangent bundle $\mathrm{T}^1\,\mathbb{H}^d$ with G/K and G/M respectively. The action of G on the hyperbolic space $\mathbb{H}^d = G/K$ extends continuously to the compactification $\mathbb{S}^{d-1} \cup \mathbb{H}^d$. If $g \in G$ corresponds to a frame $(v_0, \cdots, v_{d-1}) \in \mathrm{F}\,\mathbb{H}^d$, we define $g^+, g^- \in \mathbb{S}^{d-1}$ to be the forward and backward end points of the directed geodesic tangent to v_0 respectively. The right translation action of A on $G = \mathrm{F}\,\mathbb{H}^d$ defines the frame flow and we have $g^\pm = \lim_{t \to \pm\infty} \pi(ga_t)$ where $\pi : G = \mathrm{F}\,\mathbb{H}^d \to \mathbb{H}^d$ is the basepoint projection.

Note that $g^+ = g(\infty)$ and $g^- = g(0)$. The subgroup MA fixes both points 0 and ∞, and N fixes ∞, and the restriction of the map $g \mapsto g(0)$ to N defines an isomorphism $N \to \mathbb{R}^{d-1}$ given by $u^-(x) \mapsto x$.

For each non-trivial connected subgroup $U < N$, we denote by $H(U)$ the connected closed subgroup of G generated by U and the transpose of U. It is the smallest simple closed Lie subgroup of G containing A and U. For a subset $S \subset G$, we denote by $\mathrm{N}_G(S)$ and $\mathrm{C}_G(S)$ the normalizer of S and the centralizer of S respectively. We denote by $\mathrm{N}(S)$ and $\mathrm{C}(S)$ the identity components of $\mathrm{N}_G(S)$ and $\mathrm{C}_G(S)$ respectively.

We set $H'(U) := \mathrm{N}(H(U)) = H(U)\,\mathrm{C}(H(U))$, which is a connected reductive algebraic subgroup of G with compact center. Fix the standard basis $e_1, \cdots, e_{d-1}$ of $\mathbb{R}^{d-1}$. For $1 \le k \le d-1$, define U_k to be the connected subgroup of N spanned by $e_1, \cdots, e_k$. Then $H(U_k) = \langle U_k, U_k^t\rangle = \mathrm{SO}^\circ(k+1,1)$, $\mathrm{C}(H(U_k)) = \mathrm{SO}(d-k-1)$ and $\mathrm{N}(H(U_k)) = \mathrm{SO}^\circ(k+1,1)\mathrm{SO}(d-k-1)$. Since the adjoint action of M on N corresponds to the standard action of $\mathrm{SO}(d-1)$ on $\mathbb{R}^{d-1}$, any connected closed subgroup $U < N$ is conjugate to U_k and $H(U)$ is conjugate to $H(U_k)$ by an element of M, where $k = \dim(U)$.

We set

$$\mathrm{C}_1(U) := \mathrm{C}(H(U)) = M \cap \mathrm{C}(U), \text{ and } \mathrm{C}_2(U) := M \cap \mathrm{C}(U^\perp) \subset H(U). \tag{2.2}$$

Lemma 2.1. *We have* $\mathrm{N}(U) = NA\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$ *and* $\mathrm{C}(U) = N\,\mathrm{C}_1(U)$.

Proof. For the first claim, it suffices to show that for $U = U_k$, $\mathrm{N}(U) = NA\,\mathrm{SO}(k)\,\mathrm{SO}(d-1-k)$. It is easy to check that $Q := NA\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$ normalizes U. Let $g \in \mathrm{N}(U)$. We claim that $g \in Q$. Using the decomposition $G = KAN$, we may assume that $g \in K$. Then $Ug(\infty) = gU(\infty) = g(\infty)$ since $U(\infty) = \infty$. Since $\infty \in \mathbb{S}^{d-1}$ is the unique fixed point of U, it follows $g(\infty) = \infty$. As $M = \mathrm{Stab}_K(\infty)$, we get $g \in M$. Now $gU(0) = Ug(0) = U(0)$. As $U(0) = \mathbb{R}^k$, $g\mathbb{R}^k = \mathbb{R}^k$. Therefore, as $g \in M$, we also have $g\mathbb{R}^{d-1-k} = \mathbb{R}^{d-1-k}$, and consequently $g \in \mathrm{O}(k)\,\mathrm{O}(d-1-k)$. This shows that $NA\,\mathrm{SO}(k)\,\mathrm{SO}(d-1-k) \subset \mathrm{N}(U) \subset NA\,\mathrm{O}(k)\,\mathrm{O}(d-1-k)$. As $\mathrm{N}(U)$ is connected, this implies the claim. For the second claim, note first that $N\,\mathrm{C}_1(U) < \mathrm{C}(U)$. Now let $g \in \mathrm{C}(U)$. Since $\mathrm{C}(U) < \mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, we can write $g = ac_2nc_1 \in A\,\mathrm{C}_2(U)N\,\mathrm{C}_1(U)$. Since nc_1 commutes with U, it follows $ac_2 \in \mathrm{C}(U)$. Now the adjoint action of a on U is a dilation and the adjoint action of c_2 on U is a multiplication by an orthogonal matrix. Therefore we get $a = c_2 = e$, finishing the proof. □

Denote by $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G. Note that the product $AU^\perp\,\mathrm{C}_2(U)$ is a subgroup of G.

Lemma 2.2. *An unbounded one-parameter subsemigroup S of $AU^\perp\,\mathrm{C}_2(U)$ is either $\{\left(v\exp(t\xi_A)v^{-1}\right)\exp(t\xi_C) : t \ge 0\}$ or $\{\exp(t\xi_V)\exp(t\xi_C) : t \ge 0\}$ for some $\xi_A \in \mathrm{Lie}(A) - \{0\}, \xi_C \in \mathrm{Lie}(\mathrm{C}_2(U)), v \in U^\perp$, and $\xi_V \in \mathrm{Lie}(U^\perp) - \{0\}$.*

Proof. Let $\xi \in \mathrm{Lie}(AU^\perp\,\mathrm{C}_2(U))$ be such that $S = \{\exp(t\xi) : t \ge 0\}$. Write $\xi = \xi_0 + \xi_C$ where $\xi_0 \in \mathrm{Lie}(AU^\perp)$ and $\xi_C \in \mathrm{Lie}(\mathrm{C}_2(U))$. Since $AU^\perp$ commutes with $\mathrm{C}_2(U)$, $\exp(t\xi) = \exp(t\xi_0)\exp(t\xi_C)$ for any $t \in \mathbb{R}$. Hence we only need to show that either $\xi_0 \in \mathrm{Lie}(U^\perp)$ or

$$\{\exp(t\xi_0) : t \ge 0\} = \{v\exp(t\xi_A)v^{-1} : t \ge 0\} \tag{2.3}$$

for some $v \in U^\perp$ and $\xi_A \in \mathrm{Lie}(A)$. Now if $\xi_0 \notin \mathrm{Lie}(U^\perp)$, then writing $\xi_0 = \begin{pmatrix} a & x^t & 0 \\ 0 & 0_{d-1} & -x \\ 0 & 0 & -a \end{pmatrix} \in \mathrm{Lie}(AU^\perp)$ with $a \neq 0$, a direct computation shows that $\xi_0 = v\xi_A v^{-1}$ where $\log v = \begin{pmatrix} 0 & -x^t/a & 0 \\ 0 & 0_{d-1} & x/a \\ 0 & 0 & 0 \end{pmatrix}$ and $\xi_A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0_{d-1} & 0 \\ 0 & 0 & -a \end{pmatrix}$. □

A direct computation shows:

Lemma 2.3. *If $v_i \to \infty$ in $U^\perp$, then $\limsup_{i\to\infty} v_i A v_i^{-1}$ contains one-parameter subgroup of $U^\perp$.*

The complementary subspaces $\mathfrak{h}_U^\perp$ and $\mathfrak{h}^\perp$. If L is a reductive Lie subgroup of G with $\mathfrak{l} = \mathrm{Lie}(L)$, the restriction of the adjoint representation of G to L is completely reducible, and hence there exists an $\mathrm{Ad}(L)$-invariant complementary subspace $\mathfrak{l}^\perp$ so that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^\perp$. It follows from the inverse function theorem that the map $L \times \mathfrak{l}^\perp \to G$ given by $(g, X) \mapsto g \exp X$ is a local diffeomorphism onto an open neighborhood of e in G.

Let $U = U_k$. Denote by $\mathfrak{h}_U \subset \mathfrak{g}$ the Lie algebra of $H(U)$, by $\mathfrak{u}^\perp$ the subspace $\mathrm{Lie}(U^\perp)$, and by $(\mathfrak{u}^\perp)^t$ its transpose. Then $\mathfrak{h}_U^\perp$ can be given explicitly as follows:

$$\mathfrak{h}_U^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0 \tag{2.4}$$

where $\mathfrak{m}_0 = \left\{ m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & Z \end{pmatrix}, Z^t = -Z, Y \in \mathrm{M}_{k\times(d-1-k)}(\mathbb{R}) \right\}$.

Similarly, setting $\mathfrak{h} := \mathrm{Lie}(H'(U))$, $\mathfrak{h}^\perp$ is given by

$$\mathfrak{h}^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0' \tag{2.5}$$

where $\mathfrak{m}_0' := \left\{ m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & 0 \end{pmatrix} \right\}$.

By Lemma 2.1 and (2.5), we have:

Lemma 2.4. *If $r_i \to e$ in $\exp \mathfrak{h}^\perp - \mathrm{C}(H(U))$, then either $r_i \notin \mathrm{N}(U)$ for all i, or $r_i \notin \mathrm{N}(U^+)$ for all i, by passing to a subsequence.*

Definition 2.5. For a connected reductive subgroup $L < G$, denote by L_{nc} the maximal connected normal semisimple subgroup of L with no compact factors.

A connected reductive algebraic subgroup L of G is an almost direct product $L = L_{nc}CT$ where C is a connected semisimple compact normal subgroup of L and T is the central torus of L. If L contains a unipotent element, then L_{nc} is non-trivial, and simple, containing a conjugate of A, and the center of L is compact.

Proposition 2.6. *If $L < G$ is a connected reductive algebraic subgroup normalized by A and containing a unipotent element, then $L = H(U)C$ where $U < N$ is a non-trivial connected subgroup and C is a closed subgroup of $\mathrm{C}(H(U))$. In particular, L_{nc} and $\mathrm{N}(L_{nc})$ are equal to $H(U)$ and $H'(U)$ respectively.*

Proof. If L is normalized by A, then so is L_{nc}. Therefore it suffices to prove that a connected non-compact simple Lie subgroup $H < G$ normalized by A is of the form $H = H(U)$ where $U < N$ is a non-trivial connected subgroup.

First, consider the case when $A < H$. Let $\mathfrak{h}$ be the Lie algebra of H, and $\mathfrak{a}$ be the Lie algebra of A. Since $\mathfrak{h}$ is simple, its root space decomposition for the adjoint action of $\mathfrak{a}$ is of the form $\mathfrak{h} = \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^-$ where $\mathfrak{u}^\pm$ are the sum of all positive and negative root subspaces respectively and $\mathfrak{z}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$. Since the sum of all negative root subspaces for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is $\mathrm{Lie}(N^-)$, it follows that $U := \exp(\mathfrak{u}^-) < N^-$ and $H = H(U)$. Now for the general case, H contains a conjugate gAg^{-1} for some $g \in G$. Hence $g^{-1}Hg = H(U)$. Since $H(U)$ contains both A and $g^{-1}Ag$, they must be conjugate within $H(U)$, so $A = h^{-1}g^{-1}Agh$ for some $h \in H(U)$. Hence $gh \in \mathrm{N}_G(A) = AM$. Therefore $H = gH(U)g^{-1}$ is equal to $mH(U)m^{-1}$ for some $m \in M$. Since m normalizes N and $mH(U)m^{-1} = H(mUm^{-1})$, the claim follows. □

It is easy to deduce the following from the above proposition:

Corollary 2.7. *Any connected closed subgroup L of G generated by unipotent elements is conjugate to either U or $H(U)$ for some non-trivial connected subgroup $U < N$.*

Totally geodesic immersed planes. Let Γ be a discrete, torsion free, non-elementary, subgroup of G, and consider the associated hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^d = \Gamma\backslash G/K$. We refer to [25] for basic properties of hyperbolic manifolds. As in the introduction, we denote by Λ the limit set of Γ and by $\mathrm{core}(M)$ the convex core of M.

We denote by $\mathrm{F}\,M \simeq \Gamma\backslash G$ the bundle of all oriented orthonormal frames over M. We denote by $\pi : \Gamma\backslash G \to M = \Gamma\backslash G/K$ the base-point projection. By abuse of notation, we also denote by $\pi : G \to \mathbb{H}^d = G/K$ the base-point projection. For $g \in G$, $[g]$ denotes its image under the covering map $G \to \Gamma\backslash G$.

Fix $1 \le k \le d-2$ and let

$$H = \mathrm{SO}^\circ(k+1,1) \quad \text{and} \quad H' = \mathrm{SO}^\circ(k+1,1)\cdot \mathrm{SO}(d-k-1). \tag{2.6}$$

Let $C_0 := \mathbb{R}^k \cup \{\infty\}$ denote the unique oriented k-sphere in $\mathbb{S}^{d-1}$ stabilized by H'. Then $\tilde{S}_0 := \mathrm{hull}(C_0)$ is the unique oriented totally geodesic subspace of $\mathbb{H}^d$ stabilized by H', and $\partial\tilde{S}_0 = C_0$. We note that H' (resp. H) consists of all oriented frames $(v_0, \cdots, v_{d-1}) \in G$ (resp. $(v_0, \cdots, v_k, e_{k+1}, \cdots, e_{d-1}) \in G$) such that the $k+1$-tuple $(v_0, \cdots, v_k)$ is tangent to $\tilde{S}_0$, compatible with the orientation of $\tilde{S}_0$. The group G acts transitively on the space of all oriented k spheres in $\mathbb{S}^{d-1}$ giving rise to the isomorphisms of G/H' with $\mathcal{C}^k =$ the space of all oriented k-spheres in $\mathbb{S}^{d-1}$ as well as with the space of all oriented totally geodesic $(k+1)$-planes of $\mathbb{H}^d$.

We discuss the fundamental group of an immersed geodesic k-plane $S \subset M$. Choose a totally geodesic subspace $\tilde{S}$ of $\mathbb{H}^d$ which covers S. Then $\tilde{S} = g\tilde{S}_0$ for some $g \in G$, and the stabilizer of $\tilde{S}$ in G is equal to $gH'g^{-1}$. We have $\Gamma_{\tilde{S}} = \{\gamma \in \Gamma : \gamma\tilde{S} = \tilde{S}\} = \Gamma \cap gH'g^{-1}$ and get an immersion $\tilde{f} : \Gamma_{\tilde{S}}\backslash\tilde{S} \to M$ with image S. Consider the projection map

$$p : gH'g^{-1} \to gHg^{-1}. \tag{2.7}$$

Then p is injective on $\Gamma_{\tilde{S}}$ and $\Gamma_{\tilde{S}}\backslash\tilde{S} \simeq p(\Gamma_{\tilde{S}})\backslash\tilde{S}$ is an isomorphism, since $g\,\mathrm{C}(H)g^{-1}$ acts trivially on $\tilde{S}$. Hence $\tilde{f}$ gives an immersion

$$f : p(\Gamma_{\tilde{S}})\backslash\tilde{S} \to M \tag{2.8}$$

with image S. We say S properly immersed if f is a proper map.

The following proposition is standard:

Proposition 2.8. *Let $x \in \Gamma\backslash G$, and set $S := \pi(xH') \subset M$. Then*

1. *xH' is closed in $\Gamma\backslash G$ if and only if S is properly immersed in M.*
2. *If M is convex cocompact and S is properly immersed, then S is convex cocompact and $\partial\tilde{S} \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}}))$ for any geodesic subspace $\tilde{S} \subset \mathbb{H}^d$ which covers S.*

3. Thick return time

In this section, we study the closed $H(U)$-invariant subset $F_{H(U)} := \mathrm{RF}_+ M \cdot H(U)$ when $M = \Gamma\backslash\mathbb{H}^d$ is a convex cocompact manifold with Fuchsian ends. At the end of the section, we address the global thickness of the return time of any one-parameter subgroup of N to $\mathrm{RF}\, M$.

Definition 3.1. A convex cocompact hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^d$ is said to have non-empty *Fuchsian ends* if one of the following equivalent conditions holds:

(1) its convex core has non-empty interior and non-empty totally geodesic boundary.
(2) $\Omega := \mathbb{S}^{d-1} - \Lambda$ is a dense union $\bigcup_{i=1}^{\infty} B_i$ of infinitely many round balls with mutually disjoint closures.

In the whole section, let M be a convex cocompact hyperbolic manifold of non-empty Fuchsian ends.

Renormalized frame bundle. The renormalized frame bundle $\mathrm{RF}\, M \subset \mathrm{F}\, M$ is defined as the following AM-invariant subset

$$\mathrm{RF}\, M = \{[g] \in \Gamma\backslash G : g^{\pm} \in \Lambda\} = \{x \in \Gamma\backslash G : xA \text{ is bounded}\}.$$

Unless mentioned otherwise[2], we set $A^+ = \{a_t : t \geq 0\}$. We define

$$\mathrm{RF}_+ M = \{[g] \in \Gamma\backslash G : g^{+} \in \Lambda\} = \{x \in \Gamma\backslash G : xA^+ \text{ is bounded}\}$$

which is a closed NAM-invariant subset. As $\pi(xNA) = \pi(xG) = M$ for any $x \in \Gamma\backslash G$, we have $\pi(\mathrm{RF}_+ M) = M$.

It is easy to verify:

Lemma 3.2. *For $x \in \mathrm{RF}_+ M$, $\overline{xA^+}$ meets $\mathrm{RF}\, M$.*

$H(U)$-invariant subsets: $F_{H(U)}, F^*_{H(U)}, \partial F_{H(U)}$. Fix a non-trivial connected subgroup $U < N$, and consider the associated subgroups $H(U)$ and $H'(U)$ as defined in section 2.

We define

$$F_{H(U)} := \mathrm{RF}_+ M \cdot H(U). \tag{3.1}$$

We denote by $F^*_{H(U)}$ the interior of $F_{H(U)}$ and by $\partial F_{H(U)}$ the boundary of $F_{H(U)}$. When there is no room for confusion, we will omit the subscript $H(U)$ and simply write F, F^* and ∂F.

The closedness of F is an easy consequence of compactness of the limit set Λ. It is also $H'(U)$-invariant, since $\mathrm{RF}_+ M$ is M-invariant and $\mathrm{C}(H(U))$ is contained in M. For $g \in G$, we denote by $C_g = C_{gH(U)} \subset \mathbb{S}^{d-1}$ the sphere given by the boundary of the geodesic plane $\pi(gH(U))$. Then $\operatorname{hull} C_g = \pi(g(H(U))$, and $C_g = gH(U)^+ = gH(U)^-$ where $H(U)^{\pm} = \{h^{\pm} : h \in H(U)\}$. It follows that $F = \{[g] \in \Gamma\backslash G : C_g \cap \Lambda \neq \emptyset\}$.

Lemma 3.3. *We have $F = \{x \in \Gamma\backslash G : \pi(\overline{xH(U)}) \cap \operatorname{core} M \neq \emptyset\}$.*

Proof. Let $x \in F$. By modifying it using an element of $H(U)$, we may assume that $x \in \mathrm{RF}_+ M$. By Lemma 3.2, $\overline{xA^+}$ contains $x_0 \in \mathrm{RF}\, M$. Since $x_0 A$ is bounded, $\pi(x_0 A)$ is a bounded geodesic, and hence $\pi(x_0 A) \subset \pi(\overline{xH(U)}) \cap \operatorname{core} M$ because

[2] At certain places, we use notation A^+ for any subsemigroup of A

core M contains all bounded geodesics. This proves the inclusion $\subset$. Now suppose $x = [g] \notin F$. Then $C_g \cap \Lambda = \emptyset$, and hence C_g must be contained in a connected component, say B_i, of Ω. Hence $\pi(gH(U)) = \text{hull}(C_g)$ is contained in the interior of hull(B_i), which is disjoint from hull(Λ), by the convexity of B_i. Therefore the orbit $\Gamma\pi(gH(U))$ is a closed subset of $\mathbb{H}^d$, disjoint from hull(Λ). This proves $\supset$. $\square$

Denote by M^* the interior of the core of M. Then $F^* = \{x \in \Gamma\backslash G : \pi(xH(U)) \cap M^* \neq \emptyset\}$. Note that for $[g] \in F$, $\#C_g \cap \Lambda \geq 2$ and hence

$$F^* \subset \text{RF}\, M \cdot H(U). \tag{3.2}$$

In particular, $\text{RF}\, M \cdot H(U)$ is dense in F.

Lemma 3.4. *We have* $\text{RF}_+ M \cap F^* \subset \text{RF}\, M \cdot U$.

Proof. Let $y = [g] \in \text{RF}_+ M \cap F^*$. We need to show that $yU \cap \text{RF}\, M \neq \emptyset$. As $y \in \text{RF}_+ M$, $g^+ = g(\infty) \in \Lambda$, and hence $C_g \cap \Lambda \neq \emptyset$. If $\#C_g \cap \Lambda = 1$, then C_g must be contained in $\overline{B_i}$ for some i, which implies $[g] \notin F^*_{H(U)}$. Therefore $\#C_g \cap \Lambda \geq 2$. We note that $gU(0) \cup \{g(\infty)\} = C_g$; this is clear when $U = U_k$ for some $k \geq 1$ and $g = e$, to which a general case is reduced. Hence there exists $u \in U$ such that $gu(0) \in \Lambda$. Since $gu(\infty) = g(\infty) \in \Lambda$, we have $yu = [g]u \in \text{RF}\, M$. $\square$

We call an oriented frame $g = (v_0, \cdots, v_{d-1}) \in \text{F}\, M = G$ a boundary frame if the first $(d-1)$ vectors $v_0, \cdots, v_{d-2}$ are tangent to the boundary of core M. Set $\check{H} := H(U_{d-2}) = \text{SO}^\circ(d-1, 1)$, and denote by $\check{V}$ the one-dimensional subgroup $\mathbb{R}e_{d-1}$ of $N = \mathbb{R}^{d-1}$; note that $\check{V} = (\check{H} \cap N)^\perp$. We denote by BF M the set of all boundary frames of M; it is a union of compact $\check{H}$-orbits: $\text{BF}\, M = \bigcup_{i=1}^{k} z_i\check{H}$ such that $\pi(z_i\check{H}) = \Gamma\backslash\Gamma\,\text{hull}(B_i)$ for some component B_i of Ω.

The boundary ∂F for $U < \check{H} \cap N$. Suppose that U is contained in $\check{H} \cap N = \mathbb{R}^{d-2}$. Then there exists a one-parameter semigroup $\check{V}^+$ of $\check{V}$ such that $\partial F = \text{BF}\, M \cdot \check{V}^+ \cdot H'(U)$. We use the notation $\check{V}^- = \{v^{-1} : v \in \check{V}^+\}$. Note that

$$\partial F \cap \text{RF}\, M = \text{BF}\, M \cdot \text{C}(H(U));\ \partial F \cap \text{RF}_+ M = \text{BF}\, M \cdot \check{V}^+ \cdot \text{C}(H(U)). \tag{3.3}$$

For a general proper connected closed subgroup $U < N$, $mUm^{-1} \subset \check{H} \cap N$ for some $m \in M$, and $\partial F \cap \text{RF}\, M = \text{BF}\, Mm\, \text{C}(H(U))$ where BF Mm is now a union of finitely many $m^{-1}\check{H}m$-compact orbits.

Lemma 3.5. *Let $U < \check{H} \cap N$, $z \in \text{BF}\, M$ and $v \in \check{V} - \{e\}$. If $zv \in \text{RF}\, M$, then $zv \in F^*$.*

Proof. Let $z = [g] \in \text{BF}\, M$. Then $\partial(\pi(g\check{H})) = \partial B_j$ for some j. Let $v \in \check{V} - \{e\}$ be such that $zv \in \text{RF}\, M$. Suppose $zv \in \partial F_{H(U)}$. Then $C_{gv} \subset \overline{B_i}$ for some i. Since the sphere $C_{gv} = \{gvh(\infty) : h \in H(U)\}$ contains $g(\infty)$ which belongs to ∂B_j, we have $i = j$, as $\overline{B}_i$'s are mutually disjoint. As $zv \in \text{RF}\, M$, $C_{gv} \subset \partial B_j$. Hence $gvH(U)^+ \subset g\check{H}^+$. It follows that $gvH(U) \subset g\check{H}$, and hence $vH(U) \cap \check{H} \neq \emptyset$, which is a contradiction since $v \notin \check{H}$, and $H(U) \subset \check{H}$. $\square$

Properly immersed geodesic planes. Let $H = H(U_k)$ and $H' = H'(U_k)$ be as in (2.6), and p be the map in (2.7). In (2.8), if $p(\Gamma_{\tilde{S}})\backslash\tilde{S}$ is a convex cocompact hyperbolic k-manifold with Fuchsian ends and f is proper, then the image $S = \text{Im}(f)$ is referred to as a properly immersed convex cocompact geodesic k-plane of Fuchsian ends.

Proposition 3.6. *If xH' is closed for $x \in \mathrm{RF}\, M$, then $S = \pi(xH')$ is a properly immersed convex cocompact geodesic plane with (possibly empty) Fuchsian ends.*

Proof. Choose $g \in G$ so that $x = [g]$. Let $\tilde{S}$ and $\Gamma_{\tilde{S}}$ be as in Proposition 2.8. Set $C = \partial \tilde{S}$. By loc. cit., S is properly immersed, and $C \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}}))$. Write $C - (C \cap \Lambda) = \bigcup_{i \in I} (C \cap B_i)$ where I is the collection of all i such that $C \cap B_i \neq \emptyset$. If $C \cap \Lambda$ contains a non-empty open subset of C, then the limit set of $p(\Gamma_{\tilde{S}})$ has Hausdorff dimension equal to the dimension of C. So $p(\Gamma_{\tilde{S}})$ is a uniform lattice in gHg^{-1}, and hence S is compact. In the other case, I is an infinite set and $\bigcup_{i \in I} (C \cap B_i)$ is dense in C; so S is a a convex cocompact hyperbolic submanifold of Fuchsian ends by Definition 3.1(2). □

Lemma 3.7. *For a sphere $C \subset \mathbb{S}^{d-1}$ with $\# C \cap \Lambda \geq 2$, the intersection $C \cap \Lambda$ is Zariski dense in C.*

Proof. The claim is clear if $C \cap \Lambda$ contains a non-empty open subset of C. If not, $C \cap \Lambda$ contains infinitely many $C \cap \partial B_i$'s, each of which is an irreducible co-dimension one real subvariety of C. It follows that the Zariski closure of $C \cap \Lambda$ has dimension strictly greater than $\dim C - 1$, hence is equal to C. □

We let $\pi_1 : H' \to H$ and $\pi_2 : H' \to \mathrm{C}(H)$ denote the canonical projections.

Proposition 3.8. *Suppose that xH' is closed for $x = [g] \in \mathrm{RF}\, M$, and set $\Gamma' := g^{-1} \Gamma g \cap H'$. Then $\overline{xH} = xHC$ where $C = \overline{\pi_2(\Gamma')}$ and HC is equal to the identity component of the Zariski closure of Γ'.*

Proof. Without loss of generality, we may assume $g = e$. As H' is a direct product $H \times \mathrm{C}(H)$, we write an element of H' as (h, c) with $h \in H$ and $c \in \mathrm{C}(H)$. For all $\gamma \in \Gamma'$, $xH = [(e, e)]H = [(e, \pi_2(\gamma))]H = [(e, e)]H\pi_2(\gamma)$ and hence $xH = xH\pi_2(\Gamma')$. It follows that $xHC \subset \overline{xH}$. To show the other inclusion, let $(h_0, c_0) \in H\,\mathrm{C}(H)$ be arbitrary. If $[(h_0, c_0)] \in \overline{xH} = \overline{[(e, e)]H}$, then there exist sequences $\gamma_i \in \Gamma'$ and $h_i \in H$ such that $\gamma_i(h_i, e) \to (h_0, c_0)$ in H' as $i \to \infty$. In particular, $\pi_2(\gamma_i) \to c_0$ in $\mathrm{C}(H)$ as $i \to \infty$ and hence $c_0 \in C = \overline{\pi_2(\Gamma')}$. This proves $\overline{xH} = xHC$. Let W denote the identity component of the Zariski closure of Γ' in H'. Since any proper algebraic subgroup of G stabilizes either a point, or a proper sphere in $\mathbb{S}^{d-1}$, it follows from Proposition 2.8 and Lemma 3.7 that $\pi_1(\Gamma')$ is Zariski dense in H; so $\pi_1(W) = H$. So the quotient $W \backslash H'$ is compact. This implies that W contains a maximal real-split connected solvable subgroup, say, P of H'. Now $H \cap W$ is a normal subgroup of H, as $\pi_1(W) = H$. Since $P < H \cap W$ and H is simple, we conclude that $H \cap W = H$, i.e., $H < W$. Hence $W = H\pi_2(W)$. As any compact linear group is algebraic, C is algebraic and hence $C = \pi_2(W) = \overline{\pi_2(\Gamma')}$. Therefore $W = HC$, finishing the proof. □

Global thickness of the return time to RF M. We recall the various notions of thick subsets of $\mathbb{R}$, following [19] and [21].

Definition 3.9. Fix $k > 1$.

- A closed subset $\mathsf{T} \subset \mathbb{R}$ is locally k-thick at t if for any $\lambda > 0$, $\mathsf{T} \cap (t \pm [\lambda, k\lambda]) \neq \emptyset$.
- A closed subset $\mathsf{T} \subset \mathbb{R}$ is k-thick if T is locally k-thick at 0.
- A closed subset $\mathsf{T} \subset \mathbb{R}$ is k-thick at ∞ if $\mathsf{T} \cap (\pm[\lambda, k\lambda]) \neq \emptyset$ for all sufficiently large $\lambda \gg 1$.

- A closed subset $\mathsf{T} \subset \mathbb{R}$ is globally k-thick if $\mathsf{T} \neq \emptyset$ and T is locally k-thick at every $t \in \mathsf{T}$.

We will frequently use the fact that if T_i is a sequence of k-thick subsets, then $\limsup \mathsf{T}_i$ is also k-thick, and that if T is k-thick, so is $-\mathsf{T}$.

Proposition 3.10. *There exists a constant $k > 1$ depending only on the systole of the double of* $\operatorname{core} M$ *such that for any one-parameter subgroup $U = \{u_t : t \in \mathbb{R}\}$ of $N^{\pm}$, and any $y \in \operatorname{RF} M$, $\mathsf{T}(y) := \{t \in \mathbb{R} : yu_t \in \operatorname{RF} M\}$ is globally k-thick.*

Proof. Let $\eta > 0$ be the systole of the hyperbolic double of $\operatorname{core} M$, which is a closed hyperbolic manifold. Let $k > 1$ be given by the equation

$$d\big(\operatorname{hull}([-k,-1]),\ \operatorname{hull}([1,k])\big) = \eta/4 \tag{3.4}$$

where d is the hyperbolic distance in the upper half plane $\mathbb{H}^2$. Note that

$$\inf_{i \neq j} d(\operatorname{hull} B_i,\ \operatorname{hull} B_j) \geq \eta/2 \tag{3.5}$$

as the geodesic realizing this distance is either a closed geodesic or half of a closed geodesic in the double of $\operatorname{core} M$.

We first prove the case when $U < N$. Let $s \in \mathsf{T}(y)$ be arbitrary. To show that $\mathsf{T}(y)$ is locally k-thick at s, we may assume that $s = 0$, by replacing y with $yu_s \in \operatorname{RF} M$. We may also assume that $y = [g]$ where $g(\infty) = \infty$ and $g(0) = 0$. As $y \in \operatorname{RF} M$, this implies that $0, \infty \in \Lambda$. Since $gu_t(\infty) = g(\infty) \in \Lambda$, we have $\mathsf{T}(y) = \{t \in \mathbb{R} : gu_t(0) \in \Lambda\}$. Suppose that $\mathsf{T}(y)$ is not locally k-thick at 0. Then there exist $w \in U$ and $t > 0$ such that $([-kt,-t] \cdot w \cup [t,kt] \cdot w) \cap \Lambda = \emptyset$. Since each component of Ω is convex and $0 \notin \Omega$, it follows that $[-kt,-t] \cdot w$ and $[t,kt] \cdot w$ lie in distinct components of Ω, say B_i and B_j, $(i \neq j)$. But this yields

$$d_w(\operatorname{hull}([-kt,-t] \cdot w),\ \operatorname{hull}([t,kt] \cdot w)) \geq d(\operatorname{hull} B_i,\ \operatorname{hull} B_j) \geq \eta/2 \tag{3.6}$$

where d_w denotes the hyperbolic distance of the plane above the line $\mathbb{R}w$. Observe that the distance in (3.6) is independent of $w \in \mathbb{R}^{d-1}$ and $t > 0$, because both the dilation centered at 0 and the $(d-2)$-dimensional rotation with respect to the vertical axis above 0 are hyperbolic isometries. Therefore, we get a contradiction to (3.4). The case of $U < N^+$ is proved similarly, just replacing the role of g^+ and g^- in the above arguments. $\square$

4. Structure of singular sets

Let $\Gamma < G = \operatorname{SO}^\circ(d,1)$ be a convex cocompact torsion-free Zariski-dense subgroup. Let $U < G$ be a connected closed subgroup of G generated by unipotent elements in it. In this section, we define the singular set $\mathscr{S}(U)$ associated to U and study its structural property.

Definition 4.1 (Singular set)**.** We set

$$\mathscr{S}(U) = \left\{ x \in \Gamma\backslash G : \begin{array}{c} \text{there exists a proper connected} \\ \text{closed subgroup } W \supset U \text{ such that } xW \\ \text{is closed and } \operatorname{Stab}_W(x) \text{ is Zariski dense in } W. \end{array} \right\}.$$

Definition 4.2. We denote by $\mathscr{H}$ the collection of all *proper* connected closed subgroups $H < G$ containing a unipotent element such that $\Gamma\backslash\Gamma H$ is closed, and $H \cap \Gamma$ is Zariski dense in H.

