POLAR DECOMPOSITION FOR P-ADIC SYMMETRIC SPACES

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Abstract. Let G be the group of k-points of a connected reductive k-group and H a symmetric subgroup associated to an involution σ of G. We prove a polar decomposition $G = KAH$ for a symmetric space G/H over any local field k of characteristic not 2. Here K is a compact subset of G and A is a finite union of groups A_i which are the k-points of of a maximal (k, σ)-split torus, one for each H-conjugacy class.

This decomposition is analogous to the well-known polar decomposition $G = KAH$ for a real symmetric space G/H.

1. Introduction

We begin with a short motivation of this article. It is well known that *every non-degenerate quadratic form on* $\mathbb{R}^n$ *can be put into diagonal form by an orthogonal base change.*

This statement is an instance of the so-called polar decomposition $G = KAH$ for a real symmetric space G/H (see section 5.1), with $G = \mathrm{GL}(n, \mathbb{R})$, $H = \mathrm{O}(p, n-p)$, $1 \leq p \leq n$, $K = \mathrm{O}(n)$ and A the subgroup of diagonal matrices of G.

Let k be a non-archimedean local field. It is also well known that *if the residual characteristic of* k *is not* 2*, every non-degenerate quadratic form on* k^n *can be put into diagonal form by a base change preserving the sup norm* (see section 5.2). The aim of this paper is to prove a polar decomposition for any symmetric space G/H over a local field of characteristic not 2

Here is the precise statement of our polar decomposition. Let k be a non-archimedean local field of characteristic not 2, $\mathbf{G}$ a connected reductive k-group, σ a k-involution of $\mathbf{G}$ and $\mathbf{H}$ an open k-subgroup of the group $\mathbf{G}^\sigma$ of σ-fixed points. A k-torus $\mathbf{S}$ of $\mathbf{G}$ is said to be (k, σ)-split if it is k-split and $\sigma(g) = g^{-1}$ for all $g \in \mathbf{S}$.

We use the notation G, H, S,... to denote the groups of k-points of $\mathbf{G}$, $\mathbf{H}$, $\mathbf{S}$,...

By a theorem of Helminck and Wang [HW], there are only finitely many H-conjugacy classes of maximal (k, σ)-split tori of $\mathbf{G}$. Choose a set $\{\mathbf{A}_i : 1 \leq i \leq n\}$ of representatives of H-conjugacy class of maximal (k, σ)-split tori of $\mathbf{G}$ and set $A = \cup_{i=1}^n A_i$.

Theorem 1.1. *There exists a compact subset K of G such that we have*

$$G = KAH\ .$$

the second author is partially supported by NSF grant 0629322.

We remark that all the subgroups A_i, $1 \leq i \leq n$ are needed, up to H-conjugacy, in the above decomposition (Lemma 5.5) and that we can not in general choose K to be a compact subgroup of G.

The proof relies simultaneously on algebraic properties of the symmetric spaces G/H mostly due to Helminck and Wang, as well as on geometric properties of the Bruhat-Tits building X_G of G. In fact, we will deduce Theorem 1.1 from the following geometric statement: (see (4.2) for the definition of σ-apartment):

Theorem 1.2. *There exists a constant $C > 0$ such that any point $x \in X_G$ has distance at most C from some σ-apartment in X_G.*

This paper is organized as follows: in section 2 we recall the construction of the Bruhat-Tits building of G, as well as the polar decomposition for the group G itself which is a consequence of the Bruhat-Tits theory. After recalling in section 3 a few known facts on symmetric spaces over local fields, we prove in section 4 a geometric version of the polar decomposition (Theorem 4.6) from which we deduce Theorem 1.1. We finish in section 5, by giving a few examples and comments. It may be a good idea for a reader to start with this section which is independent of the rest of the article.

We plan to apply our polar decomposition given in Theorem 1.1 to some counting and equidistribution results, with rate of convergence, for S-integral points on symmetric spaces in a forthcoming paper [BO]. That is why we do not work here in the more general setting of a valued field.

After a lecture by the first author in CIRM on this paper, P. Delorme gave him the preprint [DS] with V. Sécherre, written simultaneously and independantly, in which they prove similar results when the residual characteristic is not 2. Their method uses the Bruhat-Tits buildings in a more algebraic way.

2. Bruhat-Tits buildings and polar decomposition

Let k be a non-archimedean local field with a valuation ω and G be the group of k-points of a reductive k-group $\mathbf{G}$. In this section we recall the definition of the Bruhat-Tits building X_G of G, which is a metric space on which G acts properly by isometries with a compact fundamental domain. This metric space is very similar to the Riemannian symmetric space associated to a real reductive group. It is a very important tool since it gives a geometric insight in understanding the algebraic properties of G. It is precisely this insight that we will use to prove our main theorem in this paper. We also recall several properties of X_G that we will need later.

The construction of X_G relies on a cautious construction of certain open compact subgroups of G. All the theory on building presented below is due to Bruhat and Tits ([BT1], [BT2]) but the readers may find references such as [Ti], [Ro2] or [La] shorter and more helpful.

2.1. **Euclidean Buildings.** We first give the axiomatic definition of a building. All our buildings will be Euclidean and discrete.

Let E be a real affine space endowed with an Euclidean distance d_E and let W be a discrete subgroup, generated by (hyperplane) reflections, of the group of affine isometries of E. Let V be the corresponding vector space and ${}^vW \subset \mathrm{GL}(V)$ the finite group given by the linear part of W. Here are a few vocabularies. A *wall* is a hyperplane of E pointwise fixed by a reflection of W. A *root* is a halfspace of E bounded by a wall. A *facet* is an equivalence class in E with respect to the relation: $x \sim y$ if and only if x and y live in the same roots. The *type* of a facet is its W-orbit. A *chamber* is an open facet. A point x in E is *special* if every wall is parallel to a wall containing x. Such a point always exists.

Buildings are always associated to such a pair (E, W). Roughly, they are obtained by gluing together copies of E along convex union of facets in such a way that they satisfy very strong geometric properties. More precisely:

A *building*, modeled on (E, W), is a metric space (X, d) endowed with a *system of apartments*, i.e., a collection of subsets $\mathcal{A}$ of X called *apartments*, and a partition of X into subsets called *facets* such that :
- for each apartment $\mathcal{A}$ there is an isometry from E to $\mathcal{A}$ sending facets to facets,
- any two points of X is contained in at least one apartment,
- for any apartments $\mathcal{A}$ and $\mathcal{A}'$, the intersection $\mathcal{A} \cap \mathcal{A}'$ is a closed convex subset of $\mathcal{A}$ which is a union of facets and there is a facet preserving isometry from $\mathcal{A}$ to $\mathcal{A}'$ which is the identity on $\mathcal{A} \cap \mathcal{A}'$.

As a metric space, every building X is CAT(0)-space (see [KL]). A *translation* of X is an isometry of X which induces a translation in each apartment $\mathcal{A} \simeq E$ of X. The building X is *essential* if the only translation of X is the identity. Denote by V_0 the vector space of translations of X. The quotient $X^{\mathrm{ess}} := V_0 \backslash X$ is naturally an essential building which is called the *essential quotient* of X and X is isometric to the Euclidean product $X^{\mathrm{ess}} \times V_0$.

2.2. **Reductive k-groups.** We now recall the construction of some important compact open subgroups of G (which are called, at least when $\mathbf{G}$ is semisimple, connected and simply-connected, *parahoric subgroups*).

For any k-split torus $\mathbf{S}$ of $\mathbf{G}$, let $X^*(S)$ be the group of k-characters of $\mathbf{S}$, $X_*(S)$ the group of k-co-characters of $\mathbf{S}$, $V(S)$ the real vector space $V(S) = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathbf{T}$ be a maximal k-split torus of $\mathbf{G}$ so that $\dim(\mathbf{T}) = k\text{-rank}(\mathbf{G})$. Let $\mathbf{N}$ and $\mathbf{Z}$ be respectively the normalizer and centralizer of $\mathbf{T}$ in $\mathbf{G}$, and Φ be the root system of $\mathbf{G}$ relative to $\mathbf{T}$. For any root $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the corresponding root subgroup. The group G is generated by N and these root groups U_α [Bo]. For any u in U_α,

$u \neq 1$, there exists a unique element

$$m(u) \in N \cap U_{-\alpha} u U_{-\alpha} \ .$$

The image of $m(u)$ in the group N/Z is the reflection s_α associated with the root α. We denote by ${}^vW$ the group generated by these reflections s_α, $\alpha \in \Phi$, and call it the Weyl group (associated to $\mathbf{T}$). When $\mathbf{G}$ is connected one has ${}^vW = N/Z$.