Proposition 4.3. *Any $H \in \mathscr{H}$ is a reductive subgroup of G, and hence is of the form $gH(U)Cg^{-1}$ for some connected subgroup $U < N$, a closed subgroup $C < \mathrm{C}(H(U))$ and $g \in G$ such that $[g] \in \mathrm{RF}\, M$.*

Proof. Suppose $H \in \mathscr{H}$ is not reductive. Then its unipotent radical is non-trivial, which we can assume to be a subgroup U of N, up to a conjugation. Now we write $H = H_{nc}CTU$ where C is a connected semisimple compact subgroup and T is a torus centralizing $H_{nc}C$. As H is contained in $\mathrm{N}(U) = NA\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, which does not contain any non-compact simple Lie subgroup, it follows that H_{nc} is trivial. Now if T were compact, then $H \cap \Gamma$ would consist of parabolic elements, which is a contradiction as Γ is convex cocompact. Hence T is non-compact. Write $T = T_0S$ where S is a split torus and T_0 is compact. Then T_0 is equal to a conjugate of A, say, $g^{-1}Ag$ for some $g \in G$. As T_0 normalizes U, and $\mathrm{N}(U)$ fixes ∞, we deduce that $g(\infty)$ is either ∞ or 0. Since $\mathrm{Stab}_G(\infty) = NAM$, $g(\infty) = \infty$ implies $g \in NAM$, and $g(\infty) = 0$ implies $jg \in NAM$ where $j \in G$ is an element of order 2 such that $j(0) = \infty$. In either case, $T_0 = v^{-1}Av$ for some $v \in N$. By replacing H with vHv^{-1}, we may assume that $T_0 = A$. Since CS is a compact subgroup commuting with A, $CS \subset M$. Therefore H is of the form M_0AU where M_0 is a closed subgroup of $M \cap \mathrm{N}(U)$; note that we used the fact that v commutes with U. Now the commutator subgroup $[H, H]$ is equal to $[M_0, M_0]U$. Since $[H \cap \Gamma, H \cap \Gamma]$ must be Zariski dense in $[H, H]$, we deduce that Γ contains an element $m_0u \in M_0U$ with u non-trivial. Since m_0u is a parabolic element of Γ, this is a contradiction to the assumption that Γ is convex cocompact. This proves that H is reductive.

By Proposition 2.6, H is of the form $gH(U)Cg^{-1}$ for some $g \in G$ and $C < \mathrm{C}(H(U))$. For some $m \in M$ and $1 \le k \le d-2$, $H(U) = mH(U_k)m^{-1}$. Hence $\Gamma\backslash\Gamma gmH(U_k)C_0$ is closed where $C_0 = m^{-1}Cm$. By Proposition 2.8, the boundary of the geodesic plane $\pi(gmH(U_k))$ contains uncountably many points of Λ, since $(gm)H(U_k)C_0(gm)^{-1} \cap \Gamma$ is Zariski dense in $(gm)H(U_k)C_0(gm)^{-1}$. Using two such limit points, we can find an element $h \in H(U_k)$ such that $(gmh)^{\pm} \in \Lambda$. Since $(gmhm^{-1})^{\pm} = (gmh)^{\pm}$ and $mhm^{-1} \in H(U)$, it follows that $[g]H(U) \cap \mathrm{RF}\, M \neq \emptyset$, and hence we can take $[g] \in \mathrm{RF}\, M$ by modifying it with an element of $H(U)$ if necessary. This finishes the proof. □

Therefore, for each $H \in \mathscr{H}$, the non-compact semisimple part H_{nc} of H is well defined.

Proposition 4.4. *For any $H \in \mathscr{H}$, $H \cap \Gamma$ is finitely generated and $[\mathrm{N}_G(H_{nc}) \cap \Gamma : H \cap \Gamma] < \infty$.*

Proof. Let p denote the projection $\mathrm{N}_G(H_{nc}) \to H_{nc}$. Note that p is an injective map on $\mathrm{N}_G(H_{nc}) \cap \Gamma$, as Γ is torsion free and the kernel of p is a compact subgroup. It follows from Proposition 4.3 that H_{nc} is co-compact in $\mathrm{N}_G(H_{nc})$. Since $H \in \mathscr{H}$, the orbit $[e]H$ is closed and hence $[e]\,\mathrm{N}_G(H_{nc})$ is closed. It follows that both $p(H \cap \Gamma)$ and $p(\mathrm{N}_G(H_{nc}) \cap \Gamma)$ are convex cocompact Zariski dense subgroups of H_{nc} by Proposition 2.8. As any convex cocompact subgroup is finitely generated [3], $p(H \cap \Gamma)$ is finitely generated. Hence $H \cap \Gamma$ is finitely generated by the injectivity of $p|_{H \cap \Gamma}$. Since $p(H \cap \Gamma)$ is a normal subgroup of $p(\mathrm{N}_G(H_{nc}) \cap \Gamma)$, it follows that $p(H \cap \Gamma)$ has finite index in $p(\mathrm{N}_G(H_{nc}) \cap \Gamma)$ by Lemma 4.5 below. Since $p|_{\mathrm{N}_G(H_{nc}) \cap \Gamma}$ is injective, it follows that $H \cap \Gamma$ has finite index in $\mathrm{N}_G(H_{nc}) \cap \Gamma$. □

Lemma 4.5. *Let Γ_1 and Γ_2 be non-elementary convex cocompact subgroups of G. If Γ_2 is a normal subgroup of Γ_1, then $[\Gamma_1 : \Gamma_2] < \infty$.*

Proof. Let Λ_i be the limit set of Γ_i for $i = 1, 2$. Since $\Gamma_2 < \Gamma_1$, $\Lambda_2 \subset \Lambda_1$. As Γ_2 is normalized by Γ_1, Λ_2 is Γ_1-invariant. Since Γ_1 is non-elementary, Λ_1 is a minimal Γ_1-invariant closed subset. Hence $\Lambda_1 = \Lambda_2$. Let $M_i := \Gamma_i\backslash\mathbb{H}^d$. Then the convex core of M_1 is equal to $\Gamma_1\backslash \operatorname{hull}(\Lambda_2)$ and covered by $\operatorname{core} M_2 = \Gamma_2\backslash \operatorname{hull}(\Lambda_2)$. Since $\operatorname{core} M_2$ is compact, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. ☐

Definition 4.6 (Definition of $\mathscr{H}^\star$)**.**

$$\mathscr{H}^\star := \{\mathrm{N}_G(H_{nc}) : H \in \mathscr{H}\}. \tag{4.1}$$

Corollary 4.7 (Countability)**.** *The collection $\mathscr{H}$ is countable, and the map $H \to \mathrm{N}_G(H_{nc})$ defines a bijection between $\mathscr{H}$ and $\mathscr{H}^\star$.*

Proof. As Γ is convex cocompact, it is finitely generated. Therefore there are only countably many finitely generated subgroups of Γ. By Proposition 4.4, there are only countably many possible $H \cap \Gamma$ for $H \in \mathscr{H}$. Since H is determined by $H \cap \Gamma$, being its Zariski closure, the first claim follows.

Since $H \cap \Gamma$ has finite index in $\mathrm{N}_G(H_{nc}) \cap \Gamma$ by Proposition 4.4, H is determined as the identity component of the Zariski closure of $\mathrm{N}_G(H_{nc}) \cap \Gamma$. This proves the second claim. ☐

In the case of a convex cocompact hyperbolic manifold of Fuchsian ends, there is a one to one correspondence between $\mathscr{H}$ and the collection of all closed $H'(U)$-orbits of points in $\mathrm{RF}\, M$ for $U < N$: if $H \in \mathscr{H}$, then $H = gH(U)Cg^{-1}$ for some $U < N$ and $g \in G$ with $[g] \in \mathrm{RF}\, M$ and $[g]H'(U)$ is closed. Conversely, if $[g]H'(U)$ is closed for some $[g] \in \mathrm{RF}\, M$, then the identity component of the Zariski closure of $\Gamma \cap gH'(U)g^{-1}$ is given by $gH(U)Cg^{-1}$ for some closed subgroup $C < \mathrm{C}(H(U))$ by Proposition 3.8, and hence $gH(U)Cg^{-1} \in \mathscr{H}$. Moreover, since the normalizer of $H(U)C$ is contained in $H'(U)$, if $g_1H(U)Cg_1^{-1} = g_2H(U)Cg_2^{-1}$, then $g_2^{-1}g_1 \in H'(U)$, so $[g_1]H'(U) = [g_2]H'(U)$. Therefore Corollary 4.7 implies the following corollary by Propositions 2.8 and 3.8.

Corollary 4.8. *Let M be a convex cocompact hyperbolic manifold with Fuchsian ends. Then*

1. *there are only countably many properly immersed geodesic planes of dimension at least 2 intersecting* $\operatorname{core} M$.
2. *For each $1 \leq m \leq d-2$, there are only countably many spheres $S \subset \mathbb{S}^{d-1}$ of dimension m, such that $\#S \cap \Lambda \geq 2$ and ΓS is closed in the space $\mathcal{C}^m$.*

Remark 4.9. In (2), we may replace the condition $\#S \cap \Lambda \geq 2$ with $\#S \cap \Lambda \geq 1$, because if $\#S \cap \Lambda = 1$, then ΓS is not closed (see Remark 10.6).

For a subgroup $H < G$, define

$$X(H, U) := \{g \in G : gUg^{-1} \subset H\}. \tag{4.2}$$

Note that $X(H, U)$ is left-$\mathrm{N}_G(H)$ and right-$\mathrm{N}_G(U)$-invariant, and for any $g \in G$, $X(gHg^{-1}, U) = gX(H, U)$. For $H \in \mathscr{H}$ and any connected unipotent subgroup $U < G$, observe that $X(H, U) = X(H_{nc}, U) = X(\mathrm{N}_G(H_{nc}), U)$; this follows since any unipotent element of $\mathrm{N}_G(H_{nc})$ is contained in H_{nc}.

Proposition 4.10. *We have* $\mathscr{S}(U) = \bigcup_{H \in \mathscr{H}^\star} \Gamma\backslash\Gamma X(H, U)$.

Proof. If $x = [g] \in \mathscr{S}(U)$, then there exists a proper connected closed subgroup W of G containing U such that $[g]W$ is closed and $\mathrm{Stab}_W(x)$ is Zariski dense in W. This means $H := gWg^{-1} \in \mathscr{H}$ and $g \in X(H,U)$. Since $X(H,U) = X(\mathrm{N}_G(H_{nc}), U)$, and $\mathrm{N}_G(H_{nc}) \in \mathscr{H}^\star$, this proves the inclusion $\subset$. Conversely, let $g \in X(\mathrm{N}_G(H_{nc}), U)$ for some $H \in \mathscr{H}$. Set $W := g^{-1}Hg$. Then $U \subset W$, $[g]W = \Gamma H g$ is closed and $\mathrm{Stab}_W([g]) = g^{-1}(\Gamma \cap H)g$ is Zariski dense in W. Hence $[g] \in \mathscr{S}(U)$. $\square$

Singular subset of a closed orbit. Let $L < G$ be a connected reductive subgroup of G containing unipotent elements. For a closed orbit x_0L of $x_0 \in \mathrm{RF}\, M$, and a connected subgroup $U_0 < L \cap N$, we define the singular set $\mathscr{S}(U_0, x_0L)$ by the following:

$$\left\{ x \in x_0 L : \begin{array}{c} \text{there exists a connected closed subgroup } W < L, \\ \text{containing } U_0 \text{ such that } \dim W_{nc} < \dim L_{nc}, \\ xW \text{ is closed and } \mathrm{Stab}_W(x) \text{ is Zariski dense in } W \end{array} \right\}. \tag{4.3}$$

It follows from Proposition 4.10 and Proposition 4.3 that the subgroup W in the definition 4.1 is conjugate to $H(\widehat{U})C$ for some $\widehat{U} < N$. Hence W being a proper subgroup of G is same as requiring $\dim W_{nc} < \dim G$. Therefore $\mathscr{S}(U_0) = \mathscr{S}(U_0, \Gamma\backslash G)$ and $\mathscr{S}(U_0, x_0L) = x_0L \cap \bigcup \Gamma\backslash\Gamma X(H, U_0)$ where the union is taken over all subgroups $H \in \mathscr{H}^\star$ such that H is a subgroup of $g_0Lg_0^{-1}$ with $\dim H_{nc} < \dim L_{nc}$ and $x_0 = [g_0]$. Equivalently,

$$\mathscr{S}(U_0, x_0L) = \bigcup_{W \in \mathscr{H}^\star_{x_0L}} x_0(L \cap X(W, U_0)) \tag{4.4}$$

where $\mathscr{H}^\star_{x_0L}$ consists of all subgroups of the form $W = g_0^{-1}Hg_0 \cap L$ for some $H \in \mathscr{H}^\star$ and $\dim W_{nc} < \dim L_{nc}$. Then the generic set $\mathscr{G}(U_0, x_0L)$ is defined by

$$\mathscr{G}(U_0, x_0L) := (x_0L \cap \mathrm{RF}_+ M) - \mathscr{S}(U_0, x_0L). \tag{4.5}$$

Definition of $\mathcal{L}_U$ and $\mathcal{Q}_U$. Fix a non-trivial connected closed subgroup $U < N$. We define the collection $\mathcal{L}_U$ of all subgroups of the form $H(\widehat{U})C$ where $U < \widehat{U} < N$ and C is a closed subgroup of $\mathrm{C}(H(\widehat{U}))$ satisfying the following:

$$\mathcal{L}_U := \left\{ L = H(\widehat{U})C : \begin{array}{c} \text{for some } [g] \in \mathrm{RF}_+ M,\ [g]L \text{ is closed in } \Gamma\backslash G \\ \text{and } L \cap g^{-1}\Gamma g \text{ is Zariski dense in } L \end{array} \right\}. \tag{4.6}$$

Observe that for $L = H(\widehat{U})C \neq G$, the condition $L \in \mathcal{L}_U$ with $[g]L$ closed is equivalent to the condition that $gLg^{-1} \in \mathscr{H}$.

Lemma 4.11. *Let L_1 and L_2 be members of $\mathcal{L}_U$ such that xL_1 and xL_2 are closed for some $x \in \mathrm{RF}\, M$. If $(L_1)_{nc} = (L_2)_{nc}$, then $L_1 = L_2$.*

Proof. If L_1 or L_2 is equal to G, then the claim is trivial. Suppose that both L_1 and L_2 are proper subgroups of G. If $x = [g]$, then both subgroups $H_1 := gL_1g^{-1}$ and $H_2 := gL_2g^{-1}$ belong to $\mathscr{H}$. Since $(H_1)_{nc} = (H_2)_{nc}$, we have $H_1 = H_2$ by Corollary 4.7. Hence $L_1 = L_2$. $\square$

We also define

$$\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U, v \in \mathrm{N}(U)\}. \tag{4.7}$$

Since $\mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$ by Lemma 2.1, and the collection $\mathcal{L}_U$ is invariant under a conjugation by an element of $AU\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, we have

$$\mathcal{Q}_U = \{vLv^{-1} : L \in \mathcal{L}_U, v \in U^\perp\}. \tag{4.8}$$

Lemma 4.12. *For $U_0 < U < N$, we have $X(H(U), U_0) = \mathrm{N}_G(H(U))\,\mathrm{N}_G(U_0)$.*

Proof. Without loss of generality, we may assume $U = U_m$ and $U_0 = U_\ell$ with $1 \le \ell \le m \le d-1$. Set $H = H(U_m)$. If $m = d-1$, then $H = G$, and the statement is trivial. Assume $m \le d-2$ below. We will prove the inclusion $X(H, U_0) \subset \mathrm{N}_G(H)\,\mathrm{N}_G(U_0)$, as the other one is clear. Let $g \in X(H, U_0)$ be arbitrary. By multiplying g by an element of $\mathrm{N}_G(H)$ on the left as well as by an element of $\mathrm{N}_G(U_0)$ on the right, we will reduce g to an element of $\mathrm{N}_G(U_0)$, which implies the claim. In view of the Iwasawa decomposition $G = KAN$, since $AN < \mathrm{N}_G(U_0)$, we may assume that $g = k \in K$. As $k \in X(H, U_0)$, we have $kU_0k^{-1} \subset H$. Hence there exists $w \in K \cap H$ such that $kU_0k^{-1} = wU_0w^{-1}$. Since $w^{-1}kU_0 = U_0w^{-1}k$, we deduce $w^{-1}k(\infty) = U_0(w^{-1}k(\infty))$. Since $\infty \in \mathbb{S}^{d-1}$ is the unique fixed point of U_0, $w^{-1}k(\infty) = \infty$. Hence $w^{-1}k \in K \cap (MAN) = M$. Since $w \in H$, we may now assume that $k \in M$. From $kU_0 \subset Hk$, we get $kU_0(0) \subset Hk(0) = H(0)$ and hence $\langle ke_1, \cdots, ke_\ell\rangle \subset \langle e_1, \cdots, e_m\rangle$. By considering the action of $H \cap K$ on space of ℓ-tuples of orthonormal vectors in the subspace $\langle e_1, \cdots, e_m\rangle$, we may assume $ke_1 = e_1, \cdots, ke_{\ell-1} = e_{\ell-1}$, and $ke_\ell = \pm e_\ell$. This implies that $k \in \mathrm{C}_1(U_0)$, or $k\omega \in \mathrm{C}_1(U_0)$ where $\omega \in M$ is an involution which fixes all e_i, $i \ne \ell, \ell+1$ and $\omega(e_i) = -e_i$ for $i = \ell, \ell+1$. As $\mathrm{N}_G(U_0)$ contains $\mathrm{C}_1(U_0)$ and ω, the proof is complete. ☐

Proposition 4.13. *Consider a closed orbit x_0L for $L \in \mathcal{Q}_U$ and $x_0 \in \mathrm{RF}\,M$. If $x \in \mathscr{S}(U_0, x_0L)$ for a connected closed subgroup $U_0 < U$, then there exists a subgroup $Q \in \mathcal{Q}_{U_0}$ such that $\dim Q_{nc} < \dim L_{nc}$, xQ is closed and $\overline{xU_0} \subset xQ$.*

Proof. If $x = [g] \in \mathscr{S}(U_0, x_0L)$, then $g \in X(H, U_0)$ for some $H \in \mathscr{H}$ such that $\dim H_{nc} < \dim L_{nc}$. Then $\overline{xU_0} \subset x(g^{-1}Hg)$. By Proposition 4.3, $H = qH(\widehat{U})Cq^{-1}$ for some $U_0 < \widehat{U} < L \cap N$ and some $[q] \in \mathrm{RF}\,M$. Note that $q^{-1}g \in X(H(\widehat{U}), U_0)$. By Lemma 4.12, we have $q^{-1}g \in \mathrm{N}_G(H(\widehat{U}))\,\mathrm{N}_G(U_0)$. Hence $g^{-1}Hg = vH(\widehat{U})Cv^{-1}$ for some $v \in \mathrm{N}_G(U_0)$, and $\overline{xU_0} \subset xvH(\widehat{U})Cv^{-1}$. It suffices to set $Q := vH(\widehat{U})Cv^{-1}$. ☐

Lemma 4.14. *Let $L = H(\widehat{U})C$ for a connected closed subgroup $\widehat{U} < N$ and closed subgroup $C < \mathrm{C}(H(\widehat{U}))$. Let $W = g^{-1}H(\tilde{U})C_0g$ be a subgroup of L where $g \in L$, $\tilde{U}$ is a proper connected closed subgroup of $\widehat{U}$ and C_0 is a closed subgroup of $H(\tilde{U})$. Then for any non-trivial closed connected subgroup $U < \widehat{U}$, $(L \cap X(W, U))H(U)$ is a nowhere dense subset of L.*

Proof. Write $g = hc \in H(\widehat{U})C$. Note that

$$\begin{aligned} L \cap X(W, U) &= L \cap X(g^{-1}H(\tilde{U})g, U) = L \cap X(h^{-1}H(\tilde{U})h, U) \\ &= h(L \cap X(H(\tilde{U}), U)) = h(H(\widehat{U}) \cap X(H(\tilde{U}), U))C. \end{aligned}$$

Hence it suffices to show that $(H(\widehat{U}) \cap X(H(\tilde{U}), U))H(U)$ is a nowhere dense subset of $H(\widehat{U})$. Without loss of generality, we may now assume $H(\widehat{U}) = G$. We observe that using Lemma 4.12,

$$\begin{aligned} &X(H(\tilde{U}), U)H(U) = \mathrm{N}_G(H(\tilde{U}))\,\mathrm{N}_G(U)H(U) \\ &= H(\tilde{U})\,\mathrm{C}_1(\tilde{U})AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)H(U) = (K \cap H(\tilde{U}))U^{\perp}H'(U). \end{aligned}$$

Let $\dim \tilde{U} = m$ and $\dim U = k$. Then $1 \leq k \leq m < d-1 = \dim N$. Now, if we view the subset $(K \cap H(\tilde{U}))U^{\perp}H'(U)/H'(U)$ in the space $\mathcal{C}^k = G/H'(U)$, this set is contained in the set of all spheres $C \in \mathcal{C}^k$ which are tangent to the m-sphere given by $S_0 := (K \cap H(\tilde{U}))(\infty)$. Since $m < d-1$, it follows that $X(H(\tilde{U}), U)H(U)/H'(U)$ is a nowhere dense subset of $\mathcal{C}^k$, and hence $X(H(\tilde{U}), U)H(U)$ is a nowhere dense subset of G. ☐

Lemma 4.15. *Let $x_0\widehat{L}$ be a closed orbit of $\widehat{L} \in \mathcal{L}_U$ with $x_0 \in \mathrm{RF}\, M$. If U is a proper subgroup of $\widehat{L} \cap N$, then $\mathscr{S}(U, x_0\widehat{L}) \cdot H(U) \cap F$ is a proper subset of $x_0\widehat{L} \cap F$.*

Proof. Choose $g_0 \in G$ so that $x_0 = [g_0]$. Let $p : G \to \Gamma\backslash G$ be the canonical projection map. Then $p^{-1}(\mathscr{S}(U, x_0\widehat{L}) \cdot H(U))$ is a countable union $\gamma g_0(\widehat{L} \cap X(W, U))H(U)$ where $\gamma \in \Gamma$ and $W \in \mathscr{H}^{\star}_{x_0\widehat{L}}$ by (4.4). Hence by Lemma 4.14, $\mathscr{S}(U, x_0\widehat{L}) \cdot H(U)$ is a countable union of nowhere dense subsets of x_0L. Since $F^* \cap x_0\widehat{L}$ is an open subset of $x_0\widehat{L}$, it follows from the Baire category theorem that $F^* \cap x_0\widehat{L} \not\subset \mathscr{S}(U, x_0\widehat{L}) \cdot H(U)$. This proves the claim. ☐

The following geometric property of a convex cocompact hyperbolic manifold of Fuchsian ends is one of its key features which is needed in the proof of our main theorems stated in the introduction.

Proposition 4.16. *Let M be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $x_0\widehat{L}$ be a closed orbit of $\widehat{L} \in \mathcal{L}_U$ with $x_0 \in \mathrm{RF}\, M$ and with $\dim(\widehat{L} \cap N) \geq 2$. Either $x_0\widehat{L}$ is compact or $\mathscr{S}(U, x_0\widehat{L})$ contains a compact orbit zL_0 with $L_0 \in \mathcal{L}_U$.*

Proof. Write $\widehat{L} = H(\widehat{U})C$ for a connected closed subgroup $U < \widehat{U} < N$. Since $x_0\widehat{L}$ is closed, $\pi(x_0\widehat{L}) = \pi(x_0 H'(\widehat{U}))$ is a properly immersed convex cocompact geodeisc plane of dimension at least 3 with Fuchsian ends by Proposition 3.6. Suppose that x_0L is not compact. Then $\pi(x_0L)$ has non-empty Fuchian ends. This means that there exist a co-dimension one subgroup U_0 of $\widehat{U}$ and $z \in \widehat{L}$ such that $zH'(U_0)$ is compact and $\pi(zH'(U_0))$ is a component of the core of $\pi(x_0\widehat{L})$. By Proposition 3.8, there exists a closed subgroup $C_0 < \mathrm{C}(H(U_0)) \cap \widehat{L}$ such that $H(U_0)C_0 \in \mathcal{L}_{U_0}$ and $zH(U_0)C_0$ is compact. Let $m \in M \cap \widehat{L}$ be an element such that $U \subset m^{-1}U_0 m$. Then $zm(m^{-1}H(U_0)C_0 m)$ is a compact orbit contained in $\mathscr{S}(U, x_0\widehat{L})$ and $m^{-1}H(U_0)C_0 m \in \mathcal{L}_U$, finishing the proof. ☐

5. Inductive search lemma

In this section, we prove a combinatorial lemma 5.4, which we call an *inductive search lemma*, and use it to prove Proposition 5.3. This proposition will be used in the proof of the avoidance theorem 6.13 in the next section.

Definition 5.1. Let $J^* \subset I$ be a pair of open subsets of $\mathbb{R}$.

- The degree of (I, J^*) is defined to be the minimal integer $\delta \in \mathbb{N}$ such that for each connected component I° of I, the number of connected components of J^* contained in I° is bounded by δ.
- For $\beta > 0$, the pair (I, J^*) is said to be *β-regular* if for any connected component I° of I, and any component J° of $J^* \cap I^\circ$, $J^\circ \pm \beta \cdot |J^\circ| \subset I^\circ$ where $|J^\circ|$ denotes the length of J°.

Definition 5.2. Let $\mathcal{X}$ be a family of countably many triples (I, J^*, J') of open subsets of $\mathbb{R}$ such that $I \supset J^* \supset J'$.

- Given $\beta > 0$ and $\delta \in \mathbb{N}$, we say that $\mathcal{X}$ is *β-regular of degree δ* if for every triple $(I, J^*, J') \in \mathcal{X}$, the pair (I, J^*) is β-regular with degree at most δ.
- Given a subset $\mathsf{T} \subset \mathbb{R}$, we say that $\mathcal{X}$ is of *T-multiplicity free* if for any two distinct triples (I_1, J_1^*, J_1') and (I_2, J_2^*, J_2') of $\mathcal{X}$, we have $I_1 \cap J_2' \cap \mathsf{T} = \emptyset$.

For a family $\mathcal{X} = \{(I_\lambda, J_\lambda^*, J_\lambda') : \lambda \in \Lambda\}$, we will use the notation

$$I(\mathcal{X}) := \bigcup_{\lambda\in\Lambda} I_\lambda, \quad J^*(\mathcal{X}) := \bigcup_{\lambda\in\Lambda} J_\lambda^* \quad \text{and} \quad J'(\mathcal{X}) := \bigcup_{\lambda\in\Lambda} J_\lambda'.$$

The goal of this section is to prove:

Proposition 5.3 (Thickness of $\mathsf{T} - J'(\mathcal{X})$)**.** *Given $n, k, \delta \in \mathbb{N}$, there exists a positive number $\beta_0 = \beta_0(n, k, \delta)$ for which the following holds: let $\mathsf{T} \subset \mathbb{R}$ be a globally k-thick set, and let $\mathcal{X}_1, \cdots, \mathcal{X}_\ell$, $\ell \le n$, be β_0-regular families of degree δ and of T-multiplicity free. Let $\mathcal{X} = \bigcup_{i=1}^{\ell} \mathcal{X}_i$. If $0 \in \mathsf{T} - I(\mathcal{X})$, then $\mathsf{T} - J'(\mathcal{X})$ is a $2k$-thick set.*

The general case reduces to the case of $\delta = 1$, by replacing m by $m\delta$. Roughly speaking, the following lemma gives an inductive argument for the search of a sequence of t_i's which is almost geometric in a sense that the ratio $|t_i|/|t_{i-1}|$ is coarsely a constant and which lands on $\mathsf{T} - J'(\mathcal{X})$ in a time controlled by n.

Lemma 5.4 (Inductive search lemma)**.** *Let $k > 1$, $n \in \mathbb{N}$ and $0 < \varepsilon < 1$ be fixed. There exists $\beta = \beta(n, k, \varepsilon) > 0$ for which the following holds: Let $\mathsf{T} \subset \mathbb{R}$ be a globally k-thick set, and let $\mathcal{X}_1, \cdots, \mathcal{X}_n$ be β-regular families of countably many triples $(I_\lambda, J_\lambda^*, J_\lambda')$ with degree 1, and of T-multiplicity free. Set $\mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_n$, and assume $0 \notin I(\mathcal{X})$. For any $t \in \mathsf{T} \cap J'(\mathcal{X})$ and any $1 \le r \le n$, we can find distinct triples $(I_1, J_1^*, J_1'), \cdots, (I_{m-1}, J_{m-1}^*, J_{m-1}') \in \mathcal{X}$ with $2 \le m \le 2^r$, and a sequence of pivots $t = t_1 \in \mathsf{T} \cap J_1'$, $t_2 \in \mathsf{T} \cap J_2', \cdots, t_{m-1} \in \mathsf{T} \cap J_{m-1}'$, $t_m \in \mathsf{T}$ which satisfy the following conditions:*

1. *either $t_m \notin J'(\mathcal{X})$, or $t_m \in J_m'$ for some $(I_m, J_m^*, J_m') \in \mathcal{X}$, which is distinct from (I_i, J_i^*, J_i') for all $1 \le i \le m-1$, and the collection $\{(I_i, J_i^*, J_i') : 1 \le i \le m\}$ intersects at least $(r+1)$ number of $\mathcal{X}_i$'s;*
2. *for all $1 \le i \le j \le m$, $|t_i - t_j| \le 2((4k)^r - 1)k \max_{1\le p\le j-1} |J_p^*|$;*
3. *for each $1 \le i \le m$, $(1-\varepsilon)^{i-1}|t_1| \le |t_i| \le (1+\varepsilon)^{i-1}|t_1|$.*

In particular, for any $t \in \mathsf{T} \cap J'(\mathcal{X})$, there exists $t' \in \mathsf{T} - J'(\mathcal{X})$ such that

$$(1-\varepsilon)^{2^n-1}|t| \le |t'| \le (1+\varepsilon)^{2^n-1}|t|.$$

Proof. We set

$$\beta = \beta(n, k, \varepsilon) = (4k)^{n+1}\varepsilon^{-1}. \tag{5.1}$$

Consider the increasing sequence $Q(r) := (4k)^r - 1$ for $r \in \mathbb{N}$. Note that $Q(1) \ge 2$ and $Q(r+1) \ge 4Q(r)k + 1$. Moreover we check that $\beta > \max((Q(n) + 4Q(n-1))k, Q(n)k\varepsilon^{-1})$. We proceed by induction on r. First consider the case when $r = 1$. There exists $(I_1, J_1^*, J_1') \in \mathcal{X}$ such that $t_1 := t \in J_1' \cap \mathsf{T}$. As T is globally k-thick, we can choose

$$t_2 \in \big(t_1 \pm Q(1)(|J_1^*|, k|J_1^*|)\big) \cap \mathsf{T}. \tag{5.2}$$

We claim that t_1, t_2 is our desired sequence with $m = 2$. In the case when $t_2 \in J'(\mathcal{X})$, there exists $(I_2, J_2^*, J_2') \in \mathcal{X}$ such that $t_2 \in J_2'$. We check:

(1): If $t_2 \in J'(\mathcal{X})$, then $t_2 \in J_2' - J_1^*$ implies that J_1^* and J_2^* are distinct. Hence (I_1, J_1^*, J_1') and (I_2, J_2^*, J_2') are distinct as well. Since $\beta > Q(1)k$, by the β-regularity of (I_1, J_1^*), we have $t_2 \in I_1$. By the T-multiplicity free condition, (I_1, J_1^*, J_1') and (I_2, J_2^*, J_2') don't belong to the same family, that is, $\{(I_1, J_1^*, J_1'), (I_2, J_2^*, J_2')\}$ intersects two of $\mathcal{X}_i$'s.

(2): By (5.2), $|t_1 - t_2| < Q(1)k|J_1^*| = (4k-1)k|J_1^*|$.

(3): Note that $0 \notin I_1$, since $0 \notin I(\mathcal{X})$. By the β-regularity of (I_1, J_1^*), we have $t_1 \pm \beta|J_1^*| \subset I_1$. Since $0 \notin I_1$ and $\beta > \varepsilon^{-1}Q(1)k$, we have $|t_1| - \varepsilon^{-1}Q(1)k|J_1^*| > 0$. On the other hand, by (5.2), $|t_2 - t_1| \le Q(1)k|J_1^*| \le \varepsilon|t_1|$. In particular, $|t_2| \le |t_1| + |t_2 - t_1| < |t_1| + Q(1)k|J_1^*| < (1+\varepsilon)|t_1|$ and $|t_2| \ge |t_1| - |t_2 - t_1| > |t_1| - Q(1)k|J_1^*| > (1-\varepsilon)|t_1|$. This proves the base case of $r = 1$.

Next, assume the induction hypothesis for r. Hence we have a sequence $t_1(= t) \in J_1'$, $t_2 \in J_2'$, $\cdots$, $t_{m-1} \in J_{m-1}'$, and t_m in T with $m \le 2^r$ together with $\{(I_i, J_i^*, J_i') : 1 \le i \le m-1\}$ satisfying the three conditions listed in the lemma. If $t_m \notin J'(\mathcal{X})$, the same sequence would satisfy the hypothesis for $r+1$ and we are done. Now we assume that $t_m \in J_m'$ for some $(I_m, J_m^*, J_m') \in \mathcal{X}$, and that $\{(I_i, J_i^*, J_i') : 1 \le i \le m\}$ intersect at least $(r+1)$ numbers of $\mathcal{X}_i$'s. We may assume that they intersect exactly $(r+1)$-number of $\mathcal{X}_i$'s, which we may label as $\mathcal{X}_1, \cdots, \mathcal{X}_{r+1}$, since if they intersect more than $(r+1)$ of them, we are already done. Choose a largest interval J_ℓ^* among $J_1^*, \cdots, J_m^*$. Again using the global k-thickness of T, we can choose

$$s_1 \in \big(t_\ell \pm Q(r+1)(|J_\ell^*|, k|J_\ell^*|)\big) \cap \mathsf{T}. \tag{5.3}$$

First, consider the case when $s_1 \notin J'(\mathcal{X})$. We will show that the points $t_1, \cdots, t_m, s_1$ give the desired sequence. Indeed, the condition (1) is immediate. For (2), observe that by the induction hypothesis for r, we have $|s_1 - t_i| \le |s_1 - t_\ell| + |t_\ell - t_i| \le (Q(r+1)k + 2Q(r)k)|J_\ell^*|$ for all $1 \le i \le m$. The conclusion follows as $Q(r+1) > 2Q(r)$. To show (3), since $\beta > \varepsilon^{-1}Q(r+1)k$ and $0 \notin I_\ell$, by applying the β-regularity to the pair (I_ℓ, J_ℓ^*), we have $|t_\ell| - \varepsilon^{-1}Q(r+1)k|J_\ell^*| > 0$. It follows that

$$|s_1| \le |t_\ell| + |s_1 - t_\ell| < |t_\ell| + Q(r+1)k|J_\ell^*| < (1+\varepsilon)|t_\ell| \le (1+\varepsilon)^m |t_1|;$$
$$|s_1| \ge |t_\ell| - |s_1 - t_\ell| > |t_\ell| - Q(r+1)k|J_\ell^*| > (1-\varepsilon)|t_\ell| \ge (1-\varepsilon)^m |t_1|.$$

This proves (3).

For the rest of the proof, we now assume that $s_1 \in J'(\mathcal{X})$. Apply the induction hypothesis for r to $s_1 \in \mathsf{T} \cap J'(\mathcal{X})$ to obtain a sequence $\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') \in \mathcal{X} : 1 \le j \le m'-1\}$ with $m' \le 2^r$ and $s_1 \in \tilde{J}_1' \cap \mathsf{T}$, $s_2 \in \tilde{J}_2' \cap \mathsf{T}$, $\cdots$, $s_{m'-1} \in \tilde{J}_{m'-1}' \cap \mathsf{T}$, and $s_{m'} \in \mathsf{T}$. Set q_0 to be the smallest $1 \le q \le m'-1$ satisfying

$$\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le q\} \not\subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_{r+1} \tag{5.4}$$

if it exists, and $q_0 := m'$ otherwise. We claim that the sequence

$$t_1, \cdots, t_m, s_1, \cdots, s_{q_0} \tag{5.5}$$

of length $m + q_0 \le 2^{r+1}$ satisfies the conditions of the lemma for $r+1$.

Claim: We have

$$|J_\ell^*| = \max_{1 \le i \le m, 1 \le j \le q_0 - 1} (|J_i^*|, |\tilde{J}_j^*|). \tag{5.6}$$

Recall that $|J_\ell^*|$ was chosen to be maximal among $|J_1^*|, \cdots, |J_m^*|$. Hence, if the claim does not hold, then we can take j to be the least number such that $|\tilde{J}_j^*| > |J_\ell^*|$. Then by the induction hypothesis for (2),

$$\begin{aligned}|t_\ell - s_j| &\le |t_\ell - s_1| + |s_1 - s_j| \le Q(r+1)k|J_\ell^*| + 2Q(r)k \max_{1\le i\le j-1} |\tilde{J}_i^*| \\ &\le (Q(r+1) + 2Q(r))k|J_\ell^*|.\end{aligned}$$

Now the collection $\{(I_i, J_i^*, J_i') : 1 \le i \le m\}$ intersects $(r+1)$ families $\mathcal{X}_1, \cdots, \mathcal{X}_{r+1}$ and $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ belongs to one of these families, as $j \le q_0 - 1$. Hence there exists a triple (I_i, J_i^*, J_i') that belongs to the same family as $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$. Recall that the induction hypothesis for $t_1, \cdots, t_m$ gives us $|t_\ell - t_i| \le 2Q(r)k|J_\ell^*|$. Since $\beta > (Q(r+1) + 4Q(r))k$, we have

$$|t_i - s_j| \le |t_i - t_\ell| + |t_\ell - s_j| \le (Q(r+1) + 4Q(r))k|J_\ell^*| < \beta|\tilde{J}_j^*|.$$

Applying the β-regularity to the pair $(\tilde{I}_j, \tilde{J}_j^*)$, we conclude that $t_i \in \tilde{I}_j \cap J_i' \cap \mathsf{T}$. Since $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ and (I_i, J_i^*, J_i') belong to the same family which is T-multiplicity free, they are equal to each other. This is a contradiction since $|\tilde{J}_j^*| > |J_\ell^*| \ge |J_i^*|$, proving the claim (5.6).

We next prove that (I_i, J_i^*, J_i') and $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ are distinct for all $1 \le i \le m$ and $1 \le j \le q_0 - 1$. It suffices to check that J_i^* and $\tilde{J}_j^*$ are distinct. Note that we have $\max\limits_{1\le i,j\le m} |t_i - t_j| < 2Q(r)k|J_\ell^*|$ and $\max\limits_{1\le i,j\le q_0} |s_i - s_j| < 2Q(r)k|J_\ell^*|$ by the induction hypothesis together with claim (5.6). Now for $t_i \in J_i^*(1 \le i \le m)$ and $s_j \in \tilde{J}_j^*(1 \le j < q_0)$, we estimate:

$$\begin{aligned}|s_j - t_i| &\ge |s_1 - t_\ell| - |t_i - t_\ell| - |s_1 - s_j| \\ &> Q(r+1)|J_\ell^*| - 2Q(r)k|J_\ell^*| - 2Q(r)k|\tilde{J}_\ell^*| \\ &= (Q(r+1) - 4Q(r)k)|J_\ell^*| \ge |J_\ell^*|.\end{aligned} \tag{5.7}$$

This in particular means that $s_j \notin J_i^*$ and $t_i \notin \tilde{J}_j^*$. Hence $J_i^* \ne \tilde{J}_j^*$.

We now begin checking the conditions (1), (2) and (3).

(1): If $s_{q_0} \notin J'(\mathcal{X})$, there is nothing to check.

Now assume that $s_{q_0} \in \tilde{J}_{q_0}'$ for some $(\tilde{I}_{q_0}, \tilde{J}_{q_0}^*, \tilde{J}_{q_0}') \in \mathcal{X}$. If $q_0 < m'$, then again there is nothing to prove, as the union

$$\{(I_i, J_i^*, J_i') : 1 \le i \le m\} \cup \{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le q_0\} \tag{5.8}$$

intersects a family other than $\mathcal{X}_1, \cdots, \mathcal{X}_{r+1}$. Hence we will assume $q_0 = m'$. By the induction hypothesis for r on the sequence $(s_1, \cdots, s_{m'})$, the family $\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\}$ consists of pairwise distinct triples intersecting at least $(r+1)$ numbers of $\mathcal{X}_i$'s. Observe that in the estimate (5.7), there is no harm in allowing $j = q_0$ in addition to $j < q_0$. This shows that $\tilde{J}_{m'}^*$ is also distinct from all J_i^*'s. Hence the the triples in (5.8) are all distinct.

Now, unless the following inclusion

$$\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\} \subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_{r+1} \tag{5.9}$$

holds, we are done. Suppose that (5.9) holds. We will deduce a contradiction. Without loss of generality, we assume that $(I_\ell, J_\ell^*, J_\ell') \in \mathcal{X}_{r+1}$. We now claim that the following inclusion holds:

$$\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\} \subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_r. \tag{5.10}$$

Note that this gives the desired contradiction, since $\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\}$ must intersect at least $(r+1)$ number of $\mathcal{X}_i$ by the induction hypothesis. In order to prove the inclusion (5.10), suppose on the contrary that $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') \in \mathcal{X}_{r+1}$ for some $1 \le j \le m'$. Using $\beta > (Q(r+1) + 2Q(r))k$ and (5.6), we deduce $|t_\ell - s_j| \le |t_\ell - s_1| + |s_1 - s_j| \le Q(r+1)k|J_\ell^*| + 2Q(r)k|J_\ell^*| < \beta|J_\ell^*|$ where we used the induction hypothesis for the sequence $(s_1, \cdots, s_{m'})$ in the second line, to estimate the term $|s_1 - s_j|$.

Next, applying the β-regularity to the pair (I_ℓ, J_ℓ^*), we conclude that $s_j \in I_\ell$. Since $s_j \in \tilde{J}_j'$, it follows that $I_\ell \cap \tilde{J}_j' \cap \mathsf{T} \ne \emptyset$. This contradicts the condition that $\mathcal{X}_{r+1}$ is of T-multiplicity free, as both $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ and $(I_\ell, J_\ell^*, J_\ell')$ belong to the same family $\mathcal{X}_{r+1}$. This completes the proof of (1).

(2): For $1 \le i \le m$ and $1 \le j \le q_0$, observe that

$$\begin{aligned} |t_i - s_j| &\le |t_i - t_\ell| + |t_\ell - s_1| + |s_1 - s_j| \\ &\le 2Q(r)k|J_\ell^*| + Q(r+1)k|J_\ell^*| + 2Q(r)k|J_\ell^*| < 2Q(r+1)k|J_\ell^*| \end{aligned}$$

as $Q(r+1) > 4Q(r)$. Hence we get the desired result by (5.6).

(3): We already have observed that the inequality $\beta > \varepsilon^{-1}Q(r+1)k$ implies that $(1-\varepsilon)^m|t_1| \le |s_1| \le (1+\varepsilon)^m|t_1|$. Combining this with the induction hypothesis, we deduce that $(1-\varepsilon)^{m+i-1}|t_1| \le |s_i| \le (1+\varepsilon)^{m+i-1}|t_1|$ for all $1 \le i \le q_0$. Finally, the last statement of the lemma is obtained from the case $r = n$, since there are only n-number of $\mathcal{X}_i$'s; hence the second possibility of (1) cannot arise for $r = n$. □

Proof of Proposition 5.3. We may assume that $\mathcal{X}_i$'s are all of degree 1, by replacing each $\mathcal{X}_i$'s with δ-number of families associated to it.

We set $\beta_0(n, k, 1) = (4k)^{n+1}\varepsilon^{-1}$ where ε satisfies $\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2^n-1} \le 2$. Note that $\beta_0(n, k, 1)$ is equal to the number given in (5.1). We may assume $x = 0$ without loss of generality. Let $\lambda > 0$. We need to find a point

$$t' \in \Big([-2k\lambda, -\lambda] \cup [\lambda, 2k\lambda]\Big) \cap \Big(\mathsf{T} - \bigcup_{i \in \Lambda} J'(\mathcal{X}_i)\Big). \tag{5.11}$$

Choose $s > 0$ such that $(1-\varepsilon)^{-(2^n-1)}\lambda \le s \le 2(1+\varepsilon)^{-(2^n-1)}\lambda$. Since T is globally k-thick, there exists $t \in ([-ks, -s] \cup [s, ks]) \cap \mathsf{T}$. If $t \notin \bigcup_{i=1}^n J'(\mathcal{X}_i)$, then by choosing $t' = t$, we are done. Now suppose $t \in \bigcup_{i=1}^n J'(\mathcal{X}_i)$. Since $0 \notin \bigcup_{i=1}^n I(\mathcal{X}_i)$, by applying Lemma 5.4 to $t \in \mathsf{T} \cap (\bigcup_{i=1}^n J'(\mathcal{X}_i))$, we obtain $t' \in \mathsf{T} - \bigcup_{i=1}^n J'(\mathcal{X}_i)$ such that $(1-\varepsilon)^{2^n-1}|t| \le |t'| \le (1+\varepsilon)^{2^n-1}|t|$. Note that $|t'| \le (1+\varepsilon)^{2^n-1}|t| \le (1+\varepsilon)^{2^n-1}ks \le 2k\lambda$. Similarly, we have $|t'| \ge (1-\varepsilon)^{2^n-1}|t| \ge (1-\varepsilon)^{2^n-1}s \ge \lambda$. This completes the proof since t' satisfies (5.11).

6. Avoidance of the singular set

Let $\Gamma < G$ be a convex cocompact non-elementary subgroup and let $U = \{u_t\} < N$ be a one-parameter subgroup. Let $\mathscr{S}(U)$, $\mathscr{G}(U)$, $X(H, U)$, and $\mathscr{H}^\star$ be as defined in section 4. In particular, $\mathscr{S}(U)$ is a countable union: $\mathscr{S}(U) = \bigcup_{H \in \mathscr{H}^\star} \Gamma\backslash\Gamma X(H, U)$. The main goal of this section is to prove the avoidance Theorem 6.13 for any convex cocompact hyperbolic manifold with Fuchsian ends. For this, we extend the linearization method developed by Dani and Margulis [12] to our setting. Via a careful analysis of the graded self-intersections of the union $\bigcup_i \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$ for finitely many groups $H_i \in \mathscr{H}^\star$ and compact subsets $D_i \subset X(H_i, U)$, we construct families of triples of subsets of $\mathbb{R}$ satisfying the conditions of Proposition 5.3 relative to the global k-thick subset of the return time to RF M under U given in Proposition 3.10.

Linearization. Let $H \in \mathscr{H}^\star$. Then H is reductive, algebraic, and is equal to $\mathrm{N}_G(H)$ by Proposition 4.3. There exists an $\mathbb{R}$-regular representation $\rho_H : G \to \mathrm{GL}(V_H)$ with a point $p_H \in V_H$, such that $H = \mathrm{Stab}_G(p_H)$ and the orbit $p_H G$ is Zariski closed [2, Theorem 3.5]. Since $\Gamma\backslash\Gamma H$ is closed, it follows that $p_H\Gamma$ is a closed (and hence discrete) subset of V_H.

Let $\eta_H : G \to V_H$ denote the orbit map defined by $\eta_H(g) = p_H g$ for all $g \in G$. As H and U are algebraic subgroups, the set $X(H, U) = \{g \in G : gUg^{-1} \subset H\}$ is Zariski closed in G. Since $p_H G$ is Zariski closed in V_H, it follows that $A_H := p_H X(H, U)$ is Zariski closed in V_H and $X(H, U) = \eta_H^{-1}(A_H)$.

Following [15], for given $C > 0$ and $\alpha > 0$, a function $f : \mathbb{R} \to \mathbb{R}$ is called (C, α)-good if for any interval $I \subset \mathbb{R}$ and $\varepsilon > 0$, we have

$$\ell\{t \in I : |f(t)| \le \varepsilon\} \le C \cdot \left(\tfrac{\varepsilon}{\sup_{t\in I}|f(t)|}\right)^{\alpha} \cdot \ell(I)$$

where ℓ is a Lebesgue measure on $\mathbb{R}$.

Lemma 6.1. *For given $C > 1$ and $\alpha > 0$, consider functions $p_1, p_2, \cdots, p_k : \mathbb{R} \to \mathbb{R}$ satisfying the (C, α)-good property. For $0 < \delta < 1$, set*

$$I = \{t \in \mathbb{R} : \max_i |p_i(t)| < 1\} \quad \textit{and} \quad J(\delta) = \{t \in \mathbb{R} : \max_i |p_i(t)| < \delta\}.$$

For any $\beta > 1$, there exists $\delta = \delta(C, \alpha, \beta) > 0$ such that the pair $(I, J(\delta))$ is β-regular (see Def. 5.2).

Proof. We prove that the conclusion holds for $\delta := ((1 + \beta)C)^{-1/\alpha}$. First, note that the function $q(t) := \max_i |p_i(t)|$ also has the (C, α)-good property. Let $J' = (a, b)$ be a component of $J(\delta)$, and I' be the component of I containing J'. Note that I' is an open interval and $(a, \infty) \cap I' = (a, c)$ for some $b \le c \le \infty$. We claim

$$J' + \beta|J'| \subset (a, \infty) \cap I' \subset I'. \tag{6.1}$$

We may assume that $c < \infty$; otherwise the inclusion is trivial. We claim that $q(c) = 1$. Since $\{t \in \mathbb{R} : q(t) < 1\}$ is open and c is the boundary point of I', we have $q(c) \ge 1$. If $q(c)$ were strictly bigger than 1, since $\{t \in \mathbb{R} : q(t) > 1\}$ is open, I' would be disjoint from an open interval around c, which is impossible. Hence $q(c) = 1$. Now that $\sup\{q(t) : t \in (a, \infty) \cap I'\} = q(c) = 1$, by applying the (C, α)-good property of q on the interval $(a, \infty) \cap I'$, we get

$$\ell(J') \le \ell\{t \in (a, \infty) \cap I' : |q(t)| \le \delta\} \le C\delta^\alpha \cdot \ell((a, \infty) \cap I').$$

Now as $J' = (a, b)$ and $(a, \infty) \cap I'$ are nested intervals with one common endpoint, it follows from the equality $C\delta^\alpha = 1/(1 + \beta)$ that $J' + \beta|J'| \subset (a, \infty) \cap I' \subset I'$, proving (6.1). Similarly, applying the (C, α)-good property of q on $(-\infty, b) \cap I'$, we deduce that $J' - \beta|J'| \subset I'$. This proves that $(I, J(\delta))$ is β-regular. □

Proposition 6.2. *Let V be a finite dimensional real vector space, $\theta \in \mathbb{R}[V]$ be a polynomial and $A = \{v \in V : \theta(v) = 0\}$. Then for any compact subset $D \subset A$ and any $\beta > 0$, there exists a compact neighborhood $D' \subset A$ of D which has a*

β-regular size with respect to D in the following sense: for any neighborhood Φ of D', there exists a neighborhood $\Psi \subset \Phi$ of D such that for any $q \in V - \Phi$ and for any one-parameter unipotent subgroup $\{u_t\} \subset \mathrm{GL}(V)$, the pair $(I(q), J(q))$ is β-regular where $I(q) = \{t \in \mathbb{R} : qu_t \in \Phi\}$ and $J(q) = \{t \in \mathbb{R} : qu_t \in \Psi\}$. Furthermore, the degree of $(I(q), J(q))$ is at most $(\deg \theta + 2) \cdot \dim V$.

Proof. Choose a norm on V so that $\|\cdot\|^2$ is a polynomial function on V. Since D is compact, we can find $R > 0$ such that $D \subset \{v \in V : \|v\| < R\}$. Then we set $D' = \{v \in V : \theta(v) = 0, \|v\| < R/\sqrt{\delta}\}$, where $0 < \delta < 1$ is to be specified later. Note that if Φ is a neighborhood of D', there exists $0 < \eta < 1$ such that $\{v \in V : \theta(v) < \eta, \|v\| < (R + \eta)/\sqrt{\delta}\} \subset \Phi$. We set $\Psi := \{v \in V : \theta(v) < \eta\delta, \|v\| < (R + \eta)\}$ and $\tilde{I}(q) := \{t \in \mathbb{R} : \theta(qu_t) < \eta, \|qu_t\| < (R + \eta)/\sqrt{\delta}\}$. Since $\tilde{I}(q) \subset I(q)$ for $0 < \delta < 1$, it suffices to find δ (and hence D' and Ψ) so that the pair $(\tilde{I}(q), J(q))$ is β-regular. If we set $\psi_1(t) := \frac{\theta(qu_t)}{\eta}$ and $\psi_2(t) := \left(\frac{\|qu_t\|\sqrt{\delta}}{R+\eta}\right)^2$, then $\tilde{I}(q) = \{\max(\psi_1(t), \psi_2(t)) < 1\}$ and $J(q) = \{\max(\psi_1(t), \psi_2(t)) < \delta\}$. As ψ_1 and ψ_2 are polynomials, they have the (C, α)-property for an appropriate choice of C and α. Therefore by applying Lemma 6.1, by choosing δ small enough, we can make the pair $(\tilde{I}(q), J(q))$ β-regular for any $\beta > 0$. Note that the degrees of ψ_1 and ψ_2 are bounded by $\deg \theta \cdot \dim V$ and $2 \dim V$ respectively. Therefore $J(q)$ cannot have more than $(\deg \theta + 2) \cdot \dim V$ number of components. Hence the proof is complete. ☐

Definition 6.3. We define $\mathcal{E} = \mathcal{E}_U$ to be the collection of all compact subsets of $\mathscr{S}(U) \cap \mathrm{RF}\, M$ which can be written as

$$E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i D_i \cap \mathrm{RF}\, M \tag{6.2}$$

where $\{H_i \in \mathscr{H}^\star : i \in \Lambda\}$ is a finite collection and $D_i \subset X(H_i, U)$ is a compact subset. In this expression, we always use the minimal index set Λ for E. When E is of the form (6.2), we will say that E is associated to the family $\{H_i : i \in \Lambda\}$.

Remark 6.4. We note that E can also be expressed as $\bigcup_{i\in\Lambda} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$ where $H_i \in \mathscr{H}$ is a finite collection, and $D_i \subset X(H_i, U)$ is a compact subset which is left $\mathrm{C}(H_i)$-invariant.

Lemma 6.5. *In the expression (6.2) for $E \in \mathcal{E}$, the collection $\{H_i : i \in \Lambda\}$ is not redundant, in the sense that no $\gamma H_j \gamma^{-1}$ is equal to H_i for all triples $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$ except for the trivial cases of $i = j$ and $\gamma \in H_i$.*

Proof. Observe that if $\gamma H_j \gamma^{-1} = H_i$ for some $\gamma \in \Gamma$, then $\Gamma H_j D_j = \Gamma H_i \gamma D_j$, and hence by replacing D_i by $D_i \cup \gamma D_j \subset X(H_i, U)$, we may remove j from the index subset Λ. This contradicts the minimality of Λ. ☐

Note that for $D_i \subset X(H_i, U)$, and $\gamma \in \Gamma$, the intersection $H_1 D_1 \cap \gamma H_2 D_2$ only depends on the $(\Gamma \cap H_1, \Gamma \cap H_2)$-double coset of γ.

Proposition 6.6. *Let $H_1, H_2 \in \mathscr{H}^\star$. Then for any compact subset $D_i \subset X(H_i, U)$ for $i = 1, 2$ and a compact subset $K \subset \Gamma\backslash G$, there exists a finite set $\Delta \subset (H_1 \cap \Gamma)\backslash\Gamma/(H_2 \cap \Gamma)$ such that*

$$\left\{K \cap \Gamma\backslash\Gamma(H_1 D_1 \cap \gamma H_2 D_2)\right\}_{\gamma\in\Gamma} = \left\{K \cap \Gamma\backslash\Gamma(H_1 D_1 \cap \gamma H_2 D_2)\right\}_{\gamma\in\Delta}$$

where the latter set consists of distinct elements.

Moreover for each $\gamma \in \Delta$, there exists a compact subset $C_0 \subset H_1D_1 \cap \gamma H_2D_2 \subset X(H_1 \cap \gamma H_2\gamma^{-1}, U)$ such that $K \cap \Gamma\backslash\Gamma(H_1D_1 \cap \gamma H_2D_2) = \Gamma\backslash\Gamma C_0$.

Proof. For simplicity, write $\eta_{H_i} = \eta_i$ and $p_i = p_{H_i}$. Let $K_0 \subset G$ be a compact set such that $K = \Gamma\backslash\Gamma K_0$. We fix $\gamma \in \Gamma$, and define for any $\gamma' \in \Gamma$, $K_{\gamma'} = \{g \in K_0 : \gamma' g \in H_1D_1 \cap \gamma H_1D_2\}$. We check that $K \cap \Gamma\backslash\Gamma(H_1D_1 \cap \gamma H_2D_2) = \Gamma\backslash\Gamma\left(\cup_{\gamma' \in \Gamma} K_{\gamma'}\right)$. If this set is non-empty, then $K_{\gamma'} \neq \emptyset$ for some $\gamma' \in \Gamma$ and $p_1\gamma' g \in p_1D_1$, and $p_2\gamma^{-1}\gamma' g \in p_2D_2$ for some $g \in K_0$. In particular,

$$p_1\gamma' \in p_1DK_0^{-1}, \quad p_2\gamma^{-1} \in p_2DK_0^{-1}\gamma'^{-1}. \tag{6.3}$$

As $p_1\Gamma$ is discrete, and $p_1D_1K_0^{-1}$ is compact, the first condition of (6.3) implies that there exists a finite set $\Delta_0 \subset G$ such that $\gamma' \in (H_1 \cap \Gamma)\Delta_0$. Writing $\gamma' = h\delta_0$ where $h \in H_1 \cap \Gamma$, and $\delta_0 \in \Delta_0$, the second condition of (6.3) implies $p_2\gamma^{-1}h \in p_2D_2K_0^{-1}\delta_0^{-1}$. As $p_2D_2K_0^{-1}\Delta_0^{-1}$ is compact and $p_2\Gamma$ is discrete, there exists a finite set $\Delta \subset G$ such that $\gamma^{-1}h \in (H_2 \cap \Gamma)\Delta$. Hence, if $K \cap \Gamma\backslash\Gamma(H_1D_1 \cap \gamma H_2D_2) \neq \emptyset$, then $\gamma \in (H_1 \cap \Gamma)\Delta(H_2 \cap \Gamma)$. This completes the proof of the first claim. For the second claim, it suffices to set $C_0 := \bigcup_{\gamma' \in \Delta} K_{\gamma'}$. □

Proposition 6.7. *Let $H_1, H_2 \in \mathscr{H}^\star$ be such that $H_1 \cap H_2$ contains a unipotent element. Then there exists a unique smallest connected closed subgroup, say H_0, of $H_1 \cap H_2$ containing all unipotent elements of $H_1 \cap H_2$ such that $\Gamma\backslash\Gamma H_0$ is closed. Moreover, $H_0 \in \mathscr{H}$.*

Proof. The orbit $\Gamma\backslash\Gamma(H_1 \cap H_2)$ is closed [29, Lem. 2.2]. Hence such H_0 exists. We need to show that $\Gamma \cap H_0$ is Zariski dense in H_0. Let L be the subgroup of H_0 generated by all unipotent elements in H_0. Note that L is a normal subgroup of H_0 and hence $(H_0 \cap \Gamma)L$ is a subgroup of H_0. If F is the identity component of the closure of $(H_0 \cap \Gamma)L$, then $\Gamma\backslash\Gamma F$ is closed. By the minimality assumption on H_0, we have $F = H_0$. Hence $\overline{(H_0 \cap \Gamma)L} = H_0$; so $\overline{[e]L} = [e]H_0$. We can then apply [29, Cor. 2.12] and deduce the Zariski density of $H_0 \cap \Gamma$ in H_0. □

Corollary 6.8. *Let $H_1, H_2 \in \mathscr{H}^\star$ and $\gamma \in \Gamma$ be satisfying that $X(H_1 \cap \gamma H_2\gamma^{-1}, U) \neq \emptyset$. Then there exists a subgroup $H \in \mathscr{H}^\star$ contained in $H_1 \cap \gamma H_2\gamma^{-1}$ such that for any compact subsets $D_i \subset X(H_i, U)$, $i = 1, 2$, there exists a compact subset $D_0 \subset X(H, U)$ such that $K \cap \Gamma\backslash\Gamma(H_1D_1 \cap \gamma H_2D_2) = K \cap \Gamma\backslash\Gamma HD_0$.*

Proof. Let $F \in \mathscr{H}$ be given by Proposition 6.7 for the subgroup $H_1 \cap \gamma H_2\gamma^{-1}$. Set $H := \mathrm{N}_G(F_{nc}) \in \mathscr{H}^\star$. Note that $X(H_1 \cap \gamma H_2\gamma^{-1}, U) = X(H, U)$. Hence, by the second claim of Proposition 6.6, there exists a compact subset $D_0 \subset H_1D_1 \cap \gamma H_2D_2$ such that

$$K \cap \Gamma\backslash\Gamma(H_1D_1 \cap \gamma H_2D_2) = \Gamma\backslash\Gamma D_0. \tag{6.4}$$

We claim that $\Gamma\backslash\Gamma D_0 = K \cap \Gamma\backslash\Gamma HD_0$. The inclusion $\subset$ is clear. Let $g := hd \in HD_0$ with $h \in H$ and $d \in D_0$, and $[g] \in K$. Then by the condition on D_0, we have $g \in H_1D_1$ and $\gamma^{-1}g \in H_2D_2$. Therefore $g \in H_1D_1 \cap \gamma H_2D_2$. By (6.4), this proves the inclusion $\supset$. □

Definition 6.9 (Self-intersection operator on $\mathcal{E}_U$)**.** We define an operator $\mathsf{s} : \mathcal{E}_U \cup \{\emptyset\} \to \mathcal{E}_U \cup \{\emptyset\}$ as follows: we set $\mathsf{s}(\emptyset) = \emptyset$. For any

$$E = \bigcup_{i \in \Lambda} \Gamma\backslash\Gamma H_iD_i \cap \mathrm{RF}\, M \in \mathcal{E}_U, \tag{6.5}$$

we define

$$\mathsf{s}(E) := \bigcup_{i,j\in\Lambda} \bigcup_{\gamma_{ij}\in\Gamma} \Gamma\backslash\Gamma(H_iD_i \cap \gamma_{ij}H_jD_j) \cap \mathrm{RF}\, M$$

where $\gamma_{ij} \in \Gamma$ ranges over all elements of Γ satisfying $\dim(H_i \cap \gamma_{ij}H_j\gamma_{ij}^{-1})_{nc} < \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}$.

By Proposition 6.6 and Corollary 6.8, we have:

Corollary 6.10. (1) *For $E \in \mathcal{E}_U$, we have $\mathsf{s}(E) \in \mathcal{E}_U$.*
(2) *For $E_1, E_2 \in \mathcal{E}_U$, we have $E_1 \cap E_2 \in \mathcal{E}_U$.*

Hence for $E \in \mathcal{E}_U$ as in (6.5), $\mathsf{s}(E)$ is of the form $\mathsf{s}(E) = \bigcup_{i\in\Lambda'} \Gamma\backslash\Gamma H_iD_i \cap \mathrm{RF}\, M$ where Λ' is a (minimal) finite index set, $H_i \in \mathscr{H}$ with $X(H_i, U) \neq \emptyset$ and $\max\{\dim(H_i)_{nc} : i \in \Lambda'\} < \max\{\dim(H_i)_{nc} : i \in \Lambda\}$. Hence, s maps $\mathcal{E}_U$ to $\mathcal{E}_U \cup \{\emptyset\}$ and for any $E \in \mathcal{E}_U$, $\mathsf{s}^{\dim G}(E) = \emptyset$.

Definition 6.11. For a compact subset $K \subset \Gamma\backslash G$ and $E \in \mathcal{E}_U$, we say that K does not have any self-intersection point of E, or simply say that K is *E-self intersection-free*, if $K \cap \mathsf{s}(E) = \emptyset$.

Proposition 6.12. *Let $E = \bigcup_{i\in\Lambda} \Gamma\backslash\Gamma H_iD_i \cap \mathrm{RF}\, M \in \mathcal{E}$ where $D_i \subset X(H_i, U)$ is a compact subset and Λ is a finite subset. Let $K \subset \mathrm{RF}\, M$ be a compact subset which is E-self intersection-free. Then there exists a collection of open neighborhoods Ω_i of D_i, $i \in \Lambda$, such that for $\mathcal{O} := \bigcup_{i\in\Lambda} \Gamma\backslash\Gamma H_i\Omega_i$, the compact subset K is $\mathcal{O}$-self intersection free, in the sense that, if $\dim H_i = \dim H_j$ and $K \cap \Gamma\backslash\Gamma(H_i\Omega_i \cap \gamma H_j\Omega_j) \neq \emptyset$ for some $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$, then $i = j$ and $\gamma \in H_i \cap \Gamma$.*

Proof. For each $k \in \mathbb{N}$ and $i \in \Lambda$, let $\Omega_i(k)$ be the $1/k$-neighborhood of the compact subset D_i. Since Λ is finite, if the proposition does not hold, by passing to a subsequence, there exist $i, j \in \Lambda$ with $\dim H_i = \dim H_j$ and a sequence $\gamma_k \in \Gamma$ such that $K \cap \Gamma\backslash\Gamma(H_i\Omega_i(k) \cap \gamma_kH_j\Omega_j(k)) \neq \emptyset$ and

$$(i, j, \gamma_k) \notin \{(i, i, \gamma) : i \in \Lambda, \gamma \in H_i \cap \Gamma\}. \tag{6.6}$$

Hence there exist $g_k = h_kw_k \in H_i\Omega_i(k)$ and $g'_k = h'_kw'_k \in H_j\Omega_j(k)$ such that $g_k = \gamma_kg'_k$ where $[g_k] \in K$. Now as $k \to \infty$, we have $w_k \to w \in D_i$ and $w'_k \to w' \in D_j$. There exists $\delta_k \in \Gamma$ such that $\delta_kg_k \in \tilde{K}$ where $\tilde{K}$ is a compact subset of G such that $K = \Gamma\backslash\Gamma\tilde{K}$, so the sequence δ_kg_k converges to g_0 as $k \to \infty$. Since ΓH_i and ΓH_j are closed, we have $\delta_kh_k \to \delta_0h_i$ and $\delta_k\gamma_kh'_k \to \delta'_0h_j$ where $\delta_0, \delta'_0 \in \Gamma$, $h_i \in H_i$ and $h_j \in H_j$. As $\Gamma[H_i]$ and $\Gamma[H_j]$ are discrete in the spaces G/H_i and G/H_j respectively, we have

$$\delta_0^{-1}\delta_k \in H_i \quad \text{and} \quad (\delta'_0)^{-1}\delta_k\gamma_k \in H_j \tag{6.7}$$

for all sufficiently large k. Therefore $g_0 = \delta_0h_iw = \delta'_0h_jw' \in \delta_0(H_iD_i \cap \delta_0^{-1}\delta'_0H_jD_j)$ and $[g_0] \in K$. Hence $K \cap \Gamma\backslash\Gamma(H_iD_i \cap \delta_0^{-1}\delta'_0H_jD_j) \neq \emptyset$. Set $\delta := \delta_0^{-1}\delta'_0 \in \Gamma$. Since $K \cap \mathsf{s}(E) = \emptyset$, this implies that $\mathrm{RF}\, M \cap \Gamma\backslash\Gamma(H_iD_i \cap \delta H_jD_j) \not\subset \mathsf{s}(E)$. By the definition of $\mathsf{s}(E)$, $\dim(H_i \cap \delta H_j\delta^{-1})_{nc} = \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}$. Since $H_i = \mathrm{N}_G(H_i) = \mathrm{N}_G((H_i)_{nc})$, and similarly for H_j, we have $H_i \cap \delta H_j\delta^{-1}$ is either H_i or $\delta H_j\delta^{-1}$. Since $\dim H_i = \dim H_j$, $\delta H_j\delta^{-1} = H_i$ or $H_i = \delta H_j\delta^{-1}$. By Lemma 6.5, this implies that $i = j$ and $\delta \in \mathrm{N}_G(H_i) \cap \Gamma$. It follows from (6.7) that $\gamma_k \in \mathrm{N}_G(H_i) \cap \Gamma = H_i \cap \Gamma$ for all large k. This is a contradiction to (6.6), completing the proof. ∎

In the rest of this section, we assume that $M = \Gamma\backslash\mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends, and let k be as given by Proposition 3.10.

Theorem 6.13 (Avoidance theorem I)**.** *Let $U = \{u_t\} < N$ be a one-parameter subgroup. For any $E \in \mathcal{E}_U$, there exists $E' \in \mathcal{E}_U$ such that the following holds: If $F \subset \mathrm{RF}\, M$ is a compact set disjoint from E', then there exists a neighborhood $\mathcal{O}^\diamond$ of E such that for all $x \in F$, the set $\{t \in \mathbb{R} : xu_t \in \mathrm{RF}\, M - \mathcal{O}^\diamond\}$ is $2k$-thick. Moreover, if E is associated to $\{H_i : i \in \Lambda\}$, then E' is also associated to the same family $\{H_i : i \in \Lambda\}$ in the sense of Definition 6.3.*

Proof. ♠1. **The constant** β_0**:** We write $\mathscr{H}^\star = \{H_i\}$. For simplicity, set $V_i = V_{H_i}$ and $p_i = p_{H_i}$. Let θ_i be the defining polynomial of the algebraic variety A_{H_i}. Set $m := \dim(G)^2$ and $\delta := \max_{H_i \in \mathscr{H}^\star} (\deg \theta_i + 2) \dim V_i$.

Note that if H_i is conjugate to H_j, then θ_i and θ_j have same degree and $\dim V_i = \dim V_j$. Since there are only finitely many conjugacy classes in $\mathscr{H}^\star$ by Proposition 4.3, the constant δ is finite. Now let $\beta_0 := \beta_0(m\delta, k, 1) = (4k)^{m\delta+1}\varepsilon^{-1}$ be given as in Proposition 5.3 where $\varepsilon = \varepsilon_{m\delta}$ satisfies $\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2^{m\delta}-1} \le 2$.