The normalizer N acts linearly on the real vector space $V(T)$. Choose an N-invariant euclidean structure on $V(T)$. There is a unique morphism ν of the group Z into the vector space $V(T)$ given by the formula

$$\chi(\nu(z)) = -\omega(\chi(z)) \ ,$$

for all $z \in Z$ and $\chi \in X^*(Z)$ where on the left-hand side χ is seen as a linear form on $V(T)$. Hence Z acts by translation via ν on the affine space E underlying $V(T)$. Since N/Z is a finite group, one has $H^2(N/Z, V(T)) = 0$ and hence, this action extends to a morphism $\nu : N \to \mathrm{Aff}(E)$. Such an extension is unique up to a translation. The hyperplane $H(u)$ fixed by $\nu(m(u))$ is defined by the equation $\alpha(x) + \varphi_\alpha(u) = 0$ for some $\varphi_\alpha(u) \in \mathbb{R}$. For $x \in E$, we introduce the set

$$U_{\alpha,x} := \{u \in U_\alpha \ \mid \ u = 1 \ \text{ or } \ \alpha(x) + \varphi_\alpha(u) \geq 0\} \ .$$

By [BT2], $U_{\alpha,x}$ is in fact a subgroup. We also set $N_x := \{n \in N \ \mid \ \nu(n)(x) = x\}$, which is clearly a subgroup.

Let K_x denote the subgroup of G generated by N_x and $U_{\alpha,x}$, $\alpha \in \Phi$. The subgroup K_x is a compact open subgroup of G.

2.3. **Bruhat-Tits Buildings.** The Bruhat-Tits building X_G of G is now defined to be the quotient of $G \times E$ by the equivalence relation :

$$(g, x) \sim (h, y) \Longleftrightarrow \exists n \in N \text{ such that } y = \nu(n)x \ \text{ and } \ g^{-1}hn \in K_x \ .$$

This building is modeled on (E, W) where W is the *affine Weyl group* i.e. the group generated by the affine reflections $\nu(m(u))$ with $u \in U_\alpha$, $u \neq 1$ and $\alpha \in \Phi$.

The apartments of X_G are parametrized by maximal k-split tori of G : the apartment $\mathcal{A}_{T'}$ of X_G corresponding to the torus $\mathbf{T}' = g\mathbf{T}g^{-1}$ for some $g \in G$ is the image of $g \times E$ in X_G.

The construction of X_G does not depend on the choice of a maximal k-split torus $\mathbf{T}$.

The G-action by left translations on the first factor of the product $G \times E$ induces an isometric action of G on X_G, which is proper and co-compact. The stabilizers in G of points in X_G are the conjugates of the subgroups K_x, $x \in E$. Moreover, the kernel of this action is precisely the maximal normal compact subgroup of G.

According to ([Ti] 2.1), the Bruhat-Tits building can be characterized as in the following proposition:

Proposition 2.1. [Ti] *The Bruhat-Tits building X_G of G is the unique G-set containing a subset $\mathcal{A}$ normalized by N such that*
- as an N-set, $\mathcal{A}$ is isomorphic to E,
- every G-orbit in X meets $\mathcal{A}$,
- for each $x \in \mathcal{A} \simeq E$, the stabilizer of a point x contains K_x.

Uniqueness means that any G-set satisfying the above three properties is G-isomorphic to X_G. The G-isomorphism is unique only up to a translation of X_G. The vector space V_0 of translations of X_G is equal to $V(S_0)$ where $\mathbf{S_0}$ is the maximal k-split torus of the center of $\mathbf{G}$.

2.4. **Polar decomposition for G.** We now recall the polar decomposition (also called Cartan decomposition) for the group G. This decomposition is an algebraic corollary of the following geometric fact : G acts strongly transitively on the building X_G i.e., G acts transitively on the set of all pairs $(C, \mathcal{A})$ where C is a chamber and $\mathcal{A}$ is an apartment containing C.

Choose a positive root system Φ^+ in Φ and set

$$Z^+ := \{z \in Z \ | \ \omega(\alpha(z)) \leq 0 \ , \ \forall \alpha \in \Phi^+\}.$$

We also choose a special point x contained in the apartment $\mathcal{A}_T$.

Proposition 2.2. [BT1] *For any non-archimedean local field k and any reductive k-group $\mathbf{G}$, we have*

$$G = K_x Z^+ K_x.$$

The special case of the above decomposition when $G = \mathrm{GL}(n, k)$ has the following interpretation, when we consider the action of G on the set of all ultra-metric norms on k^n.