♠2. **Definition of E_n and E'_n:** We write $E = \bigcup_{i\in\Lambda_0} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$ for some finite minimal set Λ_0. Set $\ell := \max_{i\in\Lambda_0} \dim(H_i)_{nc}$. We define $E_n, E'_n \in \mathcal{E}_U$ for all $1 \le n \le \ell$ inductively as follows: set $E_\ell := E$ and $\Lambda_\ell := \Lambda_0$. For each $i \in \Lambda_\ell$, let D'_i be a compact subset of $X(H_i, U)$ containing D_i such that $p_i D'_i$ has a β_0-regular size with respect to $p_i D_i$ as in Proposition 6.2. Set $E'_\ell := \bigcup_{i\in\Lambda_\ell} \Gamma\backslash\Gamma H_i D'_i \cap \mathrm{RF}\, M$. Suppose that $E_{n+1}, E'_{n+1} \in \mathcal{E}_U$ are given for $n \ge 1$. Then, define $E_n := E \cap \mathsf{s}(E'_{n+1})$. Then by Corollary 6.10, E_n belongs to $\mathcal{E}_U$ and hence can be written as $E_n = \bigcup_{i\in\Lambda_n} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$ where D_i is a compact subset of $X(H_i, U)$, so that Λ_n is a minimal index set. For each $i \in \Lambda_n$, let D'_i be a compact subset of $X(H_i, U)$ containing D_i such that $p_i D'_i$ has a β_0-regular size with respect to $p_i D_i$ as in Proposition 6.2. Set $E'_n := \bigcup_{i\in\Lambda_n} \Gamma\backslash\Gamma H_i D'_i \cap \mathrm{RF}\, M$. Hence we get a sequence of compact (possibly empty) subsets of E: $E_1, E_2, \cdots, E_{\ell-1}, E_\ell = E$, and a sequence of compact sets $E'_1, E'_2, \cdots, E'_{\ell-1}, E'_\ell = E'$. Note that $\mathsf{s}(E_1) = \mathsf{s}(E'_1) = \emptyset$ by the dimension reason.[3]

♠3. **Outline of the plan:** Let $F \subset \mathrm{RF}\, M$ be a compact set disjoint from E'. For $x \in F$, we set $\mathsf{T}(x) := \{t \in \mathbb{R} : xu_t \in \mathrm{RF}\, M\}$ which is a globally k-thick set by Proposition 3.10. We will construct

- a neighborhood $\mathcal{O}'$ of E' disjoint from F, and
- a neighborhood $\mathcal{O}^\diamond$ of E

such that for any $x \in \mathrm{RF}\, M - \mathcal{O}'$, we have $\{t \in \mathbb{R} : xu_t \in \mathrm{RF}\, M - \mathcal{O}^\diamond\} \supset \mathsf{T}(x) - J'(\mathcal{X})$ where $\mathcal{X} = \mathcal{X}(x)$ is the union of at most m-number of β_0-regular families $\mathcal{X}_i$ of triples $(I(q), J^*(q), J'(q))$ of subsets of $\mathbb{R}$ with degree δ and of $\mathsf{T}(x)$-multiplicity free. Once we do that, the theorem is a consequence of Proposition 5.3. Construction of such $\mathcal{O}'$ and $\mathcal{O}^\diamond$ requires an inductive process on E_n's.

♠4. **Inductive construction of K_n, $\mathcal{O}'_{n+1}$, $\mathcal{O}_{n+1}$, and $\mathcal{O}^\star_{n+1}$:** Let $K_0 := \mathrm{RF}\, M$. For each $i \in \Lambda_1$, there exists a neighborhood Ω'_i of D'_i such that for $\mathcal{O}'_1 := \bigcup_{i\in\Lambda_1} \Gamma\backslash\Gamma H_i \Omega'_i$, the compact subset K_0 is $\mathcal{O}'_1$-self intersection free by Lemma 6.12, since $\mathsf{s}(E'_1) = \emptyset$. By Proposition 6.2, there exists a neighborhood Ω_i of D_i such

[3] In fact $E_{\ell-i} = \emptyset$ for all $i \ge d-1$, but we won't use this information

that the pair $(I(q), J(q))$ is β_0-regular for all $q \in V_i - p_i\Omega_i'$ where

$$I(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i'\} \quad \text{and} \quad J(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i\}. \tag{6.8}$$

Set $\mathcal{O}_1 := \bigcup_{i\in\Lambda_1} \Gamma\backslash\Gamma H_i\Omega_i$. Since $E_1 = \bigcup_{i\in\Lambda_1} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$, $\mathcal{O}_1$ is a neighborhood of $E_1 = \mathsf{s}(E_2') \cap E$. Now the compact subset $\mathsf{s}(E_2') - \mathcal{O}_1$ is contained in $\mathsf{s}(E_2') - E$, which is relatively open in $\mathsf{s}(E_2')$. Therefore we can take a neighborhood $\mathcal{O}_1^\star$ of $\mathsf{s}(E_2') - \mathcal{O}_1$ so that $\overline{\mathcal{O}_1^\star} \cap E = \emptyset$.

We will now define the following quadruple $K_n, \mathcal{O}_{n+1}', \mathcal{O}_{n+1}$ and $\mathcal{O}_{n+1}^\star$ for each $1 \le n \le \ell - 1$ inductively:

- a compact subset $K_n = K_{n-1} - (\mathcal{O}_n \cup \mathcal{O}_n^\star) \subset \mathrm{RF}\, M$,
- a neighborhood $\mathcal{O}_{n+1}'$ of E_{n+1}',
- a neighborhood $\mathcal{O}_{n+1}$ of E_{n+1} and
- a neighborhood $\mathcal{O}_{n+1}^\star$ of $\mathsf{s}(E_{n+2}') - \mathcal{O}_{n+1}$ such that $\overline{\mathcal{O}_{n+1}^\star} \cap E = \emptyset$.

Assume that the sets K_{n-1}, $\mathcal{O}_n'$, $\mathcal{O}_n$ and $\mathcal{O}_n^\star$ are defined. We define $K_n := K_{n-1} - (\mathcal{O}_n \cup \mathcal{O}_n^\star) = \mathrm{RF}\, M - \bigcup_{i=1}^n (\mathcal{O}_i \cup \mathcal{O}_i^\star)$. For each $i \in \Lambda_{n+1}$, let Ω_i' be a neighborhood of D_i' in G such that for $\mathcal{O}_{n+1}' := \bigcup_{i\in\Lambda_{n+1}} \Gamma\backslash\Gamma H_i\Omega_i'$, K_n is $\mathcal{O}_{n+1}'$-self intersection free. Since $\mathcal{O}_n \cup \mathcal{O}_n^\star$ is a neighborhood of $\mathsf{s}(E_{n+1}')$, which is the set of all self-intersection points of E_{n+1}', such collection of Ω_i', $i \in \Lambda_{n+1}$ exists by Lemma 6.12.

Since $F \subset \mathrm{RF}\, M$ is compact and disjoint from E', we can also assume $\Gamma\backslash\Gamma H_i\Omega_i'$ is disjoint from F, by shrinking Ω_i' if necessary. More precisely, writing $F = \Gamma\backslash\Gamma\tilde{F}$ for some compact subset $\tilde{F} \subset G$, this can be achieved by choosing a neighborhood Ω_i' of D_i' so that $p_i\Omega_i'$ is disjoint from $p_i\Gamma\tilde{F}$; and this is possible since $p_i\Gamma\tilde{F}$ is a closed set disjoint from a compact subset p_iD_i'. After choosing Ω_i' for each $i \in \Lambda_{n+1}$, define the following neighborhood of E_{n+1}': $\mathcal{O}_{n+1}' := \bigcup_{i\in\Lambda_{n+1}} \Gamma\backslash\Gamma H_i\Omega_i'$.

We will next define $\mathcal{O}_{n+1}$. By Lemma 6.2, there exists a neighborhood Ω_i of D_i such that the pair $(I(q), J(q))$ is β_0-regular for all $q \in V_i - p_i\Omega_i'$ where $I(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i'\}$ and $J(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i\}$. We then define the following neighborhood of $E_{n+1} = \mathsf{s}(E_{n+2}') \cap E$: $\mathcal{O}_{n+1} := \bigcup_{i\in\Lambda_{n+1}} \Gamma\backslash\Gamma H_i\Omega_i$. Since the compact subset $\mathsf{s}(E_{n+2}') - \mathcal{O}_{n+1}$ is contained in the set $\mathsf{s}(E_{n+2}') - E$, which is relatively open inside $\mathsf{s}(E_{n+2}')$, we can take a neighborhood $\mathcal{O}_{n+1}^\star$ of $\mathsf{s}(E_{n+2}') - \mathcal{O}_{n+1}$ so that $\overline{\mathcal{O}_{n+1}^\star} \cap E = \emptyset$. This finishes the inductive construction.

♠5. **Definition of $\mathcal{O}'$ and $\mathcal{O}^\diamond$:** We define: $\mathcal{O}' := \bigcup_{n=1}^\ell \mathcal{O}_n'$, $\quad \mathcal{O} := \bigcup_{n=1}^\ell \mathcal{O}_n \quad \mathcal{O}^\star := \bigcup_{n=1}^\ell \overline{\mathcal{O}_n^\star}$. Note that $\mathcal{O}'$ and $\mathcal{O}$ are neighborhoods of E' and E respectively. Since $E \cap \mathcal{O}^\star = \emptyset$, the following defines a neighborhood of E:

$$\mathcal{O}^\diamond := \mathcal{O} - \mathcal{O}^\star. \tag{6.9}$$

♠6. **Construction of β_0-regular families of $\mathsf{T}(x)$-multiplicity free:**

Fix $x \in F \subset \mathrm{RF}\, M - \mathcal{O}'$. Choose a representative $g \in G$ of x. We write each Λ_n as the disjoint union $\Lambda_n = \bigcup_{j\in\theta_n} \Lambda_{n,j}$ where $\Lambda_{n,j} = \{i \in \Lambda_n : \dim H_i = j\}$ and $\theta_n = \{j : \Lambda_{n,j} \ne \emptyset\}$. Note that $\#\theta_n < \dim G$.

Fix $1 \le n \le \ell$, $j \in \theta_n$ and $i \in \Lambda_{n,j}$. For each $q \in p_i\Gamma g$, we define $I(q) := \{t : qu_t \in p_i\Omega_i'\}$ and $J(q) := \{t : qu_t \in p_i\Omega_i\}$. In general, $I(q)$'s have high multiplicity among q's in $\bigcup_{i\in\Lambda_{n,j}} p_i\Gamma g$, but the following subset $I'(q)$'s will be multiplicity-free, and this is is why we defined K_{n-1} as carefully as above:

- $I'(q) := \{t : \text{for some } a \ge 0,\ [t, t+a] \subset I(q) \text{ and } xu_{t+a} \in K_{n-1}\}$;
- $J^*(q) := I'(q) \cap J(q)$;
- $J'(q) := \{t \in J(q) : xu_t \in K_{n-1}\}$.

Observe that $I'(q)$ and $J^*(q)$ are unions of finitely many intervals, $J'(q) \subset \mathsf{T}(x)$ and that $J'(q) \subset J^*(q) \subset I'(q)$. Now, for each $1 \le n \le \ell$ and $j \in \theta_n$, define the family

$$\mathcal{X}_{n,j} = \{(I(q), J^*(q), J'(q))) : q \in \bigcup_{i \in \Lambda_{n,j}} p_i \Gamma g\}. \tag{6.10}$$

We claim that each $\mathcal{X}_{n,j}$ is a β_0-regular family with degree at most δ and $\mathsf{T}(x)$-multiplicity free. Note for each $q \in p_i\Gamma g$, the number of connected components of $J^*(q)$ is less than or equal to that of $J(q)$. Now that $J^*(q) \subset J(q)$ and all the pairs $(I(q), J(q))$ are β_0-regular pairs of degree at most δ, it follows that $\mathcal{X}_{n,j}$'s are β_0-regular families with degree at most δ.

We now claim that $\mathcal{X}_{n,j}$ has $\mathsf{T}(x)$-multiplicity free, that is, for any distinct indices $q_1, q_2 \in \bigcup_{i\in\Lambda_{n,j}} p_i\Gamma g$ of $\mathcal{X}_{n,j}$, $I(q_1) \cap J'(q_2) = \emptyset$. We first show that $I'(q_1) \cap I'(q_2) = \emptyset$. Suppose not. Then there exists $t \in I'(q_1) \cap I'(q_2)$ for some $q_1 = p_i\gamma_1 g$ and $q_2 = p_k\gamma_2 g$, where $i, k \in \Lambda_{n,j}$. Then for some $a \ge 0$, we have $[t, t+a] \subset I(q_1) \cap I(q_2)$ and $xu_{t+a} \in K_{n-1}$. In particular, $xu_{t+a} \in \Gamma\backslash\Gamma(\gamma_1^{-1}H_i\Omega_i' \cap \gamma_2^{-1}H_k\Omega_k') \cap K_{n-1}$. Since K_{n-1} is $\mathcal{O}_n'$-self intersection free, and $\dim H_i = \dim H_k = j$, we deduce from Proposition 6.12 that this may happen only when $i = k$, and $\gamma_1\gamma_2^{-1} \in H_i \cap \Gamma$. Hence we have $q_1 = q_2$. This shows that $I'(q)$'s are pairwise disjoint. Now suppose that there exists an element $t \in I(q_1) \cap J'(q_2)$. Then by the disjointness of $I'(q_1)$ and $I'(q_2)$, it follows that $t \in (I(q_1) - I'(q_1)) \cap J'(q_2)$. By the definition of $I'(q_1)$, we have $xu_t \notin K_{n-1}$. This contradicts the assumption that $t \in J'(q_2)$.

♠7. **Completing the proof:** Let $\mathcal{X} := \bigcup_{1\le i\le\ell, j\in\theta_n} \mathcal{X}_{n,j}$. In view of Proposition 5.3, it remains to check that the condition $t \in \mathsf{T}(x) - J'(\mathcal{X})$ implies that $xu_t \notin \mathcal{O}^\diamond$ where $\mathcal{O}^\diamond$ is given in (6.9). Suppose that there exists $t \in \mathsf{T}(x) - J'(\mathcal{X})$ such that $xu_t \in \mathcal{O}^\diamond$. Write the neighborhood $\mathcal{O}^\diamond$ as the disjoint union $\mathcal{O}^\diamond = \bigcup_{n=1}^{\ell} (\mathcal{O}_n - (\cup_{i\le n-1}\mathcal{O}_i \cup \mathcal{O}^\star))$. Let $n \le \ell$ be such that $xu_t \in \mathcal{O}_n - (\bigcup_{i=1}^{n-1} \mathcal{O}_i \cup \mathcal{O}^\star)$. Since $t \in \mathsf{T}(x) - J'(\mathcal{X})$, we have $xu_t \in \mathrm{RF}\, M - K_{n-1}$. Since $K_{n-1} = \mathrm{RF}\, M - \bigcup_{i=1}^{n-1} (\mathcal{O}_i \cup \mathcal{O}_i^\star)$, $xu_t \in \bigcup_{i=1}^{n-1} \mathcal{O}_i \cup \mathcal{O}_i^\star$. This is a contradiction, since $\bigcup_{i=1}^{\ell} \mathcal{O}_i^\star \subset \mathcal{O}^\star$. ☐

As $\mathscr{H}^\star$ is countable and $X(H_i, U)$ is σ-compact, the intersection $\mathscr{S}(U) \cap \mathrm{RF}\, M$ can be exhausted by the union of the increasing sequence of $E_j \in \mathcal{E}_U$'s. Therefore, we deduce:

Corollary 6.14. *There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ with $\mathscr{S}(U) \cap \mathrm{RF}\, M = \bigcup_{j=1}^{\infty} E_j$ which satisfies the following: Let $x_i \in \mathrm{RF}\, M$ be a sequence converging to $x \in \mathscr{G}(U) \cap \mathrm{RF}\, M$. Then for each $j \in \mathbb{N}$, there exist a neighborhood $\mathcal{O}_j$ of E_j and $i_j \ge 1$ such that $\{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\, M - \mathcal{O}_j\}$ is $2k$-thick for all $i \ge i_j$.*

Proof. For each $j \ge 1$, we may assume $E_{j+1} \supset E_j'$ where E_j' is given by Theorem 6.13. For each $j \ge 1$, there exists $i_j \in \mathbb{N}$ such that $x_i \notin E_{j+1}$ for all $i \ge i_j$. Applying Proposition 6.13 to a compact subset $F = \{x_i : i \ge i_j\}$ of $\mathrm{RF}\, M$, we obtain a neighborhood $\mathcal{O}_j$ of E_j such that $\{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\, M - \mathcal{O}_j\}$ is $2k$-thick for all $i \ge i_j$. ☐

Indeed we will apply Corollary 6.14 for the sequence $\{x_i\}$ contained in a closed orbit x_0L of a proper connected closed subgroup $L < G$, which can be proved in the same way:

Theorem 6.15 (Avoidance Theorem II)**.** *Consider a closed orbit x_0L for some $x_0 \in \mathrm{RF}\, M$ and $L \in \mathcal{Q}_U$. There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ with $\mathscr{S}(U, x_0L) \cap \mathrm{RF}\, M = \bigcup_{j=1}^{\infty} E_j$, which satisfies the following: if $x_i \to x$ in $\mathrm{RF}\, M \cap x_0L$ with $x \in \mathscr{G}(U, x_0L)$, then for each $j \in \mathbb{N}$, there exist $i_j \geq 1$ and an open neighborhood $\mathcal{O}_j \subset x_0L$ of E_j such that $\{t \in \mathbb{R} : x_iu_t \in \mathrm{RF}\, M - \mathcal{O}_j\}$ is a $2k$-thick set for all $i \geq i_j$.*

7. Limits of RF M-points in F^* and generic points

In the rest of paper, let $M = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Recall that $\Lambda \subset \mathbb{S}^{d-1}$ denotes the limit set of Γ. In this section, we collect some geometric lemmas which are needed in modifying a sequence limiting on an RF M point (resp. limiting on a point in $\mathrm{RF}\, M \cap \mathscr{G}(U)$) to a sequence of RF M-points (resp. whose limit still remains inside $\mathscr{G}(U)$).

Lemma 7.1. *Let $C_n \to C$ be a sequence of convergent circles in $\mathbb{S}^{d-1}$. If $C \not\subset \overline{B}$ for any component B of Ω, then $\# \limsup_{n\to\infty} C_n \cap \Lambda \geq 2$.*

Proof. Without loss of generality, we may assume that $\infty \notin \Lambda$ and hence consider Λ as a subset of the Euclidean space $\mathbb{R}^{d-1}$. Note that there is one component, say, B_1 of Ω which contains ∞ and all other components of Ω are contained in the complement of B_1, which is a (bounded) round ball in $\mathbb{R}^{d-1}$. It follows that there are only finitely many components of Ω whose diameters are bounded from below by a fixed positive number; this follows from the fact that ΓB is closed for each component B of Ω, and that there are only finitely many Γ-orbits of components of Ω.

Let $\delta = \mathrm{diam}(C)/2$ so that we may assume $\mathrm{diam}(C_n) > \delta$ for all sufficiently large $n \gg 1$. It suffices to show that there exists $\varepsilon_0 > 0$ such that $C_n \cap \Lambda$ contains ξ_n, ξ_n' with $d(\xi_n, \xi_n') \geq \varepsilon_0$ for all sufficiently large n. Suppose not. Then for any $\varepsilon > 0$, there exists an interval $I_n \subset C_n$ such that $\mathrm{diam}(I_n) \leq \varepsilon$ and $C_n - I_n \subset \Omega$ for some infinite sequence of n's. Since $C_n - I_n$ is connected, there exists a component B_n of Ω such that $C_n \subset \mathcal{N}_\varepsilon(B_n)$, where $\mathcal{N}_\varepsilon(B_n)$ denotes the ε-neighborhood of B_n. In particular, we have $\mathrm{diam}(B_n) + \varepsilon > \delta$. Taking ε smaller than 0.5δ, this means that $\mathrm{diam}(B_n) > \delta/2$. On the other hand, there are only finitely many components of Ω whose diameters are greater than 0.5δ, say $B_1, \cdots, B_\ell$. Let $\varepsilon_0 > 0$ be such that $\mathcal{N}_{\varepsilon_0}(B_1), \cdots, \mathcal{N}_{\varepsilon_0}(B_\ell)$ are all disjoint. Then by passing to a subsequence, there exists B_i such that $C_n \subset \mathcal{N}_\varepsilon(B_i)$ for all small $0 < \varepsilon < \varepsilon_0$ and $n \geq 1$; hence $C \subset \mathcal{N}_\varepsilon(B_i)$. Since this holds for all sufficiently small $\varepsilon > 0$, we get that $C \subset \overline{B_i}$, yielding a contradiction. ☐

In the next two lemmas, we set $U^- = U$ and $U^+ = U^t$.

Lemma 7.2. *Let $U < N$ be a connected closed subgroup. Let $[g]L$ be a closed orbit for some $L \in \mathcal{L}_U$ and $[g] \in \mathrm{RF}\, M$. Let S_0 and S^* denote the boundaries of $\pi(gH(U))$ and $\pi(gL)$ respectively. If S is a sphere such that $S_0 \subset S \subsetneq S^*$ and ΓS is closed, then $[g] \in \mathscr{S}(U^{\pm}, [g]L)$.*

Proof. Write $L = H(\widehat{U})C \in \mathcal{L}_U$. Since $S_0 \subset S \subsetneq S^*$, there exists a connected proper subgroup $\tilde{U}$ of $\widehat{U}$, containing U such that S is the boundary of $\pi(gH(\tilde{U}))$. Since ΓS is closed, $[g]H'(\tilde{U})$ is closed by Proposition 2.8. Now the claim follows from Proposition 3.8 and the definition of $\mathscr{S}(U^{\pm}, [g]L)$. ☐

Lemma 7.3. *Let $U < N$ be a connected closed subgroup with dimension $m \geq 1$, and let $U_{\pm}^{(1)}, \cdots, U_{\pm}^{(m)}$ be one-parameter subgroups generating $U^{\pm}$. Consider a closed orbit yL where $L \in \mathcal{L}_U$ and $y \in F_{H(U)}^* \cap \mathrm{RF}\, M \cap \bigcap_{i=1}^m \mathscr{G}(U_{\pm}^{(i)}, yL)$. If $x_n \to y$ in yL, then, by passing to a subsequence, there exists a sequence $h_n \to h$ in $H(U)$ so that $x_n h_n \in \mathrm{RF}\, M \cap yL$ and $yh \in \mathrm{RF}\, M \cap \bigcap_{i=1}^m \mathscr{G}(U_{\pm}^{(i)}, yL)$.*

Proof. Let S^* denote the boundary of $\pi(g_0 L)$. Let $\mathcal{Q}$ be the collection of all spheres $S \subsetneq S^*$ such that $S \cap \Lambda \neq \emptyset$ and ΓS is closed in $\mathcal{C}^{\dim S}$. By Corollary 4.8 and Remark 4.9, $\mathcal{Q}$ is countable. Choose a sequence $g_n \to g_0$ in G as $n \to \infty$, so that $x_n = [g_n]$ and $y = [g_0]$. Let S_n and S_0 denote the boundaries of $\pi(g_n H(U))$ and $\pi(g_0 H(U))$ respectively so that $S_n \to S_0$ in $\mathcal{C}^m$ as $n \to \infty$. We will choose a circle $C_0 \subset S_0$ and a sequence of circles $C_n \subset S_n$ so that $C_n \to C_0$ and $\limsup (C_n \cap \Lambda)$ contains two distinct points outside of $\cup_{S \in \mathcal{Q}} S$. If $m = 1$, we set $C_0 = S_0$. When $m \geq 2$, we choose a circle $C_0 \subset S_0$ as follows. Note that S_0 is not contained in any sphere in $\mathcal{Q}$ by the assumption on y and Lemma 7.2. Hence for any $S \in \mathcal{Q}$, $S_0 \cap S$ is a proper sub-sphere of S_0. Since $y \in F_{H(U)}^*$, for any component B_i of Ω, $S_0 \not\subset \overline{B_i}$ and hence $S_0 \cap \partial B_i$ is a proper sub-sphere of S_0. Choose a circle $C_0 \subset S_0$ such that $\{g_0^+, g_0^-\} \subset C_0 \cap \Lambda$, $C_0 \not\subset S$ for any $S \in \mathcal{Q}$, and $C_0 \not\subset \partial B_i \cap S_0$ for all i. This is possible, since $\mathcal{Q}$ is countable. Since $S_n \to S_0$, we can find a sequence of circles $C_n \subset S_n$ such that $C_n \to C_0$. We claim that $\limsup_n (C_n \cap \Lambda)$ is uncountable. Since $\# C_0 \cap \Lambda \geq 2$ and $C_0 \not\subset \partial B_i$, $C_0 \not\subset \overline{B_i}$ for all i. Therefore, by Lemma 7.1, for any infinite subsequence C_{n_k} of C_n, $\# \limsup_k (C_{n_k} \cap \Lambda) \geq 2$. By passing to a subsequence, we can find two distinct points $\xi_n, \xi_n' \in C_n \cap \Lambda$ which converge to two distinct points ξ, ξ' of $C_0 \cap \Lambda$ respectively as $n \to \infty$. Choose a sequence $p_n \to p \in G$ such that $p_n^+ = \xi_n$, $p_n^- = \xi_n'$, $p^+ = \xi$ and $p^- = \xi'$. The set $T_n = \{t : [p_n] u_t \in \mathrm{RF}\, M\}$ is a global k-thick subset, and hence $\mathcal{T} := \limsup_n T_n$ is a global k-thick subset contained in the set $\{t : [p] u_t \in \mathrm{RF}\, M\}$. Then $C_n \cap \Lambda$ converges, in the Hausdorff topology, to a compact subset $L \subset C_0 \cap \Lambda$ homeomorphic to the one-point compactification of $\mathcal{T}$. Therefore L is uncountable, so is $\limsup_n (C_n \cap \Lambda)$, proving the claim.

Let $\Psi := \cup_{S \in \mathcal{Q}} C_0 \cap S$, i.e., the union of all possible intersection points of C_0 and spheres in $\mathcal{Q}$. Since $C_0 \not\subset S$ for any $S \in \mathcal{Q}$, $\# C_0 \cap S \leq 2$. Hence Ψ is countable, and hence $\limsup_n (C_n \cap \Lambda) - \Psi$ is uncountable. Note that this works for any infinite subsequence of C_n's. Therefore we can choose sequences $\xi_n^-, \xi_n^+ \in C_n \cap \Lambda$ converging to distinct points ξ^-, ξ^+ of $(C_0 \cap \Lambda) - \Psi$ respectively, by passing to a subsequence. As $\xi^-, \xi^+ \in C_0$ and $C_0 \subset S_0$, there exists a frame $g_0 h = (v_0, \cdots, v_{d-1}) \in g_0 H(U)$ whose first vector v_0 is tangent to the geodesic $[\xi^-, \xi^+]$. Setting $g := g_0 h$, we claim that $[g] \in \bigcap_i \mathscr{G}(U_{\pm}^{(i)}, yL)$. Suppose that $[g] \in \mathscr{S}(U_{\pm}^{(i)}, yL)$ for some i. We will assume $[g] \in \mathscr{S}(U_{-}^{(i)}, yL)$, as the case when $[g] \in \mathscr{S}(U_{+}^{(i)}, yL)$ can be dealt similarly, by changing the role of g^- and g^+ below. For simplicity, set $U^{(i)} := U_{-}^{(i)}$. Now by Proposition 4.13, there exist $L_0 \in \mathcal{L}_{U^{(i)}}$ and $\alpha \in N \cap L$ such that $(L_0)_{nc} \lneq L_{nc}$ and $[g]\alpha L_0$ is closed. Let S denote the boundary of $\pi(g \alpha L_0)$. Since $\alpha \in N \cap L$, we have $(g\alpha)^+ = g^+ = \xi^+ \in S \cap \Lambda \cap C_0$. Since $S \subsetneq S^*$, $S \cap \Lambda \neq \emptyset$ and ΓS is closed, we have $S \in \mathcal{Q}$. It follows that $\xi^+ \in \Psi$, contradicting the choice of ξ^+. This proves the claim.

Now choose a vector $v_0^{(n)}$ which is tangent to the geodesic $[\xi_n^-, \xi_n^+]$. We then extend $v_0^{(n)}$ to a frame $g_n h_n \in g_n H(U)$ so that $g_n h_n$ converges to $g = g_0 h$ as $n \to \infty$. Since $\{\xi_n^{\pm}\} \subset \Lambda$, we have $[g_n h_n] \in \mathrm{RF}\, M$. This completes the proof. $\square$

We will need the following lemma later.

Lemma 7.4. *Let $k \geq 1$. Let χ be a k-horosphere in $\mathbb{H}^{k+1}$ resting at $p \in \partial \mathbb{H}^{k+1}$, and $\mathcal{P}$ be a geodesic k-plane in $\mathbb{H}^{k+1}$. Let $\xi \in \partial \mathcal{P}$, δ be a geodesic joining ξ and p, and $q = \delta \cap \chi$. There exists $R_0 > 1$ such that for any $R > R_0$, if $d(\chi, \mathcal{P}) < R - 1$, then $d(q, \mathcal{P}) < R$.*

Proof. For $k = 1$, this is shown in [20, Lem. 4.2]. Now let $k \geq 2$. Consider a geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^{k+1}$ which passes through q and orthogonal to $\mathcal{P}$. Then $\chi \cap \mathbb{H}^2$ and $\mathcal{P} \cap \mathbb{H}^2$ are a horocycle and a geodesic in $\mathbb{H}^2$ respectively. As $d_{\mathbb{H}^{k+1}}(\chi, \mathcal{P}) = d_{\mathbb{H}^2}(\chi \cap \mathbb{H}^2, \mathcal{P} \cap \mathbb{H}^2)$ and $d_{\mathbb{H}^{k+1}}(q, \mathcal{P}) = d_{\mathbb{H}^2}(q, \mathcal{P} \cap \mathbb{H}^2)$, the conclusion follows from the case $k = 1$. $\square$

Lemma 7.5. *Let $U < \check{H} \cap N$ be a non-trivial connected closed subgroup. If the boundary of $\pi(gH(U))$ is contained in ∂B for some component B of Ω, then $[g] \in \mathrm{BF}\, M \cdot \mathrm{C}(H(U))$.*

Proof. As U is equal to $mU_k m^{-1}$ for some $m \in \check{H} \cap M$ and $1 \leq k \leq d-2$, the general case is easily reduced to the case when $U = U_k$. Since $g = (v_0, \cdots, v_d)$ has its first $(k+1)$-vectors tangent to the geodesic $(k+1)$-plane $\pi(gH(U_k))$ and $\partial(\pi(gH(U_k)) \subset \partial B$, we can use an element $c \in \mathrm{C}(H(U_k)) = \mathrm{SO}(d-k-2)$ to modify the next $(d-k-2)$-vectors so that gc has its first $(d-1)$-vectors tangent to $\mathrm{hull}(\partial B)$. Then $[gc] \in \mathrm{BF}\, M$, proving the claim. $\square$

Lemma 7.6. *Let $U < \check{H} \cap N$ be a non-trivial connected closed subgroup. If $x_n \in \mathrm{RF}\, M \cdot U$ is a sequence converging to some $x \in \mathrm{RF}\, M$, then passing to a subsequence, there exists $u_n \in U$ such that $x_n u_n \in \mathrm{RF}\, M$ and at least one of the following holds:*

1. *$u_n \to e$ and hence $x_n u_n \to x$, or*
2. *$x = zc$ for some $z \in \mathrm{BF}\, M$ with $c \in \mathrm{C}(H(U))$, and $x_n u_n$ accumulates on $z\check{H}c$.*

Proof. If x_n belongs to $\mathrm{RF}\, M$ for infinitely many n, we simply take $u_n = e$. So assume that $x_n \notin \mathrm{RF}\, M$ for all n. Choose a sequence $g_n \to g_0$ in G so that $x_n = [g_n]$ and $x = [g_0]$. As $x \in \mathrm{RF}\, M$, we have $\{g_0(0), g_0(\infty)\} \subset \Lambda$. As $x_n \in \mathrm{RF}_+ M - \mathrm{RF}\, M$, we have $g_n(\infty) \in \Lambda$ and $g_n(0) \in \Omega$. For each n, choose an element $u_n \in U$ so that $0 < \alpha_n := \|u_n\| \leq \infty$ is the minimum of $\|u\|$ for all $u \in U$ satisfying $g_n u(0) \in \Lambda$. Set $\alpha := \limsup_n \alpha_n$. If $\alpha = 0$, then we are in case (1). Hence we will assume $0 < \alpha \leq \infty$. Let C_n denote the boundary of $\pi(g_n H(U))$ and C_0 the boundary of $\pi(g_0 H(U))$. Then $C_n \to C_0$ in $\mathcal{C}^{\dim U}$. Recall that $B_U(r)$ denotes the ball of radius r centered at 0 inside U. Set $\mathcal{B}_n := g_n B_U(\alpha_n)(0)$ and $\mathcal{B}_0 := g_0 B_U(\alpha)(0)$. Then $\mathcal{B}_n \subset C_n \cap \Omega$, and $\partial \mathcal{B}_n \cap \Lambda \neq \emptyset$ by the choice of u_n. By passing to a subsequence, we have $\alpha_n \to \alpha$ and $\mathcal{B}_n \to \mathcal{B}_0$ as $n \to \infty$ and hence the diameter of $\mathcal{B}_n$ in $\mathbb{S}^{d-1}$ is bounded below by some positive number. Hence, passing to a subsequence, we may assume that $\mathcal{B}_n$ are all contained in the same component, say B of Ω. Consequently, $\mathcal{B}_0 \subset \overline{B}$.

We claim that $\# \overline{\mathcal{B}_0} \cap \partial B \geq 2$. First note that $g_0(0) \in \Lambda$. If $\alpha = \infty$, then $g_n u_n(0) \to g_0(\infty) \in \Lambda \cap \overline{\mathcal{B}_0}$. If $\alpha < \infty$, then u_n converges to some $u \in U$, passing to a subsequence, and $u \neq e$, as $\alpha > 0$. Now, $g_n u_n(0) \to g_0 u(0) \in \Lambda \cap \overline{\mathcal{B}_0}$. Since $\Lambda \cap \overline{B} \subset \partial B$, this proves the claim.

Therefore $\mathcal{B}_0$ is contained in ∂B, and hence so is C_0. By Lemma 7.5, this implies that $x = zc$ for some $z \in \mathrm{BF}\, M$ and $c \in \mathrm{C}(H(U))$. We proceed to show that $x_n u_n$

accumulates on $z\check{H}c$. Since $c \in \mathrm{C}(H(U))$, we may assume $c = e$ by replacing x with xc^{-1}, and x_n with $x_n c^{-1}$.

We claim that $\pi(g_n u_n)$ goes arbitrarily close to the plane $\pi(g_0\check{H})$ as $n \to \infty$. Since $x\check{H} = [g_0]\check{H}$ is compact, $g_n u_n \in g_n\check{H}$ and $\pi(g_n\check{H})$ is a geodesic plane nearly parallel to $\pi(g_0\check{H})$ for all large n, this claim implies that $[g_n]u_n$ accumulates on $z\check{H}$, completing the proof.

Now, to prove the claim, let $D_n := C_n \cap \partial B$, and $\mathcal{P}_n := \mathrm{hull}(D_n)$. Let $k = \dim U$. Since C_n is a k-sphere meeting the $(d-2)$-sphere $\partial B \subset \mathbb{S}^{d-1}$, and $C_n \not\subset \partial B$, it follows that D_n is a $(k-1)$-sphere. We set $\mathcal{H}_n := \mathrm{hull}(C_n)$, $\mathcal{H}_0 := \mathrm{hull}(C_0)$ and $\mathcal{H} := \mathrm{hull}(\partial B) = \pi(g_0\check{H})$. Then $\mathcal{H}_n \cap \mathcal{H} = \mathcal{P}_n$. Let $\varepsilon > 0$ be arbitrary, and $\mathcal{N}_\varepsilon(\mathcal{H})$ denote the ε-neighborhood of $\mathcal{H}$ in $\mathbb{H}^d$. Letting $d_{\mathcal{H}_n}(\cdot,\cdot)$ denote the hyperbolic distance in $\mathcal{H}_n$, we may write

$$\mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n = \{p \in \mathcal{H}_n : d_{\mathcal{H}_n}(p, \mathcal{P}_n) < R_n\}$$

for some $R_n > 0$. This is because $\mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n$ is convex and invariant under family of isometries, whose axes of translation and rotation are contained in $\mathcal{P}_n$. As $C_n \to C_0 \subset \partial B$ as $n \to \infty$, it follows that $R_n \to \infty$ as $n \to \infty$. Let $\chi_n \coloneqq \pi(g_n U)$, and $\chi_0 \coloneqq \pi(g_0 U)$, which are k-horospheres contained in $\mathcal{H}_n$ and $\mathcal{H}_0$ respectively.