Recall that a norm N on k^n is called *ultra-metric* if

$$N(x + y) \leq \max(N(x), N(y)) \quad \text{ for all } x,\, y \in k^n.$$

Noting that K_x is any conjugate of $\mathrm{GL}(n, \mathcal{O})$ for the valuation ring $\mathcal{O}$ of k and that Z^+ is a conjugate of the semi-group consisting of all diagonal matrices in valuation decreasing order, Proposition 2.2 for $\mathrm{GL}(n, k)$ implies that *for any two ultra-metric norms N_1 and N_2 on k^n there exists a basis of k^n with respect to which both of N_1 and N_2 are diagonal i.e. of the form $N(x_1, \ldots, x_n) = \sup_{1 \leq i \leq n}(\alpha_i |x_i|)$ with $\alpha_i > 0$.*

3. Structure of symmetric spaces

We collect in this section a few preliminary facts on symmetric spaces due to Helminck and Wang (see [HW] and also [HH]).

Let k be a field with $\mathrm{char}(k) \neq 2$, $\mathbf{G}$ a connected reductive k-group, σ a k-involution of $\mathbf{G}$ and $\mathbf{H}$ an open k-subgroup of $\mathbf{G}^\sigma$.

3.1. **Existence of (k,σ)-split tori.** Recall that a k-torus $\mathbf{S}$ of $\mathbf{G}$ is said to be (k,σ)-split if it is k-split and $\sigma(g) = g^{-1}$ for all g in $\mathbf{S}$.

Here is a criterion for the existence of a (k,σ)-split torus in $\mathbf{G}$ (Proposition 4.3 of [HW]).

Lemma 3.1. [HW] *The following are equivalent:*

(i) *Every (k,σ)-split torus of $\mathbf{G}$ is trivial.*
(ii) *Every k-split torus of $\mathbf{G}$ is pointwise fixed by σ.*
(iii) *Every minimal normal isotropic k-subgroup of $\mathbf{G}$ is pointwise fixed by σ.*

3.2. **Finiteness of (k,σ)-split tori.** We will not need the whole description of the H-conjugacy classes of maximal (k,σ)-split tori of $\mathbf{G}$.

We will only need the following proposition, which can be found in (4.5), (10.3), (6.10) and (6.16) of [HW].

Proposition 3.2. [HW]

(a) *Any maximal k-split torus of $\mathbf{G}$ containing a maximal (k,σ)-split torus of $\mathbf{G}$ is σ-invariant.*
(b) *Any two maximal (k,σ)-split tori of $\mathbf{G}$ are conjugate to each other by an element of G.*
(c) *If k is a local field, the number of H-conjugacy classes of maximal (k,σ)-split tori of $\mathbf{G}$ is finite.*

Hence the maximal (k,σ)-split tori of $\mathbf{G}$ have the same dimension, which is defined to be the k-rank of G/H.

3.3. **Finiteness of σ-invariant k-split tori.** Once again, we will not need the precise description of the H-conjugacy classes of σ-invariant maximal k-split tori of $\mathbf{G}$, but only need the following proposition.

The claim (a) below is a straightforward consequence of Proposition 2.3 of [HW] applied to the centralizer of $\mathbf{T}$. For other claims, we refer to Proposition 2.3, Lemma 2.4 and Corollary 6.16 of [HW]. We recall that every maximal k-split torus of $\mathbf{G}$ is contained in a minimal parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$.

Proposition 3.3. [HW]

(a) *Any maximal σ-invariant k-split torus $\mathbf{T}$ of $\mathbf{G}$ is a maximal k-split torus of $\mathbf{G}$.*
(b) *Any minimal parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$ contains a maximal σ-invariant k-split torus of $\mathbf{G}$, unique up to conjugacy by $H \cap R_u(P)$.*
(c) *If k is a local field, the number of H-conjugacy classes of minimal parabolic k-subgroups is finite. Hence the number of H-conjugacy classes of maximal σ-invariant k-split tori of $\mathbf{G}$ is finite as well.*

When k is local with $\mathrm{char}(k) = 0$, the claims (c) in both Propositions 3.2 and 3.3 are also consequences of the following theorem of Borel and Serre: for every k-group

$\mathbf{L}$ and its k-subgroup $\mathbf{M}$, the number of L-orbits in the k-points of $\mathbf{L}/\mathbf{M}$ is finite (th. 5 of III.4.4 in [Se]).

4. KAH decomposition

To prove in Theorem 1.1, we first interpret it in geometric terms which is possible, using the Bruhat-Tits building.

4.1. **Flats, parallelism and Levi subgroups.** We need a few more results on the geometry of Euclidean buildings.