We next show that there is a uniform upper bound for $d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n)$, $n \in \mathbb{N}$. To see this, we only need to consider those $\mathcal{P}_n$'s which are disjoint from χ_n, as $d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n) = 0$ otherwise. Since $\chi_n \to \chi_0$ and $C_n \to C_0$ as $n \to \infty$, it suffices to check that the diameters of D_n with respect to the spherical metric on $\mathbb{S}^{d-1}$ have a uniform positive lower bound. Let us write $C_n - D_n = E_n \cup E_n'$, where E_n is a connected component of $C_n - D_n$ meeting B, and E_n' is the other component. Since $C_n \to C_0$ as $n \to \infty$, a uniform lower bound for both $\mathrm{diam}(E_n)$ and $\mathrm{diam}(E_n')$ will give a uniform upper bound for $\mathrm{diam}(D_n)$. Since $\mathcal{B}_n \subset E_n$, $\mathrm{diam}(E_n) > \mathrm{diam}(\mathcal{B}_0)/2$ for all sufficiently large n. On the other hand, note that $\chi_n \subset \mathcal{H}_n$ is a horosphere resting at a point in E_n'. Since χ_n converges to χ, the condition that $\mathcal{P}_n \cap \chi_n = \emptyset$ implies that $\mathrm{diam}(E_n')$ is also bounded below by some positive constant. Since $R_n \to \infty$, we conclude that $d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n) < R_n - 1$ for all sufficiently large n. Applying Lemma 7.4 to $\mathbb{H}^{k+1} = \mathcal{H}_n$, $\chi = \chi_n$, $\mathcal{P} = \mathcal{P}_n$, $\xi = g_n^+$ and $q = \pi(g_n u_n)$, we have $d_{\mathcal{H}_n}(\pi(g_n u_n), \mathcal{P}_n) < R_n$ and hence $\pi(g_n u_n) \in \mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n$, for all sufficiently large n. As $\varepsilon > 0$ was arbitrary, this proves that $\pi(g_n u_n)$ goes arbitrarily close to $\pi(g_0\check{H})$ as $n \to \infty$. This finishes the proof. □

Lemma 7.7. *Let $U < N$ be a non-trivial connected closed subgroup. If $x_n \to x$ in $F^* \cap \mathrm{RF}_+ M$, and $x \in F^* \cap \mathrm{RF}\, M$, then there exists $u_n \to e$ in U such that $x_n u_n \in \mathrm{RF}\, M$; in particular, $x_n u_n \to x$ in $F^* \cap \mathrm{RF}\, M$.*

Proof. The general case easily reduces to the case when $U < \check{H} \cap N$. Then the claim follows from Lemma 7.6 and Lemma 3.4. □

Obtaining limits in F^*. For $\varepsilon > 0$, we set

$$\mathrm{core}_\varepsilon(M) := \big\{x \in \Gamma\backslash G : \pi(x) \in \mathrm{core}\, M \text{ and } d(\pi(x), \partial\, \mathrm{core}\, M) \geq \varepsilon\big\}. \tag{7.1}$$

We note that $\mathrm{core}_\varepsilon(M)$ is a compact of F^* for all sufficiently large $\varepsilon > 0$.

Lemma 7.8. *Let $x \in \mathrm{RF}\, M$, and $V = \{v_t : t \in \mathbb{R}\} < N$ be a one-parameter subgroup. If $\pi(xV) \not\subset \partial\, \mathrm{core}\, M$, and $xv_{t_i} \in \mathrm{RF}\, M$ for some sequence $t_i \to +\infty$, then there exists a sequence $s_i \to +\infty$ such that xv_{s_i} converges to a point in F^*.*

Proof. It suffices to show that there exists $s_i \to +\infty$ such that $xv_{s_i} \in \mathrm{core}_{\eta/3}(M)$ where η is as given in (3.5). Let $x = [g]$, and set $o = (1, 0, \cdots, 0) \in \mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. We may assume $g = (e_0, \cdots, e_{d-1})_o \in \mathrm{F}\,\mathbb{H}^d$ where e_i are standard basis vectors in $\mathrm{T}_o\,\mathbb{H}^d \simeq \mathbb{R}^d$. Note that for $V^+ = \{v_t : t > 0\}$, gV^+ is a translation of the frame g along a horizontal ray emanating from o along the V^+-direction. By the definition of η, the $\eta/3$-neighborhoods of $\mathrm{hull}\, B_i$'s are mutually disjoint. For each i, set $s_i := t_i$ if $xv_{t_i} \in \mathrm{core}_{\eta/3}(M)$. Otherwise, there exists a unique j such that $d(\pi(gv_{t_i}), \mathrm{hull}\, B_j) < \eta/3$. If $\pi(gV_{[t_i,\infty)})$ were contained in the $\eta/3$-neighborhood of $\mathrm{hull}\, B_j$, then the unique geodesic 2-plane which contains $\pi(gV_{[t_i,\infty)})$ must lie in $\partial\,\mathrm{hull}\, B_j$, and hence $\pi(xV) \subset \partial\,\mathrm{core}(M)$; this contradicts the hypothesis. Therefore there exists $t_i < s_i < \infty$ such that $d(\pi(gv_{s_i}), \mathrm{hull}\, B_j) = \eta/3$, as desired. □

Lemma 7.9. *Let $x_n L_n v_n$ be a sequence of closed orbits with $x_n \in \mathrm{RF}_+ M$, $L_n \in \mathcal{L}_U$ and $v_n \in (L_n \cap N)^\perp$. Suppose that either*

(1) *$x_n \in F^*$ for all n; or*
(2) *$x_n L_n v_n \cap \mathrm{RF}_+ M \cap F^* \neq \emptyset$ for all n.*

Then $F^* \cap \limsup_n (x_n L_n v_n \cap \mathrm{RF}_+ M) \neq \emptyset$.

Proof. We claim that if $x_n \in F^*$, then $x_n L_n v_n \cap \mathrm{RF}_+ M \cap F^* \neq \emptyset$, that is, the hypothesis (1) implies (2). Suppose not. Then, since $A \subset L_n$, $\overline{(x_n A v_n A \cap \mathrm{RF}_+ M)} \subset \mathrm{RF}_+ M - F^*$. Since the set $\mathrm{RF}_+ M - F^*$ is a closed A-invariant set and $e \in \overline{A v_n A}$, we would have $x_n \in \mathrm{RF}_+ M - F^*$, yielding a contradiction. It follows from the claim that there exists $z_n \in x_n L_n \cap \mathrm{RF}_+ M$ such that $\pi(z_n v_n U) \not\subset \partial\,\mathrm{core}(M)$ for all n. In particular, there exists $u_n \in U$ such that $z_n v_n u_n \in \mathrm{core}_{\eta/3}(M)$. Since $\mathrm{core}_{\eta/3}(M)$ is a compact subset of F^*, $z_n v_n u_n = z_n u_n v_n$ converges to a point in F^*, finishing the proof. □

Lemma 7.10. *Let $x_0 L$ be a closed orbit with $x_0 \in \mathrm{RF}\, M$ and $L \in \mathcal{L}_U$. Suppose that E is a closed U-invariant subset containing $x_0 L v_n \cap \mathrm{RF}_+ M$ for some sequence $v_n \to \infty$ in $(L \cap N)^\perp$. If $x_0 \in F^*$ or $x_0 L v_n \cap \mathrm{RF}_+ M \cap F^* \neq \emptyset$ for all n, then there exist $y \in \mathrm{RF}\, M \cap F^*$ and a one parameter subgroup $V \subset (L \cap N)^\perp$ such that $E \supset y(L \cap N)V$.*

Proof. Note that $(x_0 L v_n \cap \mathrm{RF}_+ M)(v_n^{-1} A v_n) \subset E$. By Lemma 7.9, there exists $y \in F^* \cap \limsup_{n\to\infty} (x_0 L v_n \cap \mathrm{RF}_+ M)$. Since $y \in F^* \cap \mathrm{RF}_+ M \subset \mathrm{RF}\, M \cdot U$, we may assume $y \in F^* \cap \mathrm{RF}\, M$ by modifying y using an element of U. Note that $\liminf_{n\to\infty} (x_0 L v_n \cap \mathrm{RF}_+ M) \supset y(L \cap N)$, passing to a subsequence. Since $\limsup_{n\to\infty} (v_n^{-1} A v_n)$ contains a one-parameter subgroup $V \subset (L \cap N)^\perp$ by Lemma 2.3, we obtain that $y(L \cap N)V \subset E$. □

Lemma 7.11. *If $yLv_0 \cap \mathrm{RF}\, M \cap F^* \neq \emptyset$ for some $v_0 \in N$ and $L \in \mathcal{L}_U$, then $yLv \cap F^* \cap \mathrm{RF}\, M \neq \emptyset$ for all $v \in A v_0 A$.*

Proof. Let $y_0 := y\ell v_0 \in yLv_0 \cap F^* \cap \mathrm{RF}\, M$, and $v = a v_0 b \in A v_0 A$. Then $(y\ell a^{-1})v = y\ell v_0 b \in F^* \cap \mathrm{RF}\, M$ as $F^* \cap \mathrm{RF}\, M$ is A-invariant. Since $y\ell a^{-1} v \in yLv$, the claim is proved. □

Lemma 7.12. *Let $x_0 L$ be a closed orbit with $x_0 \in \mathrm{RF}\, M$ and $L \in \mathcal{L}_U$. Suppose that E is a closed AU-invariant subset containing $x_0 L v \cap \mathrm{RF}_+ M$ for some non-trivial element $v \in (L \cap N)^\perp$. If $x_0 \in F^*$ or $x_0 L v \cap \mathrm{RF}\, M \cap F^* \neq \emptyset$, then there* *exist* $y \in F^* \cap \mathrm{RF}\, M$ *and a one parameter subgroup* $V \subset (L \cap N)^\perp$ *such that* $E \supset y(L \cap N)VA$.

Proof. Since X is A-invariant, we get $(x_0 L \cap \mathrm{RF}_+ M)AvA \subset E$. Choose a sequence $v_n := a_n v a_n^{-1} \in AvA$ tending to ∞. Note that either $x_0 \in F^*$ or for all n, $x_0 L v_n \cap \mathrm{RF}\, M \cap F^* \neq \emptyset$ by Lemma 7.11. Therefore the claim follows from Lemma 7.10. $\square$

8. Limits of unipotent blowups

Fix $k > 1$ as given by Proposition 3.10. In the whole section, we fix a non-trivial connected subgroup $U < N$. For a given sequence $g_i \to e$, and a sequence of k-thick subsets T_i of a one-parameter subgroup $U_0 < U$, we study the set $\limsup \mathsf{T}_i g_i U$ under certain conditions on the sequence g_i.

The basic tool used here is the so-called *quasi-regular map* associated to the sequence g_i introduced in the work of Margulis-Tomanov [18] to study the object $\limsup U_0 g_i U$ in the finite volume case. For our application, we need a somewhat more precise information on the shape of the set $\limsup U_0 g_i U$ as well as $\limsup \mathsf{T}_i g_i U$ than discussed in [18].

Let $U^\perp$ denote the orthogonal complement of U in $N \simeq \mathbb{R}^{d-1}$ as defined in section 2. Recall from (2.2) that $\mathrm{N}(U) = AN\, \mathrm{C}_1(U)\, \mathrm{C}_2(U)$ where $\mathrm{C}_1(U) = \mathrm{C}(H(U))$ and $\mathrm{C}_2(U) = H(U) \cap M \cap \mathrm{C}(U^\perp)$. Since $\mathrm{N}(U)$ is the identity component of $\mathrm{N}_G(U)$, for a sequence $g_i \to e$, the condition $g_i \in \mathrm{N}_G(U)$ means $g_i \in \mathrm{N}(U)$ for all sufficiently large $i \gg 1$. Note that the product $AU^\perp\, \mathrm{C}_2(U)$ is a connected subgroup of G, since $\mathrm{C}_2(U)$ commutes with $U^\perp$, and A normalizes $U^\perp\, \mathrm{C}_2(U)$.

Lemma 8.1. *For a given sequence* $g_i \to e$ *in* $G - \mathrm{N}(U)$*, there exists a one-parameter subgroup* $U_0 < U$ *such that the following holds; for any given sequence of* k*-thick subsets* $\mathsf{T}_i \subset U_0$*, there exist sequences* $t_i \in \mathsf{T}_i$*, and* $u_i \in U$ *such that as* $i \to \infty$*,* $u_i g_i u_{t_i} \to \alpha$ *for some non-trivial element* $\alpha \in AU^\perp\, \mathrm{C}_2(U) - \mathrm{C}_2(U)$*. Moreover,* α *can be made arbitrarily close to* e*.*

Proof. Set $L := AU^\perp M N^+$. Note that $\mathrm{N}(U) \cap L = AU^\perp\, \mathrm{C}_1(U)\, \mathrm{C}_2(U)$ and that the product map from $U \times L$ to G is a diffeomorphism onto a Zariski open neighborhood of e in G. Following [18], we will construct a quasi-regular map $\psi : U \to \mathrm{N}(U) \cap L$ associated to the sequence g_i. Except for a Zariski closed subset of U, the product $g_i u$ can be written as an element of UL in a unique way. We denote by $\psi_i(u) \in L$ its L-component so that $g_i u \in U\psi_i(u)$. By Chevalley's theorem, there exists an $\mathbb{R}$-regular representation $G \to \mathrm{GL}(W)$ with a distinguished point $p \in W$ such that $U = \mathrm{Stab}_G(p)$. Then pG is locally closed, and $\mathrm{N}_G(U) = \{g \in G : pgu = pg \text{ for all } u \in U\}$. For each i, the map $\tilde{\phi}_i : U \to W$ defined by $\tilde{\phi}_i(u) = pg_i u$ is a polynomial map in $U = \mathbb{R}^m$ of degree uniformly bounded, and $\tilde{\phi}_i(e)$ converges to p as $i \to \infty$. As $g_i \notin \mathrm{N}_G(U)$, $\tilde{\phi}_i$ is non-constant. Denote by $B(p, r)$ the ball of radius r centered at p, fixing a norm $\|\cdot\|$ on W. Since pG is open in its closure, we can find $\lambda_0 > 0$ such that

$$B(p, \lambda_0) \cap \overline{pG} \subset pG. \tag{8.1}$$

Without loss of generality, we may assume that $\lambda_0 = 2$ by renormalizing the norm. Now define $\lambda_i := \sup\{\lambda \geq 0 : \tilde{\phi}_i(B_U(\lambda)) \subset B(p, 2)\}$. Note that $\lambda_i < \infty$ as ϕ_i is nonconstant, and $\lambda_i \to \infty$ as $i \to \infty$, as $g_i \to e$. We define $\phi_i : U \to W$ by $\phi_i(u) := \tilde{\phi}_i(\lambda_i u)$. This forms an equi-continuous family of polynomials on U. Therefore, after passing to a subsequence, ϕ_i converges to a non-constant polynomial ϕ uniformly on every compact subset of U. Moreover $\sup\{\|\phi(u) - p\| : u \in B_U(1)\} = 1$, $\phi(B_U(1)) \subset pL$, and $\phi(0) = p$. Now the following map ψ defines a non-constant rational map defined on a Zariski open dense neighborhood of $\mathcal{U}$ of e in U: $\psi := \rho_L^{-1} \circ \phi$ where ρ_L is the restriction to L of the orbit map $g \mapsto p.g$. We have $\psi(e) = e$ and $\psi(u) = \lim_i \psi_i(\lambda_i u)$ where the convergence is uniform on compact subsets of $\mathcal{U}$ and $\psi(u) \in L \cap \mathrm{N}(U) = AU^\perp \,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$. Since ψ is non-constant, there exists a one-parameter subgroup $U_0 < U$ such that $\psi|_{U_0}$ is non-constant. Now let T_i be a sequence of k-thick sets in $U_0 \simeq \mathbb{R}$. Then T_i/λ_i is also a k-thick set, and so is $\mathsf{T}_\infty := \limsup_{i\to\infty} (\mathsf{T}_i/\lambda_i) \subset U_0$. Finally, for all $t \in \mathsf{T}_\infty$, there exists a sequence $t_i \in \mathsf{T}_i$ such that $t_i/\lambda_i \to t$ as $i \to \infty$ (by passing to a subsequence). Since $\psi_i \circ \lambda_i \to \psi$ uniformly on compact subsets, $\psi(t) = \lim_{i\to\infty}(\psi_i \circ \lambda_i)\,(t_i/\lambda_i) = \lim_{i\to\infty} \psi_i(t_i)$. By the definition of ψ_i, this means that there exists $u_i \in U$ such that $\psi(t) = \lim_{i\to\infty} u_i g_i u_{t_i}$. Since $\psi|_{U_0}$ is a non-constant continuous map, and an uncountable set T_∞ accumulates on 0, the image $\psi(\mathsf{T}_\infty)$ contains a non-trivial element α of $AU^\perp \,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$ which can be taken arbitrarily close to e.

We now claim that if α is sufficiently close to e, then it belongs to $AU^\perp \,\mathrm{C}_2(U)$. Consider $H'(U) := H(U)\,\mathrm{C}_1(U)$, and let $\mathfrak{h}$ denote its Lie algebra. Now for all i large enough, using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ in (2.5), we can write $g_i = c_i d_i r_i$ where $c_i \in \mathrm{C}_1(U)$, $d_i \in H(U)$ and $r_i \in \exp \mathfrak{h}^\perp$, Since c_i commutes with U, we can write $u_i g_i u_{t_i} = (u_i u_{t_i}) c_i (u_{t_i}^{-1} d_i u_{t_i})(u_{t_i}^{-1} r_i u_{t_i})$. On the other hand, we have $\lim_i p u_i g_i u_{t_i} = \lim_i p c_i (u_{t_i}^{-1} d_i u_{t_i})(u_{t_i}^{-1} r_i u_{t_i}) = p\alpha$. Since $c_i \to e$, $u_{t_i} d_i u_{t_i}^{-1} \in H(U)$, and $u_{t_i} r_i u_{t_i}^{-1} \in \exp \mathfrak{h}^\perp$, it follows that both sequences $u_{t_i} d_i u_{t_i}^{-1}$ and $u_{t_i} r_i u_{t_i}^{-1}$ must converge, say to $h \in H(U)$ and to $q \in \exp \mathfrak{h}^\perp$, respectively. Hence $\alpha = hq$ by replacing h by uh for some $u \in U$. On the other hand, we can write $\alpha = avc_1c_2 \in AU^\perp \,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$. So $hq = avc_1c_2$. Note that $c := c_1c_2 \in \mathrm{C}(H(U))H(U) = H'(U)$. We get

$$(a^{-1}hc^{-1})(cqc^{-1}) = v. \tag{8.2}$$

Now, when α is sufficiently close to e, all elements appearing in (8.2) are also close to e. Recall that the map $H'(U) \times \mathfrak{h}^\perp \to G$ given by $(h', X) \to h' \exp X$ is a local diffeomorphism onto a neighborhood of e. Since $(a^{-1}hc^{-1}) \in H'(U)$, and $cqc^{-1}, v \in \exp \mathfrak{h}^\perp$, we have $a^{-1}hc^{-1} = e$ and $cqc^{-1} = v$ for α sufficiently small. In particular, $a^{-1}hc_2^{-1} = c_1^{-1} \in H(U) \cap \mathrm{C}(H(U)) = \{e\}$. Hence $c_1 = e$. It follows that $\alpha \in AU^\perp \,\mathrm{C}_2(U)$, as desired.

We further claim that we can choose α outside of $\mathrm{C}_2(U)$. As $\mathrm{C}_2(U)$ is a compact subgroup, we can choose a $\mathrm{C}_2(U)$-invariant Euclidean norm $\|\cdot\|$ on W. If $\alpha = \psi(t) \in \mathrm{C}_2(U)$ for some $t \in \mathsf{T}_\infty \subset U_0$, then t is one of finitely many solutions of the polynomial equation $\|\phi(t)\|^2 = \|p\|^2$. Therefore, except for finitely many $t \in \mathsf{T}_\infty$, $\alpha = \psi(t) \in AU^\perp \,\mathrm{C}_2(U) - \mathrm{C}_2(U)$. This finishes the proof. □

The following lemma is similar to Lemma 8.1, but here we consider the case when U is the whole horospherical subgroup N. In this restrictive case, the limiting element can be taken inside A.

Lemma 8.2. *Let $\mathsf{T}_i \subset N$ be a sequence of k-thick subsets in the sense that for any one-parameter subgroup $U_0 < N$, $\mathsf{T}_i \cap U_0$ is a k-thick subset of $U_0 \simeq \mathbb{R}$. For any sequence $g_i \to e$ in $G - \mathrm{N}_G(N)$, there exist $t_i \to \infty$ in T_i and $u_i \in N$ such that $u_i g_i u_{t_i} \to a$ for some non-trivial element $a \in A$. Moreover, a can be chosen to be arbitrarily close to e.*

Proof. We first consider the case when g_i belongs to the opposite horospherical subgroup N^+. We will use the notations u^+ and u^- defined in Section 2. Write $g_i = \exp u^+(w_i)$ for some $w_i \in \mathbb{R}^{d-1}$. For $x \in \mathbb{R}^{d-1}$, set $u_x := \exp u^-(x) \in N$. Let $\varepsilon > 0$ be arbitrary. Since T_i is a k-thick subset of N, there exists $\alpha_i \in \mathbb{R}$ such that $\alpha_i w_i \in \mathsf{T}_i$ and $\varepsilon < \frac{|\alpha_i|\|w_i\|^2}{2} < k\varepsilon$. Setting $u_{x_i} := u_{\alpha_i w_i} \in \mathsf{T}_i$ and $y_i := -\alpha_i w_i \left(1 + \frac{\alpha_i \|w_i\|^2}{2}\right)^{-1}$, we compute:

$$u_{y_i} g_i u_{x_i} = \begin{pmatrix} \left(1+\frac{\alpha_i\|w_i\|^2}{2}\right)^{-2} & 0 & 0 \\ \left(1+\frac{\alpha_i\|w_i\|^2}{2}\right)^{-1} w_i & \mathrm{I}_{d-1} & 0 \\ -\frac{\|w_i\|^2}{2} & -\left(1+\frac{\alpha_i\|w_i\|^2}{2}\right) w_i^t & \left(1+\frac{\alpha_i\|w_i\|^2}{2}\right)^2 \end{pmatrix}.$$

The condition for the size of α_i guarantees that, by passing to a subsequence, the sequence $u_{x_i} g_i u_{y_i}$ converges to an element $\mathrm{diag}(\alpha, \mathrm{I}_{d-1}, \alpha^{-1}) \in A$ for $\alpha \in [(1-\varepsilon)^{-2}, (1-k\varepsilon)^{-2}] \cup [(1+k\varepsilon)^{-2}, (1+\varepsilon)^{-2}]$ as $i \to \infty$. This proves the claim when $g_i \in N^+$. Since the product map $A \times M \times N^+ \times N \to G$ is a diffeomorphism onto a Zariski-open neighborhood of e in G, we can write $g_i = a_i m_i u_i^+ u_i^-$ for some $a_i \in A$, $m_i \in M$, $u_i^+ \in N^+$ and $u_i^- \in N$ all of which converge to e as $i \to \infty$. By the previous case, we can find $u_{t_i} \in \mathsf{T}_i$ and $u_i \in N$ such that $u_i u_i^+ u_{t_i}$ converges to a non-trivial element $a \in A$. Let $\tilde{u}_i := (a_i m_i) u_i (a_i m_i)^{-1} \in N$. Then $\tilde{u}_i g_i u_{t_i} = a_i m_i u_i u_i^+ u_i^- u_{t_i} = a_i m_i (u_i u_i^+ u_{t_i}) u_i^- \to a$ as $i \to \infty$, proving the claim. ☐

Lemma 8.3. *Let L be any connected reductive subgroup of G normalized by A. Let U_0 be a one-parameter subgroup of $L \cap N$. Let $\mathsf{T}_i \subset U_0$ be a sequence of k-thick subsets. For a given sequence $r_i \to e$ in $\exp(\mathfrak{l}^\perp) - \mathrm{N}(U_0)$, there exists a sequence $t_i \in \mathsf{T}_i$ such that as $i \to \infty$, $u_{t_i}^{-1} r_i u_{t_i} \to v$ for some non-trivial element $v \in (L \cap N)^\perp$, and v can be chosen arbitrarily close to e. Moreover, for all n large enough, we can make v so that $n \le \|v\| \le 2k^2 n$.*

Proof. Without loss of generality, by Proposition 2.6, we may assume that $L_{nc} = H(U)$ for $U = U_k = \mathbb{R}^k$ some $k \ge 1$ and $U_0 := \mathbb{R}e_1$. We write $r_i = \exp(q_i)$ where $q_i \to 0$ in $\mathfrak{l}^\perp$. Using the notations introduced in section 2 and setting $\mathfrak{u}^\perp = \mathrm{Lie}(U^\perp) = \mathbb{R}^{d-1-k}$, we can write $q_i = u^-(x_i) + u^+(y_i) + m(C_i)$ where $x_i \in \mathfrak{u}^\perp$, $y_i \in (\mathfrak{u}^\perp)^t$, and $C_i = \begin{pmatrix} 0_k & B_i \\ -B_i^t & A_i \end{pmatrix}$ is a skew symmetric matrix, all of which converge to 0 as $i \to \infty$. We consider $U_0 = \mathbb{R}e_1$ as $\{u_s = se_1 \in \mathbb{R}^{d-1}\}$ and define the map $\psi_i : \mathbb{R} \to \mathfrak{l}^\perp$ by $\psi_i(s) = u_s^{-1} q_i u_s$ for all $s \in \mathbb{R}$. this is well-defined since $\mathfrak{l}^\perp$ is $\mathrm{Ad}(L)$-invariant. Then a direct computation shows

$$\psi_i(s) = u^-(x_i + sB_i^t e_1 + s^2 y_i/2) + u^+(y_i) + m(\tilde{C}_i) \tag{8.3}$$

where $\tilde{C}_i$ is a skew-symmetric matrix of the form $\tilde{C}_i = \begin{pmatrix} 0_k & B_i + se_1 y_i^t \\ -B_i^t - sy_i e_1^t & A_i \end{pmatrix}$.

Since $r_i \notin \mathrm{N}(U_0)$, it follows that either $y_i \neq 0$ or $y_i = 0$ and $B_i^t e_1 \neq 0$. Hence ψ_i is a non-constant polynomial of degree at most 2, and $\psi_i(0) \to 0$. Let $\lambda_i \in \mathbb{R}$ be defined by $\lambda_i = \sup\{\lambda > 0 : |\psi_i[-\lambda, \lambda]| \le 1\}$. Then $0 < \lambda_i < \infty$ and $\lambda_i \to \infty$. Now the rescaled polynomials $\phi_i = \psi_i \circ \lambda_i : \mathbb{R} \to \mathfrak{l}^\perp$ form an equicontinuous family of polynomials of degree at most 2 and $\lim_{i\to\infty} \phi_i(0) = 0$. Therefore ϕ_i converges to a polynomial $\phi : \mathbb{R} \to \mathfrak{l}^\perp$ uniformly on compact subsets. Since $\phi(0) = 0$ and $\sup\{|\phi(\lambda)| : \lambda \in [-1,1]\} = 1$, ϕ is a non-constant polynomial. From (8.3), it can be easily seen that $\mathrm{Im}(\phi)$ is contained $\mathrm{Lie}(N) \cap \mathfrak{l}^\perp$, by considering the two cases of

$y_i \neq 0$, and $y_i = 0$ and $B_i^t e_1 \neq 0$ separately. For a sequence T_i of k-thick subsets of U_0, set $\mathsf{T}_\infty := \limsup_{i\to\infty}(\mathsf{T}_i/\lambda_i)$, which is a k-thick subset of U_0.

Let $s \in \mathsf{T}_\infty$. By passing to a subsequence, there exists $t_i \in \mathsf{T}_i$ such that $t_i/\lambda_i \to s$ as $i \to \infty$. As $\phi_i \to \phi$ uniformly on compact subsets, it follows that $\phi(s) = \lim_{i\to\infty} \psi_i(\lambda_i \cdot t_i/\lambda_i) = \lim_{i\to\infty} u_{t_i}^{-1} q_i u_{t_i}$. Since T_∞ accumulates on 0, so does $\phi(\mathsf{T}_\infty)$. Taking the exponential map to each side of the above, the first part of the lemma follows.

The second part of the lemma holds by applying Lemma 8.4 below for the non-constant polynomial $p(s) = \|\phi(s)\|^2$ of degree at most 4. ☐

Lemma 8.4. *If $p \in \mathbb{R}[s]$ is a polynomial of degree $\delta \geq 1$ and $\mathsf{T} \subset \mathbb{R}$ is a k-thick subset, then $p(\mathsf{T})$ is $2k^\delta$-thick at ∞.*

Proof. Let C be the coefficient of s^δ term of the polynomial p. Then there exists $s_0 > 1$ such that $\frac{1}{\sqrt{2}} \leq \frac{|p(s)|}{|Cs^\delta|} \leq \sqrt{2}$ for all $|s| > s_0$. Let $r > \frac{|C| s_0^\delta}{\sqrt{2}}$. Since T is k-thick, there exists $t \in \mathsf{T}$ such that $(\sqrt{2}r/|C|)^{1/\delta} < |t| < k(\sqrt{2}r/|C|)^{1/\delta}$. We compute that $r \leq |p(t)| \leq 2k^\delta r$, proving the claim. ☐

9. Translates of relative U-minimal sets

Fix $k > 1$ as given by Proposition 3.10. In this section, we fix a non-trivial connected closed subgroup $U < N$. Unless mentioned otherwise, we let R be a compact A-invariant subset of $\operatorname{RF} M$ such that for every $x \in R$, and for any one-parameter subgroup $U_0 = \{u_t\}$ of U, the set $\{t \in \mathbb{R} : xu_t \in R\}$ is k-thick. In practice, R will be either $\operatorname{RF} M$ or a compact subset of the form $\operatorname{RF} M \cap F^*_{H(U)} \cap X$ for a closed $H(U)$-invariant subset X. The main aim of this section is to prove Propositions 9.6 and 9.9 using the results of section 8.

Definition 9.1. A U-invariant closed subset $Y \subset \Gamma\backslash G$ is U-minimal with respect to R if $Y \cap R \neq \emptyset$ and for any $y \in Y \cap R$, yU is dense in Y.

In this section, we study how to find an additional invariance of Y beyond U under certain conditions.

Lemma 9.2. *Let $Y \subset \Gamma\backslash G$ be a U-minimal subset with respect to R. For any $y \in Y \cap R$, there exists a sequence $u_n \to \infty$ in U such that $yu_n \to y$.*

Proof. The set $Z := \{z \in Y : yu_n \to z \text{ for some } u_n \to \infty \text{ in } U\}$ is U-invariant and closed. The hypothesis on Y implies that $Z = Y$. ☐

A subset S of a topological space is said to be *locally closed* if S is open in its closure $\overline{S}$.

Lemma 9.3. *Let Y be a U-minimal subset of $\Gamma\backslash G$ with respect to R, and S be a closed subgroup of $\mathrm{N}(U)$ containing U. For any $y_0 \in Y \cap R$, the orbit $y_0 S$ is not locally closed.*

Proof. Suppose that $y_0 S$ is locally closed for some $y_0 \in Y \cap R$. Since Y is U-minimal with respect to R, there exists $u_n \to \infty$ in U such that $y_0 u_n \to y_0$ by Lemma 9.2. We may assume that $y_0 = [e]$ without loss of generality. Since $y_0 S$ is locally closed, $y_0 S$ is homeomorphic to $(S \cap \Gamma)\backslash S$ (cf. [35, Thm. 2.1.14]). Therefore there exists $\delta_n \in S \cap \Gamma$ such that $\delta_n u_n \to e$ as $n \to \infty$. Since $\mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, writing $\delta_n = a_n r_n$ for $a_n \in A$ and $r_n \in N\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, it follows that $a_n \to e$.

On the other hand, note that a_n is non-trivial as Γ does not contain any elliptic or parabolic element. This is a contradiction, as there exists a positive lower bound for the translation lengths of elements of Γ, which is given by the minimal length of a closed geodesic in M. □

In the rest of this section, we use the following notation: $H = H(U)$, $H' = H'(U)$, and $F^* = F^*_{H(U)}$.

Lemma 9.4. *For every U-minimal subset $Y \subset \Gamma\backslash G$ with respect to $\mathrm{RF}\, M$ such that $Y \cap F^* \cap \mathrm{RF}\, M \neq \emptyset$, and for any $y_0 \in Y \cap F^* \cap \mathrm{RF}\, M$, there exists a sequence $g_n \to e$ in $G - \mathrm{N}(U)$ such that $y_0 g_n \in Y \cap \mathrm{RF}\, M$ for all n.*

Proof. Let $y_0 \in Y \cap F^* \cap \mathrm{RF}\, M$. As $Y = \overline{y_0 U}$, $Y \subset \mathrm{RF}_+ M$. Using Lemma 3.4 and the fact that F^* is open, we get that there exists an open neighborhood $\mathcal{O}$ of e such that

$$y_0\mathcal{O} \subset Y \cap F^* \subset Y \cap \mathrm{RF}\, M \cdot U. \tag{9.1}$$

Without loss of generality, we may assume that the map $g \mapsto y_0 g \in \Gamma\backslash G$ is injective on $\mathcal{O}$, by shrinking $\mathcal{O}$ if necessary. We claim that there exists $g_n \to e$ in $G - \mathrm{N}(U)$ such that $y_0 g_n \in Y \cap F^*$. Suppose not. Then there exists a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of e such that $y_0\mathcal{O}' \cap Y \subset y_0\,\mathrm{N}(U)$. Set $S := \{g \in \mathrm{N}(U) : Yg = Y\}$ which is a closed subgroup of $\mathrm{N}(U)$ containing U. We will show that $y_0 S$ is locally closed; this contradicts Lemma 9.3. We first claim that

$$y_0\mathcal{O}' \cap Y \subset y_0 S. \tag{9.2}$$

If $g \in \mathcal{O}'$ such that $y_0 g \in Y$, then $g \in \mathrm{N}(U)$. Therefore $\overline{y_0 g U} = \overline{y_0 U} g = Yg \subset Y$. Moreover, $Yg \cap \mathrm{RF}\, M \neq \emptyset$ by (9.1). Hence $Yg = Y$, proving that $g \in S$. Now, (9.2) implies that $y_0 S$ is open in Y. On the other hand, since $U \subset S$, we get $Y = \overline{y_0 S}$. Therefore, $y_0 S$ is locally closed.