Let X be an Euclidean building. A *flat* in X is a subset isometric to some Euclidean vector space. A *geodesic* is a one-dimensional flat. The system of apartments of X is *complete* if it is maximal or, equivalently, if every maximal flat is an apartment. In fact, every system of apartments can be extended to a complete system of apartments. Moreover, the system of apartments is complete if and only if every geodesic of X is contained in some apartment ([Pa] Prop. 2.18). In this case every flat in X is contained in some apartment.

By ([BT1] 2.8.4), the system of apartments of the Bruhat-Tits building X_G is complete.

We now suppose that the system of apartments of X is complete. Two flats $\mathcal{F}$ and $\mathcal{F}'$ of X are called *parallel* if their Hausdorff distance is finite i.e. if there exists a constant $C > 0$ such that $d(y, \mathcal{F}') \leq C$ and $d(y', \mathcal{F}) \leq C$ for every $y \in \mathcal{F}$, $y' \in \mathcal{F}'$. Parallelism is an equivalence relation among flats. According to ([KL] 2.3.3 and 4.8.1) , we have

- Two flats $\mathcal{F}$ and $\mathcal{F}'$ are parallel if and only if they are both contained in the same apartment $\mathcal{A}$ and are parallel in $\mathcal{A} \simeq E$.
- For a given flat $\mathcal{F}$, the union of flats $\mathcal{F}'$ parallel to $\mathcal{F}$, which is then the union of apartments containing $\mathcal{F}$, is a sub-building of X, denoted by $X_{\mathcal{F}}$.

The *singular hull* of a flat $\mathcal{F}$ is the largest flat $\mathcal{F}^s$ contained in every apartment containing $\mathcal{F}$. A flat is *singular* if it is equal to its singular hull. By construction, we have $X_{\mathcal{F}} = X_{\mathcal{F}^s}$.

For the Bruhat-Tits building $X = X_G$ of a reductive k-group $\mathbf{G}$, one can describe the sub-building $X_{\mathcal{F}}$ as the Bruhat-Tits building X_L associated to some Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ (cf. [Ro1] 3.10), which we explain below. A *Levi k-subgroup* $\mathbf{L}$ of $\mathbf{G}$ is the centralizer $\mathbf{Z}(\mathbf{S})$ of some k-split torus $\mathbf{S}$ of $\mathbf{G}$. A k-split torus $\mathbf{S}$ in $\mathbf{G}$ is called *singular* if it is obtained as intersection of maximal k-split tori or, equivalently, if it is equal to the maximal k-split torus of the center of some Levi subgroup of $\mathbf{G}$. This gives a bijection between the Levi k-subgroups of $\mathbf{G}$ and the singular k-split tori of $\mathbf{G}$. The *singular hull* of a k-split torus $\mathbf{S}$ is the smallest singular k-split torus containing $\mathbf{S}$.

Let $\mathbf{L}$ be a Levi k-subgroup of $\mathbf{G}$. Note that any maximal k-split torus of $\mathbf{L}$ is also a maximal k-split torus of $\mathbf{G}$. According to ([BT2] 4.2.17), the union of the apartments $\mathcal{A}_T$ of X_G associated to a maximal k-split tori $\mathbf{T}$ of $\mathbf{L}$ is a sub-building isometric to the Bruhat-Tits building X_L of L. This sub-building is also denoted by X_L by abuse of notation. One can also check this claim using the characterization (2.1) of the Bruhat-Tits buildings.

We can summarize the above discussion in the following proposition.

Proposition 4.1. ([BT2], [KL], [Ro1]) *Let k be a non-archimedean local field and $\mathbf{G}$ a reductive k-group.*

(a) *For each singular flat $\mathcal{F}$ in X_G, there exists a unique singular k-split torus $\mathbf{S}$ of $\mathbf{G}$ such that $X_{\mathcal{F}} = X_{Z(S)}$. This induces a bijection between the set of the parallelism classes of singular flats of X_G and the set of all singular k-split tori of $\mathbf{G}$.*

(b) *The parallel singular flats associated to a singular k-split torus $\mathbf{S}$ are exactly the $V(S)$-orbits in $X_{Z(S)}$.*

Remarks - Every apartment of $X_{Z(S)}$ is associated to a maximal k-split torus of the Levi subgroup $\mathbf{Z}(\mathbf{S})$. Not all of them contain $\mathcal{F}$.

- The group $Z(S)$ almost never acts transitively on the set $X_{Z(S)}/V(S)$ of flats parallel to $\mathcal{F}$, but it acts co-compactly on it, since it already acts co-compactly on the building $X_{Z(S)}$.