Hence we have $g_n \to e$ in $G - \mathrm{N}(U)$ such that $y_0 g_n \in Y \cap F^*$. Since $y_0 g_n \in F^* \cap \mathrm{RF}_+ M$ converges to $y_0 \in F^* \cap \mathrm{RF}\, M$, by Lemma 7.7, there exists a sequence $u_n \to e$ in U such that $y_0 g_n u_n \in \mathrm{RF}\, M$. Therefore, by replacing g_n with $g_n u_n$, this finishes the proof. □

Lemma 9.5. *Let Y be a U-minimal subset with respect to R, and let W be a connected closed subgroup of $\mathrm{N}(U)$. If there exists a sequence $\alpha_i \to e$ in W such that $Y\alpha_i \subset Y$, then there exists a one-parameter subsemigroup $S < W$ such that $YS \subset Y$. Moreover if W_0 is a compact Lie subgroup of W and $\alpha_i \in W - W_0$ for all i, then S can be taken so that $S \not\subset W_0$.*

Proof. The set $S_0 = \{g \in W : Yg \subset Y\}$ is a closed subsemigroup of W. Write $\alpha_i = \exp \xi_i$ for some $\xi_i \in \mathrm{Lie}(W)$. Then the sequence $v_i := \|\xi_i\|^{-1}\xi_i$ of unit vectors has a limit, say, v. It suffices to note that $S := \{\exp(tv) : t \geq 0\}$ is contained in the closure of the subsemigroup generated by α_i's. Now suppose that $\alpha_i \in W - W_0$. Set $M_0 := \{g \in W_0 : Yg = Y\}$. This is a closed Lie subgroup of W_0. Write $\mathrm{Lie}\, W = \mathfrak{m}_0 \oplus \mathfrak{m}_0^\perp$ where $\mathfrak{m}_0 = \mathrm{Lie}\, M_0$. By modifying α_i by elements of M_0, we may assume $\alpha_i = \exp \xi_i$ for $\xi_i \to 0$ in $\mathfrak{m}_0^\perp$. Letting $v \in \mathfrak{m}_0^\perp$ be a limit of $\xi_i/\|\xi_i\|$, it remains to check $v \notin W_0$. Suppose not. Since W_0 is compact, we have $\{\exp tv : t \geq 0\} = \exp \mathbb{R}v$. Hence for all $t \geq 0$, $Y \exp tv \subset Y$ as well as $Y \exp(-tv) \subset Y$. Therefore $Y \exp tv = Y$. Hence $\exp v \in M_0$. This is a contradiction, since $v \in \mathfrak{m}_0^\perp$. □

Proposition 9.6 (Translate of Y inside of Y)**.** *If Y is a U-minimal set of $\Gamma\backslash G$ with respect to $\mathrm{RF}\,M$ such that $Y\cap F^*\cap \mathrm{RF}\,M\neq\emptyset$, then there exists an unbounded one-parameter subsemigroup S inside the subgroup $AU^\perp\,\mathrm{C}_2(U)$ such that $YS\subset Y$.*

Proof. Choose $y_0\in Y\cap\mathrm{RF}\,M\cap F^*$. By Lemma 9.4, there exists $g_i\to e$ in $G-\mathrm{N}(U)$ such that $y_0g_i\in Y\cap \mathrm{RF}\,M$. Let $U_0=\{u_t\}$ be a one-parameter subgroup of U as given by Lemma 8.1, with respect to the sequence g_i.

Let $\mathsf{T}_i:=\{u_t\in U_0: y_0g_iu_t\in Y\cap\mathrm{RF}\,M\}$ which is a k-thick subset of U_0. By Lemma 8.1, there exist sequences $u_{t_i}\to\infty$ in T_i, and $u_i\in U$ such that $u_ig_iu_{t_i}\to\alpha$ for some element $\alpha\in AU^\perp\,\mathrm{C}_2(U)-\mathrm{C}_2(U)$. Note that $y_0g_iu_{t_i}\in Y\cap\mathrm{RF}\,M$ converges to some $y_1\in Y\cap\mathrm{RF}\,M$ by passing to a subsequence. Hence as $i\to\infty$, $y_0u_i^{-1}=y_0g_iu_{t_i}(u_ig_iu_{t_i})^{-1}\to y_1\alpha^{-1}$.

So $y_1\alpha^{-1}\in Y$, and hence $Y\alpha^{-1}\subset Y$, since $y_1\in Y\cap\mathrm{RF}\,M$. Since α can be made arbitrarily close to e in Lemma 8.1, the claim follows from Lemma 9.5. ◻

Proposition 9.7 (Translate of Y inside of X)**.** *Let X be a closed H'-invariant set such that $X\cap R\neq\emptyset$. Let $Y\subset X$ be a U-minimal subset with respect to R, and assume that there exists $y\in Y\cap R$ and a sequence $g_n\to e$ in $G-H'$ such that $yg_n\in X$ for all n. Then there exists some non-trivial $v\in U^\perp$ such that $Yv\subset X$.*

Proof. Let $\mathfrak{h}:=\mathrm{Lie}\,H'$. We may write $g_n=r_nh_n$ where $h_n\in H'$ and $r_n\in\exp\mathfrak{h}^\perp$. By replacing g_n with $g_nh_h^{-1}$, we may assume $g_n=r_n$. If $r_n\in U^\perp$ for some n, then the claim follows since $y_0r_n\in X$ and hence $Yr_n\subset X$. Hence we assume that $r_n\notin U^\perp$ for all n. We have from (2.5) $\mathfrak{h}^\perp\cap\mathrm{Lie}(\mathrm{N}(U))=\mathrm{Lie}\,U^\perp$. Hence $r_n\notin\mathrm{N}(U)$ for all n. Therefore there exists a one-parameter subgroup $U_0=\{u_t\}<U$ such that $r_n\notin\mathrm{N}(U_0)$. Let $\mathsf{T}=\{t\in\mathbb{R}: yu_t\in R\}$. Since $y\in R$, it follows that T is a k-thick subset of $\mathbb{R}$ by the assumption on R. Hence, by Lemma 8.3, there exists $t_n\in\mathsf{T}$ such that $u_{t_n}^{-1}r_nu_{t_n}\to v$ for some non-trivial $v\in U^\perp$. Observe $(yu_{t_n})(u_{t_n}^{-1}r_nu_{t_n})=yr_nu_{t_n}\in X$. Passing to a subsequence, $yu_{t_n}\to y_0$ for some $y_0\in Y\cap R$, and hence $y_0v\in X$. It follows $Yv\subset X$. ◻

For a one-parameter subgroup $V=\{v_t:t\in\mathbb{R}\}$ and a subset $I\subset\mathbb{R}$, the notation V_I means the subset $\{v_t:t\in I\}$.

Lemma 9.8. *Let X be a closed AU-invariant set of $\Gamma\backslash G$, and V be a one-parameter subgroup of $U^\perp$. Assume that $R:=X\cap\mathrm{RF}\,M\cap F^*$ is non-empty and compact. If $x_0V_I\subset X$ for some $x_0\in R$ and a closed interval I containing 0, then X contains a V-orbit of a point in R.*

Proof. Choose a sequence $a_n\in A$ such that $\liminf_{n\to\infty}a_nV_Ia_n^{-1}$ contains a subsemigroup V^+ of V as $n\to\infty$. Then $(x_0a_n^{-1})(a_nV_Ia_n^{-1})=x_0V_Ia_n^{-1}\subset X$. By passing to a subsequence, we have $x_0a_n^{-1}$ converges to some $x_1\in\mathrm{RF}\,M$; so $x_1V^+\subset X$. Since R is compact, so is $\overline{x_0A}\cap F^*$, which implies that $x_1\in\overline{x_0A}\cap F^*$. Since x_1 belongs to the open set F^*, it follows $x_1v_s\in F^*$ for all sufficiently small $s\in\mathbb{R}$. By Lemma 3.4, this implies that $x_1v_sU\cap\mathrm{RF}\,M\neq\emptyset$ for some $s>0$ with $v_s\in V^+$. Note that $(x_1v_sU)(v_s^{-1}V^+)=x_1UV^+\subset X$. Choose $x_2\in x_1v_sU\cap\mathrm{RF}\,M\subset X\cap\mathrm{RF}\,M\cap F^*$. Then $x_2(v_s^{-1}V^+)\subset X$. Similarly as before, let $a_n\in A$ be a sequence such that $\liminf_{n\to\infty}a_n(v_s^{-1}V^+)a_n^{-1}=V$ and such that $x_2a_n^{-1}$ converges to some $x_3\in R$. From $(x_2a_n^{-1})(a_nv_s^{-1}V^+a_n^{-1})=x_2v_s^{-1}V^+a_n^{-1}\subset X$, we conclude that $x_3V\subset X$. This finishes the proof. ◻

Proposition 9.9. *Let X be a closed H'-invariant set. Assume that $R := X \cap F^* \cap \mathrm{RF}\, M$ is a non-empty compact set, and let $Y \subset X$ be a U-minimal subset with respect to R. Suppose that there exists $y \in Y \cap R$ such that $X - yH'$ is not closed. Then there exist an element $z \in R$ and a non-trivial connected closed subgroup $V < U^\perp$ such that $zUV \subset X$.*

Proof. Since $X - yH'$ is not closed, there exists a sequence $g_n \to e$ in $G - H'$ such that $yg_n \in X$ for all $n \geq 1$. By Lemma 9.8, it suffices to find $x_0 \in R$ and a one-parameter subgroup $V < U^\perp$ such that $x_0 V_I \subset X$ for some interval $I < \mathbb{R}$ containing 0. It follows from Propositions 9.6 and 9.7 that $Yv_0 \subset X$ and $YS \subset Y$ where $v_0 \in U^\perp - \{e\}$ and S is an unbounded one-parameter subsemigroup of $AU^\perp\,\mathrm{C}_2(U)$. By Lemma 2.2, S is either of the form

(1) $S = \{\exp(t\xi_V)\exp(t\xi_C) : t \geq 0\}$, or
(2) $S = \{\big(v\exp(t\xi_A)v^{-1}\big)\exp(t\xi_C) : t \geq 0\}$

for some $\xi_A \in \mathrm{Lie}(A) - \{0\}, \xi_C \in \mathrm{Lie}(\mathrm{C}_2(U)), \xi_V \in \mathrm{Lie}(V) - \{0\}$, and $v \in U^\perp$.
Case (1)**:** Since X is $H'(U)$-invariant, we may assume $YS \subset X$ with $\xi_C = 0$; so the claim follows.
Case (2)**:** Set $Y_0 := Y\,\mathrm{C}_2(U)$. It is easy to check that Y is a $U\,\mathrm{C}_2(U)$-minimal subset of X with respect to R. First suppose that $v = e$. Let $A^+ := \{\exp(t\xi_A) : t \geq 0\}$. Since $YS \subset Y$ and $\xi_C \in \mathrm{Lie}(\mathrm{C}_2(U))$, it follows that $Y_0 A^+ \subset Y_0$. Choose $y \in Y \cap R$, and let $a_n \to \infty$ be a sequence in A^+. Since R is compact and A-invariant, ya_n converges to some $z_0 \in R$ by passing to a subsequence. Since $Y_0 A^+ \subset Y_0$, we have $z_0 \in Y_0 \cap R$. Since $\liminf a_{-n}A^+ = A$, we get $z_0 A \subset Y_0$. Since $z_0 AU\,\mathrm{C}_2(U) = z_0 U\,\mathrm{C}_2(U)A$, and Y_0 is $U\,\mathrm{C}_2(U)$-minimal with respect to R, we obtain $Y_0 A \subset Y_0$. Since v_0 commutes with $\mathrm{C}_2(U)$, we also get $Y_0 v_0 \subset X$. Therefore $Y_0 A v_0 \subset Y_0 v_0 \subset X$. By the A-invariance of X, it follows $Y_0(Av_0A) \subset X$. Since Av_0A contains some V^+, the claim follows. Next suppose $v \neq e$. Since $\mathrm{C}_2(U)$ commutes with v, it follows that $Y_0 v A^+ v^{-1} \subset Y_0$. Since X is A-invariant, we get $Y_0(vA^+v^{-1})A \subset Y_0 A \subset X$. Set $V := \exp \mathbb{R}(\log v)$. Since $vA^+v^{-1}A$ contains V_I for some interval I containing 0 for any subsemigroup A^+ of A. we get $Y_0 V_I \subset X$, finishing the proof. ☐

10. Closures of orbits inside ∂F and non-homogeneity

Let U be a connected closed subgroup of $\check{H} \cap N$ and set $H := H(U)$ as before. Then $\partial F = \mathrm{BF}\, M \cdot \check{V}^+ \cdot H'(U)$ and $\partial F \cap \mathrm{RF}\, M = \mathrm{BF}\, M \cdot \mathrm{C}(H(U))$. In this section, we classify closures of $xH(U)$ and xAU for $x \in \partial F - \mathrm{RF}\, M$ (Thm. 10.5); they are never homogeneous.

Theorem 10.1. *If $x = zc \in \mathrm{BF}\, M \cdot \mathrm{C}(H(U))$ with $z \in \mathrm{BF}\, M$ and $c \in \mathrm{C}(H(U))$. Then*

(1) *$\overline{xU} = xL$ for some $L \in \mathcal{Q}_U$ contained in $c^{-1}\check{H}c$;*
(2) *$\overline{xH(U)} = xL$ for some $L \in \mathcal{L}_U$ contained in $c^{-1}\check{H}c$, and for any $y \in \mathscr{G}(U, xL)$, $\overline{yU} = xL$;*
(3) *$\overline{xAU} = \overline{xH(U)}$.*

Proof. Since x is contained in the compact homogeneous space $xc^{-1}\check{H}c$, the claims (1) and (2) are special cases of Ratner's theorem [27], which were also proved by Shah independently [32]. So we only need to discuss the proof of (3). We show that $\overline{xAU} = xL$ where L is given by (2). If $U = L \cap N$, then the claim follows from Theorem 12.1. Suppose that U is a proper subgroup of $L \cap N$. Since $\overline{xAU}(K \cap H(U)) = \overline{xH(U)} = xL$ and $\mathscr{S}(U, xL) \cdot (K \cap H(U))$ is a proper subset of xL (cf. Lemma 4.15), there exists $y \in \overline{xAU} \cap \mathscr{G}(U, xL)$. Hence (3) follows from (2). □

Lemma 10.2. *Let $V^+ \subset N$ be a one-parameter subsemigroup which is not contained in $\check{H}$. Then $V^+H(U)$ is a closed subset of G.*

Proof. Since the product map $A \times N \to AN$ is a diffeomorphism and AN is closed, the product subset AW is closed in G for any closed subset W of N. Hence AUV^+ is a closed subset of AN. We use Iwasawa decompositions $H(U) = UA(K \cap H(U))$, and the fact that $AV^+ = V^+A$ in order to write $V^+H(U) = AUV^+(K \cap H(U))$. Hence the conclusion follows from compactness of $K \cap H(U)$. □

Lemma 10.3. *Let $V^+ \subset N$ be as in Lemma 10.2. If $g_i \in \check{H}$ is a sequence such that $g_iv_ih_i$ converges for some $v_i \in V^+$ and $h_i \in H(U)$ as $i \to \infty$, then, after passing to a subsequence, there exists $p_i \in AU$ such that g_ip_i converges to an element of $\check{H}$ as $i \to \infty$.*

Proof. We write $g_i = \tilde{k}_i\tilde{a}_i\tilde{n}_i \in (K \cap \check{H})A(N \cap \check{H})$ and $h_i = u_ia_ik_i \in UA(K \cap H(U))$. Since $K \cap \check{H}$ and $K \cap H(U)$ are compact, we may assume without loss of generality that $\tilde{k}_i = k_i = e$ for all i. Observe that $g_iv_ih_i = \tilde{a}_i\tilde{n}_iv_iu_ia_i = \tilde{a}_ia_i(a_i^{-1}\tilde{n}_iu_ia_i)(a_i^{-1}v_ia_i)$ where $\tilde{a}_ia_i \in A$, $a_i^{-1}\tilde{n}_iu_ia_i \in N \cap \check{H}$, and $a_i^{-1}v_ia_i \in V^+$. Since $g_iv_ih_i$ converges as $i \to \infty$ and the product map $A \times (N \cap \check{H}) \times V^+ \to G$ is an injective proper map, it follows that all three sequences $\tilde{a}_ia_i$, $a_i^{-1}\tilde{n}_iu_ia_i$ and $a_i^{-1}v_ia_i$ are convergent as $i \to \infty$. Noting that $g_iu_ia_i = \tilde{a}_i\tilde{n}_iu_ia_i = \tilde{a}_ia_i(a_i^{-1}\tilde{n}_iu_ia_i)$, it remains to set $p_i := u_ia_i$ to finish the proof. □

For $z \in \mathrm{BF}\, M$, $\pi(z\check{H}\check{V}^+\check{H}) = \pi(z\check{H}\check{V}^+)$ is the closure of a Fuchsian end, of the form $S_0 \times [0, \infty)$ where $S_0 = \pi(z\check{H})$.

Lemma 10.4. *Let zL be a closed orbit contained in $\mathrm{BF}\, M$ for some $L \in \mathcal{L}_U$ contained in $\check{H}$, and $V^+ \subset N$ be a one-parameter subsemigroup such that $\check{H}V^+ = \check{H}\check{V}^+$. Then both $zLV^+H(U)$ and zLV^+ are closed.*

Proof. Without loss of generality, we assume $z = [e]$. Let B denote the component of Ω such that $\mathrm{hull}(\partial B) = \pi(\check{H})$ for the projection map $\pi : G \to \mathbb{H}^d$. Since $\check{H}V^+ = \check{H}\check{V}^+$, we have $\pi(\check{H}V^+\check{H}) = \mathrm{hull}\, \overline{B}$. Note that if $\gamma(\mathrm{hull}(B)) \cap \mathrm{hull}(B) \neq \emptyset$ for $\gamma \in \Gamma$, then $\gamma \in \check{H} \cap \Gamma = \mathrm{Stab}_\Gamma(B)$. Suppose that $\gamma_i\ell_iv_ih_i$ converges to some element $g \in G$ where $\gamma_i \in \Gamma$, $\ell_i \in L$, $v_i \in V^+$ and $h_i \in H(U)$. Since $\pi(\gamma_i\ell_iv_ih_i) \in \Gamma\, \mathrm{hull}\, \overline{B}$, and $\Gamma\, \mathrm{hull}\, \overline{B}$ is a closed subset of $\mathbb{H}^d$, we have $\pi(g) \in \Gamma\, \mathrm{hull}\, \overline{B}$. Without loss of generality, we may assume $\pi(g) \in \mathrm{hull}\, \overline{B}$ by replacing γ_i by $\gamma\gamma_i$ for some $\gamma \in \Gamma$ if necessary.

We claim that by passing to a subsequence, $\gamma_i \in \check{H} \cap \Gamma$. Let $\mathcal{O}$ be a neighborhood of $\pi(g)$ such that $\mathcal{O} \cap \Gamma\, \mathrm{hull}\, \overline{B} \subset \mathrm{hull}\, \overline{B}$; such $\mathcal{O}$ exists, since $d(\mathrm{hull}(\gamma B), \mathrm{hull}(B)) \geq \eta$ for all $\gamma \in \Gamma - (\check{H} \cap \Gamma)$ where $\eta > 0$ is given in (3.5). By passing to a subsequence, we may assume that $\pi(\gamma_i\ell_iv_ih_i) \in \mathcal{O}$. Since $\pi(\ell_iv_ih_i) \in \mathrm{hull}\, \overline{B}$ for all i, it follows that $\pi(\gamma_i\ell_iv_ih_i) \in \mathrm{hull}\, \overline{B}$ for all n. Therefore $\gamma_i \in \check{H} \cap \Gamma$. Applying Lemma 10.3 to the sequence $(\gamma_i\ell_i)v_ih_i \to g$, there exists $p_i \in AU$ such that $\gamma_i\ell_ip_i \to h$ in $\check{H}$ as $i \to \infty$. Since ΓL is closed, we have $h \in \Gamma L$. Since $p_i^{-1}v_ih_i \in AUV^+H(U) = V^+H(U)$ and

$$\lim_{i \to \infty} p_i^{-1}v_ih_i = h^{-1}g, \tag{10.1}$$

we have $h^{-1}g \in V^+H(U)$ by Lemma 10.2. Therefore, $g = h(h^{-1}g) \in \Gamma L V^+ H(U)$. This proves that $\Gamma L V^+ H(U)$ is closed. Note that in the above argument, if $h_i = e$ for all i, then $h^{-1}g = \lim p_i^{-1} v_i \in AUV^+$. Hence $g = h(h^{-1}g) \in \Gamma LAUV^+ = \Gamma LV^+$. This proves that ΓLV^+ is closed. □

Note that $x \in \mathrm{RF}_+ M - \mathrm{RF}\, M \cdot H(U)$ if and only if $x \in (\mathrm{RF}_+ M \cap \partial F_{H(U)}) - \mathrm{BF}\, M \cdot \mathrm{C}(H(U))$.

Theorem 10.5. *Let $x \in \mathrm{RF}_+ M - \mathrm{RF}\, M \cdot H(U)$. Then there exist a compact orbit $zL \subset \mathrm{BF}\, M$ with $L \in \mathcal{L}_U$, an element $c \in \mathrm{C}(H(U))$ and a one-parameter subsemigroup $V^+ \subset N$ with $\check{H}V^+ = \check{H}\check{V}^+$ such that $\overline{xH(U)} = zLV^+H(U)c$ and $\overline{xAU} = zLV^+c$. Moreover the closure of the geodesic plane $\pi(xH(U))$ is diffeomorphic to a properly immersed submanifold $S \times [0, \infty)$ where $S = \pi(zL)$ is a compact geodesic plane inside* $\mathrm{BF}\, M$.

Proof. We write $x = z_0 v c$ for some non-trivial $v \in \check{V}^+$, $z_0 \in \mathrm{BF}\, M$ and $c \in \mathrm{C}(H(U))$. Without loss of generality, we may assume $c = e$. By Theorem 10.1, $\overline{z_0 U} = z_0 v_0^{-1} L v_0$ where $L \in \mathcal{L}_U$ is contained in $\check{H}$ and $v_0 \in \check{H} \cap N$. Hence $\overline{xH(U)}$ contains $zL(v_0 v)H(U)$ for $z := z_0 v_0^{-1} \in \mathrm{BF}\, M$. Set $V^+ := \{\exp t(\log(v v_0)) : t \geq 0\}$. Note that V^+ is contained in $A(v_0 v)A \cup \{e\}$, and hence $zL \cup zLv_0 v H(U) = zLV^+H(U)$ and $\check{H}V^+ = \check{H}\check{V}^+$ since $v \neq e$.

Since $\overline{xH(U)} \subset zL \cup zL(v_0 v)H(U)$, and zL lies in the closure of $zL(v_0 v)H(U)$, the claim (1) follows since $zLV^+H(U)$ is closed by Lemma 10.4. For the claim (2), note that $\overline{xAU} \supset \overline{z_0 U} vA = zLV^+$. By Lemma 10.4, zLV^+ is AU-invariant and closed. Since $x \in zLV^+$, we conclude $\overline{xAU} = zLV^+$.

To see the last claim, observe that $\pi(zLV^+H(U)) = \pi(zLV^+AU) = \pi(zLV^+)$ since $V^+AU = AUV^+$, and $AU < L$. Since $\check{H}V^+ = \check{H}\check{V}^+$, and $\pi(zL)$ is a compact geodesic plane (without boundary) in $\pi(z\check{H})$, we get $\pi(z\check{H}V^+) \simeq \pi(z\check{H}) \times [0, \infty)$ and $\pi(zLV^+) \simeq \pi(zL) \times [0, \infty)$. □

Remark 10.6. An immediate consequence of Theorem 10.5 is that if $P \subset M$ is a geodesic plane such that $P \cap \operatorname{core} M = \emptyset$ but $\overline{P} \cap \operatorname{core} M \neq \emptyset$, then P is not properly immersed in M and $\overline{P}$ is a properly immersed submanifold with non-empty boundary.

11. Density of almost all U-orbits

Let $\Gamma < G = \mathrm{SO}^\circ(d, 1)$ be a Zariski dense convex cocompact subgroup. The action of N on $\mathrm{RF}_+ M$ is minimal, and hence any N-orbit is dense in $\mathrm{RF}_+ M$ [34]. Given a non-trivial connected closed subgroup U of N, there exists a dense U-orbit in $\mathrm{RF}_+ M$ [22]. In this section, we deduce from [23] and [22] that almost every U-orbit is dense in $\mathrm{RF}_+ M$ with respect to the Burger-Roblin measure in the case of a convex cocompact hyperbolic manifold with Fuchsian ends (Cor. 11.4).

The critical exponent $\delta = \delta_\Gamma$ of Γ is defined to be the infimum $s \geq 0$ such that the Poincare series $\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))}$ converges for any $o \in \mathbb{H}^d$. It is known that δ is equal to the Hausdorff dimension of the limit set Λ and $\delta = d - 1$ if and only if Γ is a lattice in G [33]. Denote by m^{BR} the N-invariant Burger-Roblin measure supported on $\mathrm{RF}_+ M$; it is characterized as a unique locally finite Borel measure supported on $\mathrm{RF}_+ M$ (up to a scaling) by ([6], [28], [34]). We won't give an explicit formula of this measure as we will only use the fact that its support is equal to $\mathrm{RF}_+ M$, together with the following theorem: recall that a locally finite

U-invariant measure μ is ergodic if every U-invariant measurable subset has either zero measure or zero co-measure, and is conservative if for any measurable subset S with positive measure, $\int_U 1_S(xu)du = \infty$ for μ-almost all x, where du denotes the Haar measure on U.

Theorem 11.1 ([23], [22])**.** *Let $U < N$ be a connected closed subgroup, and let Γ be a convex cocompact Zariski dense subgroup of G. Then m^{BR} is U-ergodic and conservative if $\delta > \text{co-dim}_N(U)$.*

Lemma 11.2. *Suppose that $\Gamma_1 < \Gamma_2$ are convex cocompact subgroups of G with $[\Gamma_1 : \Gamma_2] = \infty$. Then $\delta_{\Gamma_1} < \delta_{\Gamma_2}$.*

Proof. Note that a convex cocompact subgroup is of divergent type ([33], [28]). Hence the claim follows from [8, Proposition 9] if we check that $\Lambda_{\Gamma_1} \neq \Lambda_{\Gamma_2}$. If $\Lambda := \Lambda_{\Gamma_1} = \Lambda_{\Gamma_2}$, then their convex hulls are the same, and hence the convex core of the manifold $\Gamma_i\backslash\mathbb{H}^d$ is equal to $\Gamma_i\backslash\text{hull}(\Lambda)$, which is compact. Since we have a covering map $\Gamma_1\backslash\text{hull}(\Lambda) \to \Gamma_2\backslash\text{hull}(\Lambda)$, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. □

Lemma 11.3. *If $\Gamma\backslash\mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends, then $\delta > d-2$.*

Proof. If Γ is a lattice, then $\Lambda = \mathbb{S}^{d-1}$ and $\delta = d-1$. If $\Gamma\backslash\mathbb{H}^d$ is a convex cocompact hyperbolic manifold with non-empty Fuchsian ends, then Γ contains a cocompact lattice Γ_0 in a conjugate of $\mathrm{SO}(d-1,1)$ whose limit set is equal to ∂B_i for some i. Now $[\Gamma : \Gamma_0] = \infty$; otherwise, $\Lambda = \partial B_i$. Hence $\delta > \delta_{\Gamma_0} = d-2$ by Lemma 11.2. □

Corollary 11.4. *Let $M = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $U < N$ be any non-trivial connected closed subgroup. Then for m^{BR}-almost every $x \in \mathrm{RF}_+ M$, $\overline{xU} = \mathrm{RF}_+ M$.*

Proof. Without loss of generality, we may assume that $U = \{u_t\}$ is a one-parameter subgroup. By Lemma 11.3 and Theorem 11.1, m^{BR} is U-ergodic and conservative. Since $\delta > (d-1)/2$, there exists a unique function $\phi_0 \in L^2(M)$ which is an eigenfunction for the Laplace operator with eigenvalue $\delta(d-1-\delta)$, up to a scalar multiple [33]. Moreover ϕ_0 is *positive*. We may regard ϕ_0 as a function on $L^2(\Gamma\backslash G)$ which is K-invariant. Then $\mathsf{m}^{\mathrm{BR}}(\phi_0) = \|\phi_0\|^2 < \infty$ (cf. [14, Lem 6.7]). Hence, applying the Hopf ratio theorem [1] we get that for almost all $x \in \mathrm{RF}_+ M$ and for any continuous function f on $\mathrm{RF}_+ M$ with compact support, $\lim_{T\to\infty} \frac{\int_0^T f(xu_t)dt}{\int_0^T \phi_0(xu_t)dt} = \frac{\mathsf{m}^{\mathrm{BR}}(f)}{\|\phi_0\|^2}$. Therefore almost all U-orbits are dense in $\mathrm{supp}(\mathsf{m}^{\mathrm{BR}}) = \mathrm{RF}_+ M$. □

Since $F^*_{H(U)} \cap \mathrm{RF}_+ M$ is a non-empty open subset, it follows that almost all U-orbits in $F^*_{H(U)} \cap \mathrm{RF}_+ M$ are dense in $\mathrm{RF}_+ M$.

12. Horospherical action in the presence of a compact factor

Fix a non-trivial connected closed subgroup U of N. Consider a closed orbit xL for $x \in \mathrm{RF}\, M$ where $L \in \mathcal{Q}_U$. The subgroup $U = L\cap N$ is a horospherical subgroup of L, which is known to act minimally on $xL \cap \mathrm{RF}_+ M$ provided $L = L_{nc}$. In this section, we extend the U-minimality on xL in the case when L has a compact factor.

Theorem 12.1. *Let $X := xL$ be a closed orbit where $x \in \mathrm{RF}_+ M$, and $L \in \mathcal{Q}_U$. Let $U := L \cap N$. Then the following holds:*

(1) *$X \cap \mathrm{RF}_+ M$ is U-minimal.*
(2) *X is L_{nc}-minimal.*
(3) *If $L \in \mathcal{L}_U$ and $x \in \mathrm{RF}\, M$, then $X \cap \mathrm{RF}\, M$ contains a dense A-orbit.*
(4) *For any non-trivial connected closed subgroup $U_0 < U$, for $\mathsf{m}_X^{\mathrm{BR}}$-almost all $x \in X$, $\overline{xU_0} = X \cap \mathrm{RF}_+ M$.*

The subgroup $L \in \mathcal{Q}_U$ is of the form $v^{-1}H(U)Cv$ where $H(U)C \in \mathcal{L}_U$ and $v \in N$. A general case can be easily reduced to the case where $L \in \mathcal{L}_U$. In the following, we assume $L = H(U)C \in \mathcal{L}_U$. As before, we set

$$H = H(U),\ H' = H'(U), \text{ and } F^* = F^*_{H(U)}$$

and let $\pi_1 : H' \to H$ and $\pi_2 : H' \to \mathrm{C}(H)$ be the canonical projections. In order to define $\mathsf{m}_X^{\mathrm{BR}}$, choose $g \in G$ so that $[g] = x$. If we identify $H \simeq \mathrm{SO}^\circ(k,1)$, then by Proposition 3.8, $S := \pi_1(g^{-1}\Gamma g \cap HC)\backslash \mathbb{H}^k$ is a convex cocompact hyperbolic manifold with Fuchsian ends. Now $\pi_1(g^{-1}\Gamma g \cap HC)\backslash H$ is the frame bundle of S, on which there exists the Burger-Roblin measure as discussed in section 11. In the above statement, the notation $\mathsf{m}_X^{\mathrm{BR}}$ means the C-invariant lift of this measure to $X = xHC$.

We first prove the following, which is a more concrete version of Proposition 9.6 in the case at hand:

Proposition 12.2. *Let X be as in Theorem 12.1. Any U-minimal set Y of X with respect to $\mathrm{RF}\, M$ such that $Y \cap F^* \cap \mathrm{RF}\, M \neq \emptyset$ is A-invariant.*

Proof. Let Y be a U-minimal set of X with respect to $\mathrm{RF}\, M$. Let $y_0 \in Y \cap F^* \cap \mathrm{RF}\, M$. By Lemma 9.4, there exists a sequence $g_i \to e$ in $HC - \mathrm{N}(U)$ such that $y_0 g_i \in Y \cap \mathrm{RF}\, M$ for all $i \geq 1$. Since U is a horospherical subgroup of H and C commutes with H, we can apply Lemma 8.2 to the sequence g_i^{-1} and the sequence of k-thick sets $\mathsf{T}_i := \{u \in U : y_0 g_i u \in Y \cap \mathrm{RF}\, M\}$ of U. This gives us sequences $u_{t_i} \to \infty$ in T_i and $u_i \in U$ such that as $i \to \infty$, $u_{t_i}^{-1} g_i u_i \to a$ for some non-trivial element $a \in A$. Since $y_0 u_{t_i}$ converges to some $y_1 \in Y \cap \mathrm{RF}\, M$ by passing to a subsequence, we have $y_1 a = \lim (y_0 u_{t_i})(u_{t_i}^{-1} g_i u_i) \in Y$. Since $\overline{y_1}U = Y$, we get $Ya \subset Y$. Since a can be made arbitrarily close to e by Lemma 8.2, there exists a subsemigroup A_+ of A such that $YA_+ \subset Y$ by Lemma 9.5. Moreover, for any $a \in A_+$, $Ya \cap \mathrm{RF}\, M \neq \emptyset$ as $\mathrm{RF}\, M$ is A-invariant. Therefore, $Ya = Y$. It follows that $Ya^{-1} = Y$ as well. Hence Y is A-invariant. ☐

Proof of Theorem 12.1. First suppose that $xL \cap F^* \neq \emptyset$. We may then assume $x \in F^* \cap \mathrm{RF}\, M$. Let Y be a U-minimal set of X with respect to $\mathrm{RF}\, M$. If Y were contained in ∂F, then $Y \subset \partial F \cap \mathrm{RF}\, M$. Since $\mathrm{Stab}_L(x)$ is Zariski dense in L by the definition of $\mathcal{L}_U$, it follows from [5, Lemma 4.13] that $X \cap \mathrm{RF}_+ M$ is AU-minimal. Therefore we have $\overline{YA} = X \cap \mathrm{RF}_+ M$ and hence X has to be contained in the closed A-invariant subset $\partial F \cap \mathrm{RF}\, M$ as well, yielding a contradiction. Therefore, $Y \cap F^* \cap \mathrm{RF}\, M \neq \emptyset$.

Hence, by Proposition 12.2, Y is A-invariant. Therefore the claim (1) follows from the AU-minimality of $X \cap \mathrm{RF}_+ M$ if $x \in F^*$. Now suppose $xL \subset \partial F$. In this case, it suffices to consider the case when U is a proper subgroup of N; otherwise $L = G$ and has no compact factor. Hence we may assume without loss of generality that $U \subset \check{H} \cap N$. As xL is closed, Theorem 10.5 implies that $xL \subset \mathrm{BF}\, M \cdot \mathrm{C}(H(U))$. Hence by modifying x by an element of $\mathrm{C}(H(U))$, we may assume that X is contained in a compact homogeneous space of $\check{H} = \mathrm{SO}^\circ(d-1,1)$, which is the frame bundle of a convex cocompact hyperbolic manifold with empty Fuchsian ends. Therefore the claim (1) follows from the previous case of $x \in F^*$, since $F^* = \mathrm{RF}\, M$ in the finite volume case.

Claim (2) follows from (1) since $\mathrm{RF}_+ MH$ is closed, and $X \subset \mathrm{RF}_+ MH$.

For the claim (3), it suffices to show that the A action on $X \cap \mathrm{RF}\, M$ is topologically transitive (cf. [7]). Let x, $y \in X \cap \mathrm{RF}\, M$ be arbitrary, and $\mathcal{O}$, $\mathcal{O}'$ be open neighborhoods of e in H. The set $UU^tA(M \cap H)$ is a Zariski open neighborhood of e in H where U^t is the expanding horospherical subgroup of H for the action of A. Choose an open neighborhood Q_0 of e in U, and an open neighborhood P_0 of e in $U^tA(M \cap H)$ such that $Q_0P_0 \subset \mathcal{O}$.

We claim that $xQ_0A \cap y\mathcal{O}' \neq \emptyset$, which implies $x\mathcal{O}A \cap y\mathcal{O}' \neq \emptyset$. Suppose that this is not true. Then $xQ_0A \subset \Gamma\backslash G - y\mathcal{O}'$ where the latter is a closed set. Now, choose a sequence $a_n \in A$ such that $a_nQ_0a_n^{-1} \to U$ as $n \to \infty$, and observe $xa_n^{-1}(a_nQ_0a_n^{-1}) = xQ_0a_n^{-1} \subset \Gamma\backslash G - y\mathcal{O}'$. Passing to a subsequence, $xa_n^{-1} \to x_0$ for some $x_0 \in \mathrm{RF}\, M$, and we obtain that x_0U is contained in the closed subset $\Gamma\backslash G - y\mathcal{O}'$. This contradicts the U-minimality of $X \cap \mathrm{RF}_+ M$, which is claim (1). This proves (3). For the claim (4), note that by Corollary 11.4, almost all U_0-orbits in $\pi_1(g^{-1}\Gamma g \cap HC)\backslash H$ are dense in the corresponding $\mathrm{RF}_+ M$-set. It follows that for almost all x, the closure $\overline{xU_0}$ contains a U-orbit of X. Hence (4) follows from the claim (1).

13. Orbit closure theorems: beginning of the induction

Let $G = \mathrm{SO}^\circ(d, 1)$ and $U < N$ be a non-trivial connected proper closed subgroup, and $H(U)$ be its associated simple Lie subgroup of G. Let $\mathcal{L}_U$ and $\mathcal{Q}_U$ be as defined in (4.6) and (4.7). The remainder of the paper is devoted to the proof of the next theorem from which Theorem 1.1 follows:

Theorem 13.1. (1) *For any $x \in \mathrm{RF}\, M$, $\overline{xH(U)} = xL \cap F_{H(U)}$ where xL is a closed orbit of some $L \in \mathcal{L}_U$.*

(2) *Let $x_0\widehat{L}$ be a closed orbit for some $\widehat{L} \in \mathcal{L}_U$ and $x_0 \in \mathrm{RF}\, M$.*
   (a) *For any $x \in x_0\widehat{L} \cap \mathrm{RF}_+ M$, $\overline{xU} = xL \cap \mathrm{RF}_+ M$ where xL is a closed orbit of some $L \in \mathcal{Q}_U$.*
   (b) *For any $x \in x_0\widehat{L} \cap \mathrm{RF}\, M$, $\overline{xAU} = xL \cap \mathrm{RF}_+ M$ where xL is a closed orbit of some $L \in \mathcal{L}_U$.*

(3) *Let $x_0\widehat{L}$ be a closed orbit for some $\widehat{L} \in \mathcal{L}_U$ and $x_0 \in \mathrm{RF}\, M$. Let $x_iL_i \subset x_0\widehat{L}$ be a sequence of closed orbits intersecting $\mathrm{RF}\, M$ where $x_i \in \mathrm{RF}_+ M$, $L_i \in \mathcal{Q}_U$. Assume that no infinite subsequence of x_iL_i is contained in a subset of the form y_0L_0D where y_0L_0 is a closed orbit of $L_0 \in \mathcal{L}_U$ with $\dim L_0 < \dim \hat{L}$ and $D \subset \mathrm{N}(U)$ is a compact subset. Then $\lim_{i\to\infty} (x_iL_i \cap \mathrm{RF}_+ M) = x_0\widehat{L} \cap \mathrm{RF}_+ M$.*

We will prove (1), (2), and (3) of Theorem 13.1 by induction on the co-dimension of U in N and the co-dimension of U in $\widehat{L} \cap N$, respectively.

For simplicity, let us say $(1)_m$ holds, if (1) is true for all U satisfying $\text{co-dim}_N(U) \leq m$. We will say $(2)_m$ (resp. $(2.a)_m$, $(2.b)_m$) holds, if (2) (resp. (a) of (2), (b) of (2)) is true for all U and $\widehat{L}$ satisfying $\text{co-dim}_{\widehat{L}\cap N}(U) \leq m$ and similarly for $(3)_m$.

Base case of $m = 0$. Note that the bases cases $(1)_0$, and $(3)_0$ are trivial, and that $(2)_0$ follows from Theorem 12.1. We will deduce $(1)_{m+1}$ from $(2)_m$ and $(3)_m$

in section 15, and $(2)_{m+1}$ from $(1)_{m+1}$, $(2)_m$, and $(3)_m$ in section 16, and finally deduce $(3)_{m+1}$ from $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_m$ in section 17.

Remark 13.2. When $\text{co-dim}_{\widehat{L}\cap N}(U) \geq 1$ and $\widehat{L} \in \mathcal{L}_U$, we may assume without loss of generality that $U \subset \widehat{L} \cap N \cap \check{H}$ by replacing U and $\widehat{L}$ by their conjugates using an element $m \in M$.

Remark 13.3. In the case when $x \in \partial F_{H(U)}$, Theorem 13.1 (1) and (2) follow from Theorem 10.1, and if $x_0 \in \partial F_{H(U)}$, (3) follows from the work of Mozes-Shah [22]. So the main new cases of Theorem 13.1 are when $x, x_0 \in F^*_{H(U)}$.

Singular U-orbits under the induction hypothesis. Recall the notation $\mathscr{S}(U, x\widehat{L})$ and $\mathscr{G}(U, x\widehat{L})$ from (4.5).

Lemma 13.4. *Suppose that $(2.a)_m$ is true and that for $x \in \mathrm{RF}\, M$, xU is contained in a closed orbit $x\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_U$.*

1. *If $\text{co-dim}_{\widehat{L}\cap N}(U) \leq m+1$, then for any $x_0 \in \mathscr{S}(U, x\widehat{L}) \cap \mathrm{RF}_+ M$, $\overline{x_0U} = x_0L \cap \mathrm{RF}_+ M$ where x_0L is a closed orbit of some subgroup $L < \widehat{L}$ contained in $\mathcal{Q}_U$, satisfying $\dim L_{nc} < \dim \widehat{L}_{nc}$.*
2. *If $\text{co-dim}_{\widehat{L}\cap N}(U) \leq m$, then for any $x_0 \in \mathscr{G}(U, x\widehat{L})$, $\overline{x_0U} = x_0\widehat{L} \cap \mathrm{RF}_+ M$.*

Proof. Suppose that $\text{co-dim}_{\widehat{L}\cap N}(U) \leq m+1$ and that $x_0 \in \mathscr{S}(U, x\widehat{L}) \cap \mathrm{RF}_+ M$. By Proposition 4.13, we get $\overline{x_0U} \subset x_0Q$ for some closed orbit x_0Q where $Q \in \mathcal{Q}_U$ satisfies $\dim Q_{nc} < \dim \widehat{L}_{nc}$. Now $Q = vL_0v^{-1}$ for some $L_0 \in \mathcal{L}_U$ and $v \in U^{\perp}$. We have $x_0Uv = x_0vU \subset x_0vL_0$. Since $\text{co-dim}_{N\cap L_0}(U) = \text{co-dim}_{N\cap Q}(U) \leq m$, by applying $(2)_m$, we get $\overline{x_0vU} = x_0vL \cap \mathrm{RF}_+ M$ for some closed orbit x_0vL where $L \in \mathcal{Q}_U$ is contained in L_0. Therefore $\overline{x_0U} = x_0vLv^{-1} \cap \mathrm{RF}_+ M$. As $vLv^{-1} \in \mathcal{Q}_U$ and $\dim L_{nc} \leq \dim Q_{nc} < \dim \widehat{L}_{nc}$, the claim (1) is proved.

To prove (2), note that by $(2.a)_m$, we get $\overline{x_0U} = x_0L \cap \mathrm{RF}_+ M$ for some closed orbit x_0L with $L \in \mathcal{Q}_U$ such that $L \subset \widehat{L}$. Since $x_0 \in \mathscr{G}(U, x\widehat{L})$, we have $\dim L_{nc} = \dim \widehat{L}_{nc}$. Since $L \subset \widehat{L}$, $L \cap N$ is a horospherical subgroup of $\widehat{L}$. By Theorem 12.1, $L \cap N$ acts minimally on $x\widehat{L}$, and hence $L = \widehat{L}$. ☐

14. Uniform recurrence and additional invariance

The primary goal of this section is to prove Propositions 14.1 and 14.2 in obtaining additional invariances using a sequence converging to a generic point of an intermediate closed orbit; the main ingredient is Theorem 6.15 (Avoidance theorem II).

In this section, we let $U < N$ be a non-trivial connected closed subgroup. We suppose that

- $(2)_m$ and $(3)_m$ are true;
- $x\widehat{L}$ is a closed orbit for some $x \in \mathrm{RF}\, M$, and $\widehat{L} \in \mathcal{L}_U$;
- $\text{co-dim}_{\widehat{L}\cap N}(U) \leq m+1$.

We let $\{U^{(i)}\}$ be a collection of one-parameter subgroups generating U. In the next two propositions, we let X be a closed U-invariant subset of $x_0\widehat{L}$ such that $X \supset xL \cap \mathrm{RF}_+ M$ for some closed orbit xL where $L \in \mathcal{Q}_U$ is a proper subgroup of $\widehat{L}$ and $x \in \bigcap_i \mathscr{G}(U^{(i)}, xL) \cap \mathrm{RF}\, M$.

Proposition 14.1 (Additional invariance I)**.** *Suppose that there exists a sequence $x_i \to x$ in X where $x_i = x\ell_i r_i$ with $x\ell_i \in xL \cap \mathrm{RF}\, M$ and $r_i \in \exp \mathfrak{l}^\perp - \mathrm{N}(U)$. Then there exists a sequence $v_n \to \infty$ in $(L\cap N)^\perp$ such that $xLv_n \cap \mathrm{RF}_+ M \subset X$.*

Proof. Since $r_i \notin \mathrm{N}(U)$, we can fix a one-parameter subgroup $U_0 = \{u_t : t \in \mathbb{R}\}$ in the family $\{U^{(i)}\}$ such that $r_i \notin \mathrm{N}(U_0)$ by passing to a subsequence.

Let E_j, $j \in \mathbb{N}$, be a sequence of compact subsets in $\mathscr{S}(U_0, xL) \cap \mathrm{RF}\, M$ given by Theorem 6.15. Set $z_i := x\ell_i \in xL \cap \mathrm{RF}\, M$. Fix $j \in \mathbb{N}$ and $n \gg 1$. Since $z_i \to x$ and $x \in \mathscr{G}(U_0, xL)$, there exist $i_j \geq 1$ and an open neighborhood $\mathcal{O}_j$ of E_j such that for each $i \geq i_j$, the set $\mathsf{T}_i = \{t \in \mathbb{R} : z_i u_t \in \mathrm{RF}\, M - \mathcal{O}_j\}$, is $2k$-thick by loc. cit. We apply Lemma 8.3 to the sequence T_i. We can find a sequence $t_i = t_i(n) \in \mathsf{T}_i$, $i \geq i_j$ and elements $y_j = y_j(n), v_j = v_j(n)$ satisfying that as $i \to \infty$,

- $z_i u_{t_i} \to y_j \in (\mathrm{RF}\, M \cap xL) - \mathcal{O}_j$;
- $u_{t_i}^{-1} r_i u_{t_i} \to v_j \in (L \cap N)^\perp$ with $n \leq \|v_j\| \leq (2k^2)n$.

So as $i \to \infty$, $x_i u_{t_i} = z_i r_i u_{t_i} \to y_j v_j$ in X. Note that since L is a proper subgroup of $\widehat{L}$, we have co-$\dim_{L\cap N}(U) \leq m$ by Lemma 4.11. If y_j belongs to $\mathscr{G}(U, xL)$, then $\overline{y_j U} v_j = xL \cap \mathrm{RF}_+ M$ by Lemma 13.4(2), and hence $X \supset \overline{y_j v_j U} = \overline{y_j U} v_j = xLv_j \cap \mathrm{RF}_+ M$. Hence the claim follows if $y_j(n) \in \mathscr{G}(U, xL)$ for an infinite subsequence of n's. Now we may suppose that for all $n \geq 1$ and $j \geq 1$, $y_j(n) \in \mathscr{S}(U, xL) \cap \mathrm{RF}_+ M$, after passing to a subsequence. Fix n, and set $y_j = y_j(n)$ and $v_j = v_j(n)$. Then, since $\dim_{L\cap N} U \leq m$, by $(2)_m$, we have

$$\overline{y_j U} = y_j L_j \cap \mathrm{RF}_+ M \tag{14.1}$$

for some closed $y_j L_j$ where $L_j \in \mathcal{Q}_U$ is contained in $\widehat{L}$ and $\dim(L_j)_{nc} < \dim \widehat{L}_{nc}$. Write $L_j = w_j^{-1} L_j' w_j$ for $L_j' \in \mathcal{L}_U$ and $w_j \in U^\perp$. We claim that the sequence $y_j L_j = y_j w_j^{-1} L_j' w_j$ satisfies the hypothesis of $(3)_m$. It follows from the condition $y_j \in (\mathrm{RF}\, M \cap xL) - \mathcal{O}_j$ for all j that no infinite subsequence of $y_j L_j$ is contained in a subset of the form $y_0 L_0 D \subset \mathscr{S}(U, xL)$ where $y_0 L_0$ is closed, $L_0 \subset \mathcal{Q}_U$ and $D \subset \mathrm{N}(U)$ is a compact subset. Hence, by $(3)_m$, we have $\limsup_j y_j L_j \cap \mathrm{RF}_+ M = xL \cap \mathrm{RF}_+ M$. Therefore for each fixed $n \gg 1$ and $y_j = y_j(n)$, $\limsup_j \overline{y_j U} = xL \cap \mathrm{RF}_+ M$. By passing to a subsequence, there exists $u_j \in U$ such that $y_j u_j$ converges to x. As $n \leq \|v_j(n)\| \leq (2k^2)n$, the sequence $v_j(n)$ converges to some $v_n \in (L \cap N)^\perp$ as $j \to \infty$, after passing to a subsequence. Therefore $\limsup_j \overline{y_j(n) v_j(n) U} = \limsup_j \overline{y_j(n) U} v_j(n) \supset \overline{xU} v_n = xLv_n \cap \mathrm{RF}_+ M$ where the last equality follows from Lemma 13.4(2), since co-$\dim_{L\cap N}(U) \leq m$. $\square$

Note that in the above proposition, $y_i = x\ell_i r_i$ is not necessarily in $\mathrm{RF}\, M$, and hence we cannot apply the avoidance theorem 6.15 to the sequence y_i directly. We instead applied it to the sequence $x\ell_i$.

In the proposition below, we will consider a sequence $x_i \to y$ inside $\mathrm{RF}\, M$, and apply Theorem 6.15 to the sequence x_i.

Proposition 14.2 (Additional invariance II)**.** *Suppose that there exists a sequence $x_i \in X \cap \mathrm{RF}\, M - xL \cdot \mathrm{N}(U)$, converging to x as $i \to \infty$. Then there exists a sequence $v_j \to \infty$ in $(N \cap L)^\perp$ such that*

$$xLv_j \cap \mathrm{RF}_+ M \subset X \quad \text{and} \quad xLv_j \cap \mathrm{RF}\, M \neq \emptyset.$$

The same works for $x_i \in \mathrm{RF}\, M - xL \cdot \mathrm{N}(U)$ such that $\limsup x_i U \subset X$.

Proof. Let $x_i \in \mathrm{RF}\, M - xL \cdot \mathrm{N}(U)$ be a sequence converging to x such that $\limsup x_i U \subset X$. Write $x_i = xg_i$ for $g_i \to e$ in $\widehat{L}$. Since L is reductive, we can write $g_i = \ell_i r_i$ where $\ell_i \to e$ in L and $r_i \to e$ in $\exp \mathfrak{l}^\perp$ as $i \to \infty$. By the assumption on x_i, there exists a one-parameter subgroup $U_0 = \{u_t : t \in \mathbb{R}\}$ among $U^{(i)}$ such that $r_i \notin \mathrm{N}(U_0)$ by passing to a subsequence.

For $R > 0$, we set $B(R) := \{v \in (L \cap N)^\perp \cap \widehat{L} : \|v\| \leq R\}$. Fix j and $n \in \mathbb{N}$. Let $E_j, \mathcal{O}_j$ be given by Theorem 6.15 for xL with respect to U_0. Then E_j is of the form $E_j = \bigcup_{i \in \Lambda_j} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$ where $H_i \in \mathscr{H}^\star$ satisfies $\dim(H_i)_{nc} < \dim L_{nc}$ and D_i is a compact subset of $X(H_i, U_0) \cap L$. As $B(2k^2 n) \subset \mathrm{C}(U_0)$, we have $D_j^* := D_j B(2k^2 n)$ is a compact subset of $X(H_i, U_0)$. Hence the following set $\tilde{E}_j := \bigcup_{i \in \Lambda_j} \Gamma\backslash\Gamma H_i D_i^* \cap \mathrm{RF}\, M$ belongs to $\mathcal{E}_{U_0}$ and is associated to the family $\{H_i : i \in \Lambda_j\}$, as defined in (6.3).

Let $\tilde{E}_j' \in \mathcal{E}_{U_0}$ be a compact subset given by Theorem 6.13, which is also associated to the same family $\{H_i : i \in \Lambda_j\}$. Note that for any $z \in \tilde{E}_j'$, the closure $\overline{zU_0}$ is contained in $\Gamma\backslash\Gamma H_i D_i^*$ for some $i \in \Lambda_j$. In particular, $\tilde{E}_j'$ is a compact subset disjoint from $\mathscr{G}(U_0, xL)$. Since $x_i \to x$ and $x \in \mathscr{G}(U_0, xL)$, there exists $i_j \geq 1$ such that $x_i \notin \tilde{E}_j'$ for all $i \geq i_j$. By Theorem 6.13, there exists a neighborhood $\tilde{\mathcal{O}}_j$ of $\tilde{E}_j$ such that for each $i \geq i_j$, the set $\mathsf{T}_i = \{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\, M - \tilde{\mathcal{O}}_j\}$ is $2k$-thick. Applying Lemma 8.3 to T_i, and $r_i \to e$, we can find $t_i = t_i(n) \in \mathsf{T}_i$ such that $u_{t_i}^{-1} r_i u_{t_i} \to v_j$ for some $v_j = v_j(n) \in (L \cap N)^\perp$, with $n \leq \|v_j\| \leq 2k^2 \cdot n$. Passing to a subsequence, $x_i u_{t_i}$ converges to some $\tilde{x}_j(n) \in \mathrm{RF}\, M - \tilde{\mathcal{O}}_j$ as $i \to \infty$. Set $z_i := x\ell_i$, and $\mathcal{O}_j := \tilde{\mathcal{O}}_j B(2k^2 n) \cap xL$. Since $x_i u_{t_i} = z_i u_{t_i}(u_{t_i}^{-1} r_i u_{t_i})$, we have $z_i u_{t_i} \to y_j \in (\mathrm{RF}_+ M \cap xL) - \mathcal{O}_j$ where $y_j = y_j(n) := \tilde{x}_j(n) v_j^{-1}$. We check that $E_j \subset \mathcal{O}_j$ as $B(2k^2 n) B(2k^2 n)$ contains e. It follows that $y_j \notin E_j$. Since $\tilde{x}_j(n) \in \overline{y_j U} v_j \subset X$, we have $\overline{y_j U} v_j \cap \mathrm{RF}\, M \neq \emptyset$. Given these, we can now repeat verbatim the proof of Proposition 14.1 to complete the proof. $\square$

15. $H(U)$-orbit closures: proof of $(1)_{m+1}$

We fix a non-trivial connected proper subgroup $U < N$. Without loss of generality, we may assume $U < N \cap \check{H}$ using a conjugation by an element of M. We set $H = H(U)$, $H' = H'(U)$, $F = F_{H(U)}$, $F^* = F^*_{H(U)}$, and $\partial F = \partial F_{H(U)}$. By the assumption $U < N \cap \check{H}$, we have $\partial F \cap \mathrm{RF}\, M = \mathrm{BF}\, M \cdot \mathrm{C}(H)$.

Lemma 15.1. *Let $x_1 L_1$ and $x_2 L_2$ be closed orbits where $x_1, x_2 \in \mathrm{RF}\, M$, $L_1 \in \mathcal{Q}_U$ and $L_2 \in \mathcal{L}_U$. If $x_1 L_1 \cap \mathrm{RF}\, M \subset x_2 L_2$, then $L_1 \subset L_2$ and $x_1 L_1 \subset x_2 L_2$.*

Proof. Since L_2 contains H, we get that $x_1 L_1 \cap \mathrm{RF}\, M \cdot H \subset x_2 L_2$. Suppose that $x_1 L_1 \cap F^* \neq \emptyset$. We may assume $x_1 \in F^*$. Since $F^* \subset \mathrm{RF}\, MH$, we have $x_1 L_1 \cap F^* \subset x_2 L_2$. Since F^* is open, there exist $g_1, g_2 \in G$ such that $[g_i] = x_i$, and $g_1 L_1 \cap \mathcal{O} \subset g_2 L_2$ for some open neighborhood $\mathcal{O}$ of g_1. It follows that $L_1 \cap g_1^{-1}\mathcal{O} \subset g_1^{-1} g_2 L_2$. Since $e \in g_1^{-1} g_2 L_2$, we have $g_1^{-1} g_2 L_2 = L_2$. Since L_1 is topologically generated by $L_1 \cap g_1^{-1}\mathcal{O}$, we deduce $L_1 \subset L_2$. Since $x_1 L_1 \cap x_2 L_2 \neq \emptyset$, it follows that $x_1 L_1 \subset x_2 L_2$.

Now consider the case when $x_1 L_1 \cap F^* = \emptyset$. In this case, $x_1 L_1 \cap \mathrm{RF}\, M \subset \mathrm{RF}\, M \cap \partial F$. By Theorem 12.1(4), we can assume that $\overline{x_1 U} = x_1 L_1 \cap \mathrm{RF}_+ M$. As x_1 is contained in $\mathrm{BF}\, M \cdot \mathrm{C}(H)$, so is $\overline{x_1 U}$. It follows that $x_1 L_1$ is compact and hence is contained in $\mathrm{RF}\, M$. Hence the hypothesis implies that $x_1 L_1 \subset x_2 L_2$, which then implies $L_1 \subset L_2$ by the same argument in the previous case. $\square$

Lemma 15.2. *Let y_1L_1 and y_2L_2 be closed orbits where $y_1 \in \mathrm{RF}\, M$, $y_2 \in \mathrm{RF}_+ M$, $L_1 \in \mathcal{Q}_U$ and $L_2 \in \mathcal{L}_U$. If $y_1L_1 \subset y_2L_2D$ for some subset $D \subset \mathrm{N}(U)$, then there exists $d \in D$ such that $L_1 \subset d^{-1}L_2d$ and $y_1L_1 \subset y_2L_2d$.*

Proof. By Theorem 12.1(4), we may assume $\overline{y_1U} = y_1L_1 \cap \mathrm{RF}_+ M$. By the assumption, $y_1 = y_2\ell_2 d$ for some $\ell_2 \in L_2$ and $d \in D$. Since $y_2\ell_2 = y_1d^{-1}$ and $\mathrm{N}(U)$ preserves $\mathrm{RF}_+ M$, $y_2\ell_2 \in \mathrm{RF}_+ M$. Hence we may replace y_2 by $y_2\ell_2$, and hence assume that $y_1 = y_2d$. Since

$$y_1L_1 \cap \mathrm{RF}_+ M = \overline{y_2dU} = \overline{y_2U}d \subset y_2L_2d, \tag{15.1}$$

and $F^* \subset \mathrm{RF}_+ MH$, we get $y_1L_1d^{-1} \cap F^* \subset y_2L_2$.

If $y_1L_1d^{-1} \cap F^* \neq \emptyset$, using the openness of F^*, the conclusion follows as in the first part of the proof of Lemma 15.1. Now consider the case when $y_1L_1d^{-1} \cap F^* = \emptyset$. In particular, $y_2 = y_1d^{-1}$ belongs to $\mathrm{RF}_+ M - F^* \subset \mathrm{BF}\, M \cdot \mathrm{N}(U)$ by (3.3). It follows from Theorem 10.1 that $\overline{y_2U} = y_2L_2'$ for some $L_2' \in \mathcal{Q}_U$ contained in L_2. In view of (15.1), we get $y_1L_1 \cap \mathrm{RF}_+ M = y_1d^{-1}L_2'd$. Therefore $d^{-1}L_2'd \subset L_1$. Since $y_1L_1 \cap \mathrm{RF}_+ M$ is $A(L_1 \cap N)$-invariant, it follows that $d^{-1}L_2'd \in \mathcal{L}_U$ and $d^{-1}L_2'd \cap N = L_1 \cap N$. As a result, $(L_1)_{nc} = d^{-1}(L_2')_{nc}d$. By Lemma 4.11, we get that $L_1 = d^{-1}L_2'd \subset d^{-1}L_2d$ and that $y_1L_1 = y_2L_2'd \subset y_2L_2d$. □

The following proposition says that the classification of H'-orbit closures yields the classification of H-orbit closures:

Proposition 15.3. *Let $x \in \mathrm{RF}\, M$, and assume that there exists $U < \tilde{U} < N$ such that $xH'(\tilde{U})$ is closed, and $\overline{xH'} = xH(\tilde{U}) \cdot \mathrm{C}(H) \cap F$. Then there exists a closed subgroup $C < \mathrm{C}(H(\tilde{U}))$ such that $\overline{xH} = xH(\tilde{U})C \cap F$.*

Proof. By Proposition 3.8 and Theorem 12.1(2), there exists a closed subgroup $C < \mathrm{C}(H(\tilde{U}))$ such that $H(\tilde{U})C \in \mathcal{L}_U$ and $X := xH(\tilde{U})C$ is a closed $H(\tilde{U})$-minimal subset. In particular, $\overline{xH} \subset X \cap F$. Now, by Theorem 12.1(3), there exists $y \in X$ such that $\overline{yA} = X \cap \mathrm{RF}\, M$. Since C is contained in $\mathrm{C}(H)$ and $\overline{xH} \cdot \mathrm{C}(H) = \overline{xH'} = xH(\tilde{U}) \cdot \mathrm{C}(H) \cap F$, there exists $c_0 \in \mathrm{C}(H)$ such that $yc_0 \in \overline{xH}$. Since $\overline{yA}c_0 = \overline{yc_0A} \subset \overline{xH}$ and $c_0 \in \mathrm{C}(H)$, it follows $Xc_0 \cap \mathrm{RF}\, M \subset \overline{xH} \subset X$. Applying Lemma 15.1, we get $Xc_0 = \overline{xH} = X$. □

In the rest of this section, fix $m \in \mathbb{N} \cup \{0\}$ and assume that $1 \leq \text{co-dim}_N(U) = m + 1$. In order to describe the closure of $xH(U)$, in view of Theorem 10.1, we assume that $x \in F^* \cap \mathrm{RF}\, M$. By Proposition 15.3, it suffices to show that

$$\overline{xH'} = xL\, \mathrm{C}(H) \cap F \tag{15.2}$$

for some closed orbit xL for some $L \in \mathcal{L}_U$.

In the rest of this section, we set $X := \overline{xH'}$ and assume that xH' is not closed, i.e., $X \neq xH'$. We also assume that $(2)_m$ holds in the entire section.

Lemma 15.4 (Moving from $\mathcal{Q}_U$ to $\mathcal{L}_U$)**.** *If $x_0L \cap \mathrm{RF}_+ M \subset X$ for some closed orbit x_0L with $x_0 \in \mathrm{RF}\, M$, and $L \in \mathcal{Q}_U - \mathcal{L}_U$, then $x_1\widehat{L} \cap \mathrm{RF}_+ M \subset X$ for some closed orbit $x_1\widehat{L}$ with $x_1 \in \mathrm{RF}\, M$, and $\widehat{L} \in \mathcal{L}_U$ with $\dim(\widehat{L} \cap N) > \dim(L \cap N)$. Moreover, x_1 can be taken to be any element of the set $\limsup_{t\to+\infty} x_0ua_{-t}$ for any $u \in U$.*

Proof. By (4.8), we can write $L = v^{-1}\widehat{L}v$ for some $\widehat{L} \in \mathcal{L}_U$ and $v \in (\widehat{L} \cap N)^{\perp}$. As $L \notin \mathcal{L}_U$, we have $v \neq e$. Set $\widehat{U} := \widehat{L} \cap N$. Note that $x_0v^{-1}\widehat{U}Av \subset x_0L \cap \mathrm{RF}_+ M$, as $\widehat{U}A < \widehat{L}$. Since X is A-invariant, $x_0 v^{-1}\widehat{U}AvA \subset X$. Let V^+ be the unipotent one-parameter subsemigroup contained in AvA, and let V be the one-parameter subgroup containing V^+. Then $x_0 v^{-1}V^+\widehat{U} \subset X$. Since $x_0 A \subset \mathrm{RF}\, M$ and $\mathrm{RF}\, M$ is compact, $\limsup_{t\to+\infty} x_0 a_{-t}$ is not empty. Now let x_1 be any limit of $x_0 u a_{-t_n}$ for some sequence $t_n \to \infty$ and $u \in U$. Since $v^{-1}V^+$ is an open neighborhood of e in V, $\liminf_{n\to\infty} a_{t_n} v^{-1}V^+ a_{-t_n} = V$. Note that as $u \in \widehat{U}$, $x_0 u a_{-t_n}(a_{t_n} v^{-1}\widehat{U}V^+ a_{-t_n}) = x_0 v^{-1}\widehat{U}V^+ a_{-t_n} \subset X$. As a result, we obtain that $x_1\widehat{U}V \subset X$ and hence $x_1\widehat{U}VA \subset X$. Since $\text{co-dim}_N(\widehat{U}V) \le m$, the claim follows from by $(2.a)_m$. □

Proposition 15.5. *If $R := X \cap F^* \cap \mathrm{RF}\, M$ accumulates on ∂F, i.e., there exists $x_n \in R$ converging to a point in ∂F, then $X \supset x_0 L \cap \mathrm{RF}_+ M$ for some closed orbit $x_0 L$ with $x_0 \in F^* \cap \mathrm{RF}\, M$ and $L \in \mathcal{L}_U$ such that* $\dim(L\cap N) > \dim U$.

Proof. There exists $x_n \in R$ which converges to some $z \in \mathrm{BF}\, M \cdot \mathrm{C}(H)$ as $n \to \infty$. We may assume $z \in \mathrm{BF}\, M$ without loss of generality, since R is $\mathrm{C}(H)$-invariant. We claim that $X \cap R$ contains $z_1 v$ where $z_1 \in \mathrm{BF}\, M$ and $v \in \check{V} - \{e\}$. Write $x_n = z h_n r_n$ for some $h_n \in \check{H}$ and $r_n \in \exp \check{\mathfrak{h}}^\perp$, where $\check{\mathfrak{h}}^\perp$ denotes the $\mathrm{Ad}(\check{H})$-complementary subspace to $\mathrm{Lie}(\check{H})$ in $\mathfrak{g}$. Since $x_n \in F^*$ and $z \in \mathrm{BF}\, M$, it follows that $r_n \notin \mathrm{C}(H)$ for all large n. By (2.1) and (2.5), we have $\mathrm{N}(U) \cap \exp(\check{\mathfrak{h}}^\perp \cap \mathcal{O}) \subset \check{V}\,\mathrm{C}(H)$ for a small neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$. Therefore, if $r_n \in \mathrm{N}(U)$ for some n, then the $\check{V}$-component of r_n should be non-trivial. Hence by Theorem 10.1, $X \supset \overline{z h_n U} r_n = z h_n L r_n$ for some $L \in \mathcal{Q}_U$ contained in $\check{H}$. Note that $x_n = z h_n r_n \in F^*$ and that $r_n^{-1} L r_n \in \mathcal{Q}_U - \mathcal{L}_U$, since $r_n \in \check{V} - \{e\}$. Hence the claim follows from Lemma 15.4. Now suppose that $r_n \notin \mathrm{N}(U)$ for all n. Then there exists a one-parameter subgroup $U_0 = \{u_t\} < U$ such that $r_n \notin \mathrm{N}(U_0)$. Applying Lemma 8.3, with a sequence of k-thick subsets $\mathsf{T}(x_n) := \{t \in \mathbb{R} : x_n u_t \in \mathrm{RF}\, M\}$, we get a sequence $t_n \in \mathsf{T}(x_n)$ such that $u_{t_n}^{-1} r_n u_{t_n}$ converges to non-trivial element $v \in \check{V}$. Since $z h_n u_{t_n} \in z\check{H}$ and $z\check{H}$ is compact, the sequence $z h_n u_{t_n}$ converges to some $z_1 \in z\check{H}$, after passing to a subsequence. Then

$$z_1 v = \lim (z h_n u_{t_n})(u_{t_n}^{-1} r_n u_{t_n}) \in X \cap \mathrm{RF}\, M. \tag{15.3}$$

Since $z_1 \in \mathrm{BF}\, M$ and $v \in \check{V} - \{e\}$, $z_1 v \in \mathrm{RF}\, M$ implies that $z_1 v \in F^*$, and hence $z_1 v \in R$. This proves the claim.