4.2. **σ-flats.** For the rest of section 4, we suppose as in our initial setting that k is a non-archimedean local field with $\mathrm{char}(k) \neq 2$, $\mathbf{G}$ is a connected reductive k-group, σ is a k-involution of $\mathbf{G}$ and that $\mathbf{H}$ is an open k-subgroup of $\mathbf{G}^\sigma$.

We first remark that the involution σ induces an involution, also denoted by σ, of the Bruhat-Tits building X_G. In fact, let $\mathbf{G}' = \langle\sigma\rangle \ltimes \mathbf{G}$ be the semi-direct product of the group $\langle\sigma\rangle$ of order 2 with $\mathbf{G}$. This is a reductive k-group whose Bruhat-Tits building $X_{G'}$ is the same as X_G. That is why G' acts on X_G.

The following definition allows us to restate Theorem 1.1 in a geometric way.

Definition 4.2. *A σ-flat in X_G is a σ-invariant flat on which σ has only one fixed point. A σ-flat in X_G of maximal dimension is called a σ-apartment.*

We will see below that the maximal dimension of a σ-flat in X_G is equal to the k-rank of G/H. Note that a maximal σ-flat is not always a σ-apartment: for instance, if k-rank$(G) = k$-rank(H), there exists a chamber in X_G of which every point is σ-fixed. Certainly each point in this chamber is a maximal σ-flat but not of maximal dimension unless the k-rank of G/H is zero.

This shows that the union of all σ-apartments of X_G is not equal to X_G in general. However, we will see in Theorem 4.6 that this is not far from being true.

We will need the following two lemmas. The first one is analogous to Lemma 3.1

Lemma 4.3. (a) *The following are equivalent:*

(i) *The* k-rank *of G/H is equal to* 0.

(ii) *Every σ-apartment of X_G is a point.*

(iii) *The involution σ acts trivially on X_G.*

(b) *Any point $x \in X_G$ with $\sigma(x) \neq x$ lies in a one-dimensional σ-flat.*

Proof $(i) \Rightarrow (ii)$ Let $\mathcal{F}$ be a σ-flat of dimension 1. Note that the singular hull $\mathcal{F}^s$ of $\mathcal{F}$ is a σ-invariant flat of X_G. By Proposition 4.1, there exists a singular k-split torus $\mathbf{S}$ such that $X_{\mathcal{F}^s} = X_{Z(S)}$. This torus $\mathbf{S}$ and its centralizer $\mathbf{L}$ is σ-invariant. The building X_L is isometric to a product $X_L^{\text{ess}} \times V(S)$. The group S acts on this product by translations on the Euclidean factor $V(S)$. Since the action of σ on $V(S) \simeq \mathcal{F}^s$ is non-trivial, the action of σ on S is also non-trivial. It follows that the k-rank of G/H is non-zero.

$(ii) \Rightarrow (iii)$ For $x \in X_G$, suppose $\sigma(x) \neq x$. We want to prove that there exists a one dimensional σ-flat $\mathcal{F}$ containing x. This will also prove the claim (b). Since there exists an apartment containing both x and $\sigma(x)$, there exists a geodesic $\mathcal{F}_0$ containing both x and $\sigma(x)$. The points x and $\sigma(x)$ cut this geodesic into three pieces : two rays $\mathcal{F}_-$ and $\mathcal{F}_+$ and a segment $[x, \sigma(x)]$. Let

$$\mathcal{F} := \mathcal{F}_- \cup [x, \sigma(x)] \cup \sigma(\mathcal{F}_-) \ .$$

This path $\mathcal{F}$ is indeed a geodesic since it is locally a geodesic and X_G is a CAT(0)-space. By construction, $\mathcal{F}$ is also σ-invariant. Since $\sigma(x) \neq x$, $\mathcal{F}$ is a one-dimensional σ-flat, as desired.

$(iii) \Rightarrow (i)$ Let $\mathbf{S}$ be a maximal (k, σ)-split torus of $\mathbf{G}$ and $\mathbf{T}$ a maximal k-split torus of $\mathbf{G}$ containing $\mathbf{S}$. By Proposition 3.2 (a), the torus $\mathbf{T}$ is σ-invariant. Hence the corresponding apartment $\mathcal{A}_T$ is also σ-invariant. If the k-rank of G/H is non-zero, the action of σ on $\mathbf{T}$ is non-trivial. Hence the action of σ on $\mathcal{A}_T$ is also non-trivial. This proves the claim. □

4.3. **σ-apartments.** Note that a maximal (k, σ)-split torus of $\mathbf{G}$ may not be a singular k-split torus. Nevertheless, we have the following assertion analogous to Proposition 4.1.

Lemma 4.4. (a) *For any σ-apartment $\mathcal{F}$ of X_G, there exists a unique maximal (k, σ)-split tori $\mathbf{R}$ of $\mathbf{G}$ such that $X_{\mathcal{F}} = X_{Z(R)}$. This induces a bijection between the set of all parallelism classes of σ-apartments of X_G and the set of all maximal (k, σ)-split tori of $\mathbf{G}$*

(b) *For a maximal (k, σ)-split torus $\mathbf{R}$ of $\mathbf{G}$, the parallel σ-apartments $\mathcal{F}$ associated to $\mathbf{R}$ are exactly the $V(R)$-orbits in the building $X_{Z(R)}$ and the subgroup $H \cap Z(R)$ acts co-compactly on the quotient building $X_{Z(R)}/V(R)$.*

(c) *Every apartment of X_G containing a σ-apartment is σ-invariant.*

As an immediate corollary, we obtain:

Corollary 4.5. *The k-rank of G/H is equal to the dimension of the σ-apartments of X_G.*

Proof of Lemma 4.4 We will prove (a), (b) and (c) simultaneously.

Let $\mathcal{F}$ be a σ-apartment of X_G. Let $\mathcal{F}^s$ be the singular hull of $\mathcal{F}$ and $\mathbf{S}$ the unique singular k-split torus of $\mathbf{G}$ such that $X_{Z(S)} = X_{\mathcal{F}^s}$ (see Proposition 4.1). Note that the torus $\mathbf{S}$ and its centralizer $\mathbf{L} := \mathbf{Z}(\mathbf{S})$ are also σ-invariant. Let $\mathbf{R}$ be the maximal (k, σ)-split torus of $\mathbf{S}$. Again by Proposition 4.1, the vector space $V(S)$ acts simply transitively on $\mathcal{F}^s$. Since this action is σ-equivariant, and $\mathcal{F}$ is a σ-flat of maximal dimension in $\mathcal{F}^s$, $\mathcal{F}$ is a $V(R)$-orbit.

Since $\mathcal{F}^s$ is the singular hull of $\mathcal{F}$, the torus $\mathbf{S}$ is also the singular hull of $\mathbf{R}$. It follows that $\mathbf{L}$ is equal to the centralizer of $\mathbf{R}$ and hence $X_{Z(R)} = X_{\mathcal{F}}$.

By the maximality of the σ-flat $\mathcal{F}$, the quotient building $X_L/V(R)$ does not contain any one-dimensional σ-flat. Hence, by Lemma 4.3 (i), $\mathbf{L}/\mathbf{R}$ does not contain any non-trivial (k, σ)-split torus and hence $\mathbf{R}$ is a maximal (k, σ)-split torus in $\mathbf{L}$. Since $\mathbf{L}$ is the centralizer of $\mathbf{R}$, it follows that $\mathbf{R}$ is a maximal (k, σ)-split torus of $\mathbf{G}$. To show the uniqueness, if $\mathbf{R}'$ is any maximal (k, σ)-split torus of $\mathbf{G}$ such that $X_{Z(R')} = X_{\mathcal{F}}$, then $\mathbf{R}'$ is contained in $\mathbf{S}$ and hence in $\mathbf{R}$ as well. By the maximality of $\mathbf{R}'$, $\mathbf{R}' = \mathbf{R}$. This shows the first part of the claim (a).

By Lemma 3.1 (iii), the group $H \cap L$ acts co-compactly on $X_L/V(R)$. This, together with the previous discussion, proves (b). By Lemma 4.3 (iii), the action of σ on the building $X_L/V(R)$ is trivial. Therefore every flat containing $\mathcal{F}$ is σ-invariant. This proves (c).

To finish the proof of (a), letting $\mathbf{R}$ be a maximal (k, σ)-split torus of $\mathbf{G}$, we want to prove that there exists a σ-apartment $\mathcal{F}$ of X_G, unique to parallelism, such that $X_{Z(R)} = X_{\mathcal{F}}$.

The singular hull $\mathbf{S}$ of $\mathbf{R}$ and its centralizer $\mathbf{L} = \mathbf{Z}(\mathbf{S})$, being equal to $\mathbf{Z}(\mathbf{R})$, are σ-invariant. The associated building X_L is also σ-invariant and the action of the vector space $V(S)$ on X_L commutes with σ. Hence the $V(R)$-orbits in X_L are σ-flats any two of which are parallel to each other. Let $\mathcal{F}$ be one of these σ-flats. Since $\mathbf{R}$ is maximal, the reductive k-group $\mathbf{L}/\mathbf{R}$ does not contain any non-trivial (k, σ)-split torus.