Now by Theorem 10.1, $\overline{z_1 U} = z_1 L$ for some $L \in \mathcal{Q}_U$ contained in $\check{H}$, and hence $X \supset \overline{z_1 v U} = \overline{z_1 U} v = (z_1 v)(v^{-1} L v)$. Since $v \in \check{V} - \{e\}$, $v^{-1} L v \notin \mathcal{L}_U$. Therefore, by Lemma 15.4, it suffices to prove that there exists $u \in U$ such that

$$(F^* \cap \mathrm{RF}\, M) \cap \limsup_{t\to+\infty} z_1 u v a_{-t} \neq \emptyset. \tag{15.4}$$

Let $g_1 \in G$ be such that $z_1 = [g_1]$, and set $A_{(-\infty,-t]} := \{a_{-s} : s \ge t\}$ for $t > 0$. Since $z_1 v \in F^* \cap \mathrm{RF}\, M$, the sphere $(gvU)^- \cup g^+$ intersects $\Lambda - \bigcup_i \overline{B_i}$ non-trivially. Let $u \in U$ be an element such that $(gvu)^- \in \Lambda - \bigcup_i \overline{B_i}$. As $z_1 v u \in \mathrm{RF}\, M$, $\pi(zuvA) \subset \mathrm{core}\, M$. Take $\varepsilon > 0$ small enough so that the ε-neighborhoods of $\mathrm{hull}\, B_j$'s are mutually disjoint. If (15.4) does not hold for $z_1 u v$, then there exists $t > 1$ such that the geodesic ray $\pi(z_1 v u A_{(-\infty,-t]})$ is contained in the ε-neighborhood of $\partial\, \mathrm{core}\, M$ (cf. proof of Lemma 7.8). As $\pi(g_1 u v A_{(-\infty,-t]})$ is connected, there exists B_j such that $\pi(g_1 u v A_{(-\infty,-t]})$ is contained in the ε-neighborhood of $\mathrm{hull}\, B_j$. This implies that $(g_1 u v)^- \in \partial B_j$, yielding a contradiction. This proves (15.4). □

Proposition 15.6. *The orbit xH' is not closed in F^*.*

Proof. Suppose that xH' is closed in F^*. Since we are assuming that xH' is not closed in F, $\overline{xH'}$ contains some point $y \in \partial F$. Since $\partial F = \operatorname{BF} M\check{V}^+ \operatorname{C}(H)$, we may assume $y \in \operatorname{BF} M\check{V}^+$. Write $y = zv$ where $z \in \operatorname{BF} M$ and $v \in \check{V}^+$. If $v \neq e$, $\overline{zvH'}$ intersects $\operatorname{BF} M$ by Theorem 10.5. Therefore $\overline{xH'}$ always contains a point of $\operatorname{BF} M$, say z. Let $x_n \in xH'$ be a sequence converging to a point z. Since $xH' \subset F^*$, there exist $k_n \in H \cap K$ converging to some $k \in H \cap K$ such that $x_n k_n \in xH' \cap \operatorname{RF}_+ M$ and $x_n k_n \to zk$. Then $zk \in \operatorname{BF} M \cdot H' = \operatorname{BF} M \operatorname{C}(H)$. Since $x_n k_n \in \operatorname{RF} M \cdot U$ by Lemma 3.4, there exists $u_n \in U$ such that $x_n k_n u_n$ belongs to $\operatorname{RF} M$ and converges to a point in ∂F by Lemma 7.6. Hence $X \cap F^* \cap \operatorname{RF} M$ accumulates on ∂F. Now the claim follows from Proposition 15.5. □

This proposition implies that

$$(X - xH') \cap (F^* \cap \operatorname{RF} M) \neq \emptyset. \tag{15.5}$$

Roughly speaking, our strategy in proving $(1)_{m+1}$ is first to find a closed L-orbit x_0L such that $x_0L \cap F$ is contained in X for some $L \in \mathcal{L}_U$. If $X \neq x_0L\operatorname{C}(H) \cap F$, then we enlarge x_0L to a bigger closed orbit $x_1\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ for some $\widehat{U}$ properly containing U, such that $x_1\widehat{L} \cap F$ is contained in X.

It is in the enlargement step where Proposition 14.1 (Additional invariance I) is a crucial ingredient of the arguments. In order to find a sequence x_i accumulating on a generic point of x_0L satisfying the hypothesis of the proposition, we find a closed orbit x_0L with a base point x_0 *in* $F^* \cap \operatorname{RF} M$, and enlarge it to a bigger closed orbit, again based at a point *in* $F^* \cap \operatorname{RF} M$. The advantage of having a closed orbit xL with $x \in F^* \cap \operatorname{RF} M$ is that any U_0-generic point in $xL \cap \operatorname{RF} M$ can be approximated by a sequence of $\operatorname{RF} M$-points in $F^* \cap xL$ by Lemma 7.3. The enlargement process must end after finitely many steps because of dimension reason.

Finding a closed orbit of $L \in \mathcal{L}_U$ in X.

Proposition 15.7. *There exists a closed orbit x_0L with $x_0 \in F^* \cap \operatorname{RF} M$ and $L \in \mathcal{L}_U$ such that $x_0L \cap \operatorname{RF}_+ M \subset X$.*

Proof. Let $R := X \cap F^* \cap \operatorname{RF} M$. If R is non-compact, the claim follows from Proposition 15.5. Now suppose that R is compact. By $(2.a)_m$, it is enough to show that X contains an orbit $z\widehat{U}$, and hence $z\widehat{U}A$, for some $\widehat{U} < N$ properly containing U and $z \in R$. By Proposition 9.9, it suffices to find a U-minimal subset $Y \subset X$ with respect to R and a point $y \in Y \cap R$ such that $X - yH'$ is not closed. If xH' is not locally closed, then take any U-minimal subset Y of X with respect to R. If $Y \cap R \subset xH'$, then choose any $y \in Y \cap R$. Then $X - yH' = X - xH'$ cannot be closed, as xH' is not locally closed. If $Y \cap R \not\subset xH'$, then choose $y \in (Y \cap R) - xH'$. Then $X - yH'$ contains xH' and hence cannot be closed. If xH' is locally closed, then $X - xH'$ is a closed H'-invariant subset which intersects R non-trivially. So we can take a U-minimal subset $Y \subset X - xH'$ with respect to R. Take any $y \in Y \cap R$. Then $X - yH'$ is not closed. □

Enlarging a closed orbit of $L \in \mathcal{L}_U$ in X.

Proposition 15.8. *Assume that $(3)_m$ holds as well. Suppose that there exists a closed orbit x_0L for some $x_0 \in F^* \cap \operatorname{RF} M$ and $L \in \mathcal{L}_U$ such that*

$$x_0L \cap \operatorname{RF}_+ M \subset X \text{ and } X \neq x_0L \cdot \operatorname{C}(H) \cap F. \tag{15.6}$$

Then there exists a closed orbit $x_1\widehat{L}$ *for some* $x_1 \in F^* \cap \mathrm{RF}\, M$*, and* $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ *for some* $\widehat{U} < N$ *with* $\dim \widehat{U} > \dim(L \cap N)$ *such that* $x_1\widehat{L} \cap \mathrm{RF}_+ M \subset X$.

Proof. Note that if $X \subset x_0L \cdot \mathrm{C}(H)$, then $X = x_0L \cdot \mathrm{C}(H) \cap F$. Therefore we assume that $X \not\subset x_0L \cdot \mathrm{C}(H)$. First note that the hypothesis implies that $L \neq G$, and hence $\text{co-dim}_{L\cap N}(U) \leq m$. Let $U_-^{(1)}, \cdots, U_-^{(\ell)}$ be one-parameter subgroups generating U. Similarly, let $U_+^{(1)}, \cdots, U_+^{(\ell)}$ be one-parameter subgroups generating U^+. By Theorem 12.1, $\bigcap_{i=1}^{\ell} \mathscr{G}(U_\pm^{(i)}, x_0L) \neq \emptyset$. Therefore without loss of generality, we can assume

$$x_0 \in \bigcap_{i=1}^{\ell} \mathscr{G}(U_\pm^{(i)}, x_0L). \tag{15.7}$$

Let us write $L = H(\tilde{U})C$ for some $\tilde{U} < N$ and a closed subgroup C of $\mathrm{C}(H(\tilde{U}))$. Note from the hypothesis that we have $(x_0L \cap \mathrm{RF}_+ M) \cdot H' \subset X$. Observe that (15.6) implies that $x \notin x_0L \cdot H' = x_0L \cdot \mathrm{C}(H)$. Since $C < \mathrm{C}(H)$, we have $x \notin x_0H(\tilde{U})$. Now choose a sequence $w_i \in H'$ such that $xw_i \to x_0$, as $i \to \infty$. Write $xw_i = x_0g_i$ where $g_i \to e$ in $G - LH'$. Let us write $g_i = \ell_i r_i$ where $\ell_i \in L$, and $r_i \in \exp \mathfrak{l}^\perp$. In particular, $r_i \notin \mathrm{C}(H)$. Let $x_i = x_0\ell_i$, so that $x_ir_i \in X$.

We claim that we can assume that $x_i \in \mathrm{RF}\, M \cap x_0L$, $r_i \notin \mathrm{C}(H)$, and $x_ir_i \in X$. Since $x_0 \in F^*$, by Lemma 7.3, we can find $w_i' \to w' \in H$ such that $x_0\ell_iw_i' \in \mathrm{RF}\, M$, and $x_0w' \in \bigcap_{i=1}^{\ell} \mathscr{G}(U_\pm^{(i)}, x_0L)$; hence $\overline{x_0w'U} = x_0L \cap \mathrm{RF}_+ M$. Writing $x_i' = x_0\ell_iw_i'$ and $r_i' = w_i'^{-1}r_iw_i'$, we have $x_i'r_i' = xw_iw_i' \in X$, where $x_i' \to x_0w'$ in $x_0L \cap \mathrm{RF}\, M$, and $r_i' \to e$ in $\exp \mathfrak{l}^\perp$. Since F^* is H'-invariant, we have $x_0w' \in F^*$. Since F^* is open and $x_0w' \in F^*$, it follows that $x_i' \in X \cap \mathrm{RF}\, M \cap F^*$ for sufficiently large i. Note that $r_i' \notin \mathrm{C}(H)$, as $r_i \notin \mathrm{C}(H)$. This proves the claim.

We may assume $r_i \notin \mathrm{N}(U)$ for all i, up to switching the roles of U and U^+, by Lemma 2.4. Note that $x_i \to x_0$ in $\mathrm{RF}\, M \cap x_0L$ and x_0 satisfies (15.7). As we are assuming $(2)_m$, and $(3)_m$, we may now apply Proposition 14.1 to the sequence $x_0\ell_ir_i \to x_0$ to obtain a non-trivial element $v \in \tilde{U}^\perp$ such that $x_0Lv \cap \mathrm{RF}_+ M \subset X$. Since $x_0 \in F^* \cap \mathrm{RF}\, M$, it follows from Lemma 7.12 that there exist $x_2 \in F^* \cap \mathrm{RF}\, M$ and a connected closed subgroup $\widehat{U} < N$ properly containing $L \cap N$ such that $x_2\widehat{U}A \subset X$. Since $\text{co-dim}_N(\widehat{U}) \leq m$, it remains to apply $(2.a)_m$ to finish the proof of the proposition. □

Proof of $(1)_{m+1}$**.** Combining Propositions 15.7 and 15.8, we now prove:

Theorem 15.9. *If* $(2)_m$ *and* $(3)_m$ *are true, then* $(1)_{m+1}$ *is true.*

Proof. Recall that we only need to consider the case $X = \overline{xH'}$ where $x \in F^*$ and xH' is not closed in F^*. By Proposition 15.7, there exists $x_0 \in F^* \cap \mathrm{RF}\, M$ and $L \in \mathcal{L}_U$ such that x_0L is closed and $x_0L \cap \mathrm{RF}_+ M \subset X$. Since X is H'-invariant, it follows

$$(x_0L \cap \mathrm{RF}_+ M) \cdot H' \subset X. \tag{15.8}$$

Note that $(x_0L \cap \mathrm{RF}_+ M) \cdot H' = x_0L \cdot \mathrm{C}(H) \cap F$ is a closed set. We may assume the inclusion in (15.8) is proper, otherwise we have nothing further to prove. Then by Proposition 15.8, there exists $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ for some $\widehat{U} < N$ properly containing $L \cap N$, and a closed orbit $x_1\widehat{L}$ with $x_1 \in F^* \cap \mathrm{RF}\, M$ such that $x_1\widehat{L} \cap \mathrm{RF}_+ M \subset X$. If $(x_1\widehat{L} \cap \mathrm{RF}_+ M) \cdot \mathrm{C}(H) \neq X$, then we can apply Proposition 15.8 on $x_1\widehat{L} \cap \mathrm{RF}_+ M \subset$

X, as $\mathcal{L}_{\widehat{U}} \subset \mathcal{L}_U$. Continuing in this fashion, the process terminates in a finite step for a dimension reason, and hence $X = (x_1\tilde{L} \cap \mathrm{RF}_+ M) \cdot H' = x_1\tilde{L} \cdot \mathrm{C}(H) \cap F$ for some $\tilde{L} \in \mathcal{L}_U$, completing the proof. □

16. U and AU-orbit closures: proof of $(2)_{m+1}$

In this section, we fix a closed orbit $x_0\widehat{L}$ for $x_0 \in F^*$ and $\widehat{L} \in \mathcal{L}_U$. Let $U < \widehat{L} \cap N$ be a connected closed subgroup with $\text{co-dim}_{\widehat{L}\cap N} U \leq m+1$. By replacing U and $\widehat{L}$ by their conjugates using an element $m \in M$, we may assume that $U \subset \widehat{L} \cap \check{H} \cap N$. We keep the same notation $H, F, \partial F, F^*$ etc from section 15. If $x \in \mathrm{RF}_+ M \cap \partial F$ (resp. if $x \in \mathrm{RF}\, M \cap \partial F$), then $(2.a)$ (resp. $(2.b)$) follows from Theorem 10.1.

We fix $x \in \mathrm{RF}\, M \cap x_0\widehat{L} \cap F^*$, and set

$$X := \overline{xU} \text{ and assume that } X \neq x_0\widehat{L} \cap \mathrm{RF}_+ M. \tag{16.1}$$

This assumption implies that U is a proper connected closed subgroup of $\widehat{L} \cap N$ and hence $\dim(\widehat{L} \cap N) > \dim U \geq 1$.

By Proposition 4.16, either $x_0\widehat{L}$ is compact or $\mathscr{S}(U, x_0\widehat{L})$ contains a compact orbit zL_0 with $L_0 \in \mathcal{L}_U$. If $x_0\widehat{L}$ is compact, then $(2)_{m+1}$ follows from Theorem 10.1. Therefore we assume in the rest of the section that

$$\mathscr{S}(U, x_0\widehat{L}) \text{ contains a compact orbit } zL_0 \text{ with } L_0 \in \mathcal{L}_U. \tag{16.2}$$

Lemma 16.1. *If $(1)_{m+1}$ and $(2)_m$ hold, then $\overline{xAU} \cap \mathscr{S}(U, x_0\widehat{L}) \neq \emptyset$.*

Proof. Since $(1)_{m+1}$ is true, we have $\overline{xH} = xQ \cap F$ for some $Q \in \mathcal{L}_U$ such that xQ is closed. By Lemma 15.1, $Q < \widehat{L}$. It follows from Lemma 4.11 that either $Q = \widehat{L}$ or $\dim(Q \cap N) < \dim(\widehat{L} \cap N)$. Suppose that $Q = \widehat{L}$. By (16.2), there exists a compact orbit $zL_0 \subset \mathscr{S}(U, x_0\widehat{L})$ for some $L_0 \in \mathcal{L}_U$. On the other hand, $x_0\widehat{L} \cap F = \overline{xH} = \overline{xAU}(K \cap H)$. Hence for some $k \in K \cap H$, $zk \in \overline{xAU}$. Since $H \subset L_0$, $zk \in zL_0$. So $\overline{xAU}$ intersects zL_0, proving the claim. If $\dim(Q \cap N) < \dim(\widehat{L} \cap N)$. then $\overline{xAU} \subset xQ \subset \mathscr{S}(U, x_0\widehat{L})$. □

Lemma 16.2. *If $(1)_{m+1}$ and $(2)_m$ hold, then $\overline{xU} \cap \mathscr{S}(U, x_0\widehat{L}) \neq \emptyset$.*

Proof. Since

$$\Big(x_0\widehat{L} \cap \mathrm{RF}_+ M\Big) - F^* \subset \mathscr{S}(U, x_0\widehat{L}), \tag{16.3}$$

it suffices to consider the case when $X := \overline{xU} \subset F^*$. Let $Y \subset X$ be a U-minimal set with respect to $\mathrm{RF}\, M$. Since $Y \subset F^*$, by Proposition 9.6, there exists an unbounded one-parameter subsemigroup S inside $AU^\perp\, \mathrm{C}_2(U) \cap \widehat{L}$ such that $YS \subset Y$. In view of Lemma 2.2, we could remove $\mathrm{C}_2(U)$-component of S so that S is either of the following

- $v^{-1}A^+v$ for a one-parameter semigroup $A^+ \subset A$ and $v \in U^\perp \cap \widehat{L}$;
- V^+ for a one-parameter semigroup $V^+ \subset U^\perp \cap \widehat{L}$,

and $YS \subset X(\mathrm{C}_2(U) \cap \widehat{L})$. Since $\mathscr{S}(U, x_0\widehat{L})$ is invariant by $N\,\mathrm{C}_2(U) \cap \widehat{L}$, it suffices to show that $X(N\,\mathrm{C}_2(U) \cap \widehat{L}) \cap \mathscr{S}(U, x_0\widehat{L}) \neq \emptyset$. If $S = v^{-1}A^+v$, then $Yv^{-1}A^+ \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L})$. Choose $y \in Y$. We may assume that $yv^{-1} \in F^*$ by (16.3). Then, replacing y with an element in yU if necessary, we may assume $yv^{-1} \in \mathrm{RF}\, M \cap F^*$. Choose a sequence $a_n \to \infty$ in A^+. Then $yv^{-1}a_n$ converges to some $y_0 \in \mathrm{RF}\, M$ by passing to a subsequence. Since $\liminf a_n^{-1}A^+ = A$, and $(yv^{-1}a_n)(a_n^{-1}A^+) \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L})$, we obtain that $y_0A \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L})$. Since $\overline{y_0AU} \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L})$ and $\overline{y_0AU}$ meets $\mathscr{S}(U, x_0\widehat{L})$ by Lemma 16.1, the claim follows. Next, assume that $S = V^+$, so that $YV^+ \subset \ X\,\mathrm{C}_2(U) \cap \widehat{L}$. Let $v_n \to \infty$ be a sequence in V^+. We have $Yv_n \subset X \subset F^*$. Together with the fact Yv_n is U-invariant, this implies Yv_n meets $\mathrm{RF}\, M$. Note that $Yv_n(v_n^{-1}V^+) \subset X(\mathrm{C}_2(U) \cap \widehat{L})$. Choose $y_n \in Yv_n \cap \mathrm{RF}\, M$. As $\mathrm{RF}\, M$ is compact, y_n converges to some $y_0 \in \mathrm{RF}\, M$, by passing to a subsequence, and hence $y_0UV \subset X(\mathrm{C}_2(U) \cap \widehat{L})$. Since $\text{co-dim}_N(UV) \le m$, the conclusion follows from $(2)_m$. □

Lemma 16.3. *If $(1)_{m+1}$ and $(2)_m$ hold, then $\overline{xU} \cap \mathscr{S}(U, x_0\widehat{L}) \cap F^* \neq \emptyset$.*

Proof. By Lemma 16.2, there exists $y \in \overline{xU} \cap \mathscr{S}(U, x_0\widehat{L})$. Hence by $(2)_m$, $\overline{yU} = yL \cap \mathrm{RF}_+ M \subset \overline{xU}$ for some $L \in \mathcal{Q}_U$ properly contained in $\widehat{L}$. Consider the collection of all subgroups $L \in \mathcal{Q}_U$ such that $yL \subset \overline{xU}$ for some $y \in \mathrm{RF}_+ M$. Choose L from this collection so that $L \cap N$ has maximal dimension. If $yL \cap F^* \neq \emptyset$, then the claim follows.

Now suppose that $yL \subset \partial F$. As $y \in \mathrm{RF}_+ M \cap \partial F$, we have $y = zv_0c_0$ for some $z \in \mathrm{BF}\, M$, $v_0 \in \check{V}^+$ and $c_0 \in \mathrm{C}(H)$. Since $y \in \overline{xU}$, there exists $u_i \in U$ such that xu_i converges to y as $n \to \infty$. Set $z_i := xu_ic_0^{-1}v_0^{-1} \in \overline{xU}c_0^{-1}v_0^{-1}$ so, $z_i \to z$. As $v_0 \in \check{V}^+$ and hence $v_0^{-1} \in \check{V}^-$ and $xu_i \in F^*$, we have $z_i \in F^* \cap \mathrm{RF}_+ M \subset \mathrm{RF}\, M \cdot U$. By Lemma 7.6, we may modify z_i by elements of U so that $z_i \in \mathrm{RF}\, M$ and z_i converges to some $z_0 \in z\check{H}$. Write $z_i = z_0\ell_ir_i$ for some $\ell_i \in \check{H}$ and $r_i \in \exp \check{\mathfrak{h}}^\perp$ converging to e. Since $z_i \in F^*$ and $z_0\ell_i \in \partial F$, we have $r_i \neq e$. By Theorem 10.1, we have $\overline{z_0\ell_iU} = z_0\ell_iL_i$ for some $L_i \in \mathcal{Q}_U$ contained in $\check{H}$.

Case 1: $r_i \in \mathrm{N}(U)$ **for some** i. Then $\overline{xU} = \overline{z_0\ell_ir_iv_0c_0U} = \overline{z_0\ell_iU}(r_iv_0c_0) = z_0\ell_iL_i(r_iv_0c_0)$. As $\overline{xU} \neq x_0\widehat{L}$ by the hypothesis, it follows that $x \in \mathscr{S}(U, x_0\widehat{L}) \cap F^*$, proving the claim.

Case 2: $r_i \notin \mathrm{N}(U)$ **for all** i. Then there exists a one-parameter subgroup $U_0 < U$ such that $r_i \notin \mathrm{N}(U_0)$ for all i, by passing to a subsequence.

By Lemma 8.3, we can find $u_{t_i} \to \infty$ in U_0 so that $z_iu_{t_i} \in \mathrm{RF}\, M$ and $u_{t_i}^{-1}r_iu_{t_i}$ converges to a non-trivial element $v \in \check{V}$, whose size is strictly bigger than $\|v_0\|$. As $z_0\ell_iu_{t_i}$ is contained in the compact subset $z_0\check{H}$, we may assume that $z_0\ell_iu_{t_i}$ converges to some $z' \in z_0\check{H}$. Hence $z_iu_{t_i} = z_0\ell_iu_{t_i}(u_{t_i}^{-1}r_iu_{t_i}) \to z'v \in \mathrm{RF}\, M \cap \overline{xU}c_0^{-1}v_0^{-1}$. Since $z' \in \mathrm{BF}\, M$ and $z'v \in \mathrm{RF}\, M$, we have $v \in \check{V}^-$.

By Theorem 10.1, $\overline{z'U} = z'Q_1$ for some $Q_1 \in \mathcal{Q}_U$. Since $z'vv_0c_0 \in \overline{xU}$, we get $\overline{xU} \supset z'Q_1(vv_0)c_0$. Since the size of v is larger than the size of v_0, then vv_0 is a non-trivial element of $\check{V}^-$. Since $z'Q_1 \subset \mathrm{BF}\, M$, the closed orbit $z'Q_1(vv_0)c_0$ meets F^*. Hence the claim follows. □

Theorem 16.4. *If $(1)_{m+1}$, $(2)_m$, and $(3)_m$ are true, then $(2)_{m+1}$ is true.*

Proof. We first show $(2.a)_{m+1}$ holds for $X = \overline{xU}$. By Lemma 16.3 and $(2)_m$, there exists a closed orbit yL with $y \in F^*$ and $L \in \mathcal{Q}_U$ such that $\overline{xU} \supset yL \cap \mathrm{RF}_+ M$ and $L \cap N \neq \widehat{L} \cap N$. We choose $L \in \mathcal{Q}_U$ so that $\dim(L \cap N)$ is maximal. Note that $\text{co-dim}_{L\cap N} U \le m$. By Theorem 12.1, we can assume that

$$y \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, yL) \cap F^* \cap \mathrm{RF}\, M \tag{16.4}$$

where $U^{(1)}, \cdots, U^{(\ell)}$ are one-parameter subgroups generating U. As $y \in \overline{xU}$, there exists $u_i \in U$ such that $xu_i \to y$ as $i \to \infty$. Since $y \in F^*$, we can assume $xu_i \in \mathrm{RF}\, M$ after possibly modifying u_i by Lemma 7.6. We will write $xu_i = y\ell_i r_i$ where $\ell_i \in L$ and $r_i \in \exp \mathfrak{l}^\perp \cap \widehat{L}$.
Case 1: $r_i \in \mathrm{N}(U)$ **for some** i. Then $y\ell_i \in \mathrm{RF}_+ M$ and $X = \overline{xu_iU} = \overline{y\ell_iU}r_i$. Since $y\ell_iU \subset yL$, and $\text{co-dim}_{L\cap N}(U) \le m$, we have $X = \overline{y\ell_iUr_i} = y\ell_iL'r_i \cap \mathrm{RF}_+ M$ for some $L' \in \mathcal{Q}_U$, proving the claim.
Case 2: $r_i \notin \mathrm{N}(U)$ **for all** i. By (16.4), we can apply Proposition 14.2 to the sequence $xu_i \to y$ and obtain a sequence $v_j \to \infty$ in $(L\cap N)^\perp$ such that $yLv_j \cap \mathrm{RF}_+ M \subset X$. Since $y \in F^*$, by Lemma 7.10, there exists a one-parameter subgroup $V \subset (L \cap N)^\perp$ such that $y_1(L\cap N)V \subset X$ for some $y_1 \in F^* \cap \mathrm{RF}\, M$. Hence, by $(2)_m$, we get a contradiction to the maximality of $L \cap N$; this proves $(2.a)_{m+1}$.

Now we show $(2.b)_{m+1}$ for the closure $\overline{xAU}$. By $(1)_{m+1}$, we have $\overline{xH} = xL \cap F$ for some $L \in \mathcal{L}_U$ contained in $\widehat{L}$. Hence $\overline{xAU} \subset xL \cap \mathrm{RF}_+ M$. It suffices to show that

$$\overline{xAU} = xL \cap \mathrm{RF}_+ M. \tag{16.5}$$

If $U = L \cap N$, then $\overline{xU} = xL \cap \mathrm{RF}_+ M$ by Theorem 12.1, which implies (16.5). So, suppose that U is a proper closed subgroup of $L \cap N$. Since $\overline{xAU}(K \cap H) = \overline{xH} = xL \cap F$, it follows from Lemma 4.15 that we can choose $y \in \overline{xAU} \cap \mathscr{G}(U, xL)$. By $(2.a)_{m+1}$ and Lemma 13.4, we have $\overline{yU} = xL \cap \mathrm{RF}_+ M$, finishing the proof. □

17. Topological equidistribution: proof of $(3)_{m+1}$

In this section, we prove $(3)_{m+1}$. Let $U < N$ be a non-trivial connected closed subgroup. Let $x_0\widehat{L}$ be a closed orbit for $x_0 \in F^* \cap \mathrm{RF}\, M$ and $\widehat{L} \in \mathcal{L}_U$ such that $\text{co-dim}_{\widehat{L}\cap N}(U) = m+1$. As before we may assume that $U \subset \widehat{L} \cap \check{H} \cap N$. Let $x_iL_i \subset x_0\widehat{L}$ be a sequence of closed orbits intersecting $\mathrm{RF}\, M$ where $x_i \in \mathrm{RF}_+ M$, $L_i \in \mathcal{Q}_U$. We write x_iL_i as $y_iL_iv_i$ where $y_i \in \mathrm{RF}_+ M$, $L_i \in \mathcal{L}_U$, and $v_i \in (L_i \cap N)^\perp \cap \widehat{L}$. Assume that no infinite subsequence of $y_iL_iv_i$ is contained in a subset of the form $y_0L_0D \subset \mathscr{S}(U, x_0\widehat{L})$ where y_0L_0 is a closed orbit for some $L_0 \in \mathcal{L}_U$ and $D \subset \mathrm{N}(U)$ is a compact subset. Let $E = \limsup\limits_{i\to\infty} (y_iL_iv_i \cap \mathrm{RF}_+ M)$. Note that $\liminf_{i\to\infty}(y_iL_iv_i \cap \mathrm{RF}_+ M)$ coincides with the intersection of the subsets $\limsup(y_{i_k}L_{i_k}v_{i_k} \cap \mathrm{RF}_+ M)$ for all infinite subsequences $\{i_k : k \in \mathbb{N}\}$ of $\mathbb{N}$. If the hypothesis of $(3)_{m+1}$ holds for a given sequence $y_iL_iv_i$, then it also holds for all subsequences. Hence to prove $(3)_{m+1}$, it suffices to show that $E = \mathrm{RF}_+ M \cap x_0\widehat{L}$. We note that by $(3)_m$, we may assume that $L_i \cap N = U$ for all i. This in particular implies that each $y_iL_iv_i \cap \mathrm{RF}_+ M$ is U-minimal by Theorem 12.1.

Lemma 17.1. *Assume that $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_m$ are true. Then there exist $y \in F^* \cap \mathrm{RF}\, M$ and $L \in \mathcal{Q}_U$ with $\dim(L \cap N) > \dim U$ such that yL is closed and $E \supset yL \cap \mathrm{RF}_+ M$.*

Proof. By $(2)_m$, it suffices to show that there exist $y_0 \in F^* \cap \mathrm{RF}\, M$ and $\widehat{U} < N$ properly containing U such that $E \supset y_0\widehat{U}$. Suppose that $y_iL_iv_i \subset \partial F$ for infinitely many i. Since $y_iL_iv_i \cap \mathrm{RF}\, M \ne \emptyset$, we may assume $y_iv_i \in z_i\check{H}\,\mathrm{C}(H)$ for some $z_i \in \mathrm{BF}\, M$ by (3.3). Since $L_i \cap N = U$, we get $y_iL_iv_i = \overline{y_iU} \subset z_i\check{H}\,\mathrm{C}(H)$ by Theorem 10.1. This contradicts the hypothesis on $y_iL_iv_i$'s. Therefore by passing to a subsequence, for all i, $y_iL_iv_i \cap \mathrm{RF}_+ M \cap F^* \ne \emptyset$. Since $AU < L_i$ for all i, it follows that $E = \limsup_{i\to\infty}(y_iL_iv_i \cap \mathrm{RF}_+ M)(v_i^{-1}AUv_i)$ By Lemma 7.9, there exists $y_0 \in \limsup_i(y_iL_iv_i \cap \mathrm{RF}_+ M) \cap F^*$. Hence, after passing to a subsequence,

$$y_0 \limsup_{i\to\infty}(v_i^{-1}AUv_i) \subset E. \tag{17.1}$$

If $v_i \to \infty$, then $\limsup_i(v_i^{-1}AUv_i)$ contains $A\widehat{U}$ for some $\widehat{U}$ properly containing U by Lemma 2.3. Therefore, we get the conclusion $y_0\widehat{U} \subset E$ from (17.1). Now suppose that, by passing to a subsequence, v_i converges to some $v \in N \cap \widehat{L}$. Then (17.1) gives $y_0v^{-1}AUv \subset E$. Then by $(2)_{m+1}$, $\overline{y_0v^{-1}AU}$ is of the form $y_0v^{-1}L_0 \cap \mathrm{RF}_+ M$ for some $L_0 \in \mathcal{L}_U$. Hence $E \supset y_0L \cap \mathrm{RF}_+ M$ where $L := v^{-1}L_0v$. If $L \cap N$ contains U properly, this proves the claim. So we suppose that $L \cap N = U$. By Theorem 12.1, we can assume that $y_0 \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, y_0L) \cap F^* \cap \mathrm{RF}\, M$, where $U^{(1)}, \cdots, U^{(\ell)}$ are one-parameter subgroups generating U. By replacing y_i by an element of $y_iL \cap \mathrm{RF}_+ M$, we may assume that $y_iv_i \to y_0$. Furthermore, as $y_0 \in F^* \cap \mathrm{RF}\, M$, for all i sufficiently large, $y_iv_i \in F^* \cap \mathrm{RF}_+ M \subset \mathrm{RF}\, M \cdot U$ (as F^* is open). Hence we can also assume $y_iv_i \in \mathrm{RF}\, M$ by Lemma 7.7. Therefore we may write $y_iv_i = y_0\ell_ir_i$ for some $\ell_i \to e$ in L and non-trivial $r_i \to e$ in $\exp \mathfrak{l}^\perp$.

Suppose that r_i belongs to $\mathrm{N}(U)$ for infinitely many i. Then

$$y_iL_iv_i \cap \mathrm{RF}_+ M = \overline{y_iv_iU} = \overline{y_0\ell_iU}r_i = y_0Lr_i \cap \mathrm{RF}_+ M.$$

Hence $y_iL_iv_ir_i^{-1} \cap \mathrm{RF}_+ M = y_0L \cap \mathrm{RF}_+ M$. In particular, $y_iL_iv_ir_i^{-1} \cap \mathrm{RF}\, M$ is non-empty (as it contains y_0) and is contained in y_0L. By Lemma 15.1, this implies that $y_iL_iv_i \subset y_0Lr_i$. As $r_i \to e$, this contradicts the hypothesis on $y_iL_iv_i$'s. Therefore $r_i \notin \mathrm{N}(U)$ for all i but finitely many. We may now apply Proposition 14.2 and Lemma 7.10 to deduce that E contains an orbit $z_0\widehat{U}$ for some $\widehat{U} < \widehat{L} \cap N$ containing U properly and for some $z_0 \in \mathrm{RF}_+ M \cap F^*$. This proves the claim. □

Theorem 17.2. *If $(1)_{m+1}$, $(2)_{m+1}$, and $(3)_m$ are true, then $(3)_{m+1}$ is true.*

Proof. We claim that $x_0\widehat{L} \cap \mathrm{RF}_+ M = E$. By Lemmas 17.1, we can take a maximal $\widehat{U}$ such that $E \supset y\widehat{U}$ for some $y \in F^* \cap \mathrm{RF}\, M$. By $(2)_m$, we get a closed orbit yL for some $L \in \mathcal{Q}_{\widehat{U}}$ such that $yL \cap \mathrm{RF}_+ M \subset E$. If $L = \widehat{L}$, then the claim is clear. Now suppose that L is a proper subgroup of $\widehat{L}$. This implies that $L \cap N$ is a proper subgroup of $\widehat{L} \cap N$, since $\widehat{L} \cap N$ acts minimally on $x_0\widehat{L} \cap \mathrm{RF}_+ M$ as $\widehat{L} \in \mathcal{L}_U$. By Theorem 12.1, we can assume that $y \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, yL) \cap F^* \cap \mathrm{RF}\, M$, where $U^{(1)}, \cdots, U^{(\ell)}$ are one-parameter subgroups generating U. As $y \in E$, there exists a sequence $x_i \in y_iL_iv_i \cap \mathrm{RF}_+ M$ converging to y, by passing to a subsequence. Since $U = v_i^{-1}L_iv_i \cap N$, we have $x_i \in \mathrm{RF}\, M \cdot U$. By Lemma 7.7, by replacing x_i with x_iu_i for some $u_i \to e$ in U, we may assume $x_i \in \mathrm{RF}\, M$.

We claim that $x_i \notin yL\,\mathrm{N}(U)$. Suppose not, i.e., $x_i = y\ell_ir_i$ for some $\ell_i \in L$ and $r_i \in \mathrm{N}(U)$. Then $y_iL_iv_i \cap \mathrm{RF}_+ M = \overline{x_iU} = \overline{y\ell_iU}r_i \subset yLr_i$. By the assumption on $y_iL_iv_i$'s, this cannot happen as r_i's are bounded. On the other hand, $\dim(L_i \cap N)$ is strictly smaller than $\dim(L \cap N)$, since $L_i \cap N = U$ and $\widehat{U} < L \cap N$, yielding a contradiction. Hence $x_i \notin yL\,\mathrm{N}(U)$.

We can now apply Proposition 14.2 and Lemma 7.10 and deduce that E contains $y_1\widehat{U}V$ for some $y_1 \in F^* \cap \mathrm{RF}\, M$. This is a contradiction to the maximality assumption on $\dim \widehat{U}$. □

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