By Lemma 4.3 (iii), σ acts trivially on the quotient building $X_L/V(R)$. Hence $\mathcal{F}$ is a σ-flat of maximal dimension, i.e., a σ-apartment. Since the torus $\mathbf{S}$ is the singular hull of $\mathbf{R}$, the $V(S)$-orbit $\mathcal{F}^s$ containing $\mathcal{F}$ is the singular hull of $\mathcal{F}$ and we have $X_{Z(R)} = X_{\mathcal{F}}$.

Finally, if $\mathcal{F}'$ is any σ-apartment such that $X_{Z(R)} = X_{\mathcal{F}'}$, then $\mathcal{F}'$ is contained in a $V(S)$-orbit and hence in a $V(R)$-orbit in X_L. Therefore since $\mathcal{F}'$ is of maximal dimension, $\mathcal{F}'$ is equal to a $V(R)$-orbit in X_L. Hence $\mathcal{F}'$ is parallel to $\mathcal{F}$. This completes the proof of (a).

4.4. **Geometric interpretation.** Here is a geometric reformulation of Theorem 1.1.

For $C > 0$, a subset Y of a metric space X is *C-dense* if every point of X has distance at most C from Y. A subset Y is *quasi-dense* in X if it is C-dense for some $C > 0$.

Theorem 4.6. *Let k be a non-archimedean local field with $\mathrm{char}(k) \neq 2$, $\mathbf{G}$ a connected reductive k-group and σ a k-involution of $\mathbf{G}$. The union of all σ-apartments is quasi-dense in the Bruhat-Tits building X_G.*

We begin by proving the following lemma:

Lemma 4.7. *Either σ acts trivially on X_G or the set $\{x \in X_G \mid \sigma(x) \neq x\}$ is quasi-dense in X_G.*

Remark When the residual characteristic of k is not 2, according to Prasad and Yu [PY], the set X_G^σ of σ-fixed points can be identified with the Bruhat-Tits building X_H of H, when σ-action is non-trivial on X_G. This gives a proof of Lemma 4.7 in this case. We present below a short proof of the above lemma assuming only that char $(k) \neq 2$.

Proof of Lemma 4.7 Suppose that σ-action on X_G is non-trivial. We will show that there exists a constant $C > 0$ such that, for every $x \in X_G$, the ball $B(x, C)$ is not pointwise fixed by σ.

Suppose not. Then there exists a sequence $\{x_n \in X_G\}$ such that the balls $B(x_n, n)$ are pointwise fixed by σ. We may assume that for a fixed $x_0 \in X_G$, $x_n = g_n x_0$ for some $g_n \in G$. Then the element $\sigma_n := g_n \sigma g_n^{-1}$ is an involutive automorphism of G which acts trivially on $B(x_0, n)$. Hence the sequence σ_n sub-converges to the identity in $\mathrm{Aut}(G)$. Hence any compact open subgroup of $\mathrm{Aut}(G)$ contains σ_n for all sufficiently large n. On the other hand, if $\mathrm{char}(k) = 0$, then $\mathrm{Aut}(G)$ contains a compact open subgroup which is torsion free. If the residual characteristic of k is $p \neq 2$, the group $\mathrm{Aut}(G)$, as any group G' of k-points of a k-group $\mathbf{G}'$, contains a compact open subgroup which is a pro-p-group. Since a pro-p-group has no elements of order 2, the order of σ_n is 2, and we obtain a contradiction in either case. □

Proof of Theorem 4.6 We want to prove that there exists a constant $C_G > 0$ such that every point $x \in X_G$ has distance at most C_G from some σ-apartment $\mathcal{F}$ of X_G.

We argue by induction on the dimension of $\mathbf{G}$. Writing, as in section 2, $X_G \simeq X_G^{\mathrm{ess}} \times V(S_0)$ where $\mathbf{S}_0$ is the maximal k-split torus of the center of $\mathbf{G}$, we may assume that $\mathbf{G}$ is semisimple.

If σ acts trivially on X_G then every point of X_G is a σ-apartment in which case our claim trivially holds. Hence we assume that σ does not act trivially on X_G. By Lemma 4.7 and Lemma 4.3 (b), we may then assume that x lies in a one-dimensional σ-flat $\mathcal{F}$. Then x belongs to the sub-building $X_{\mathcal{F}^s}$ where $\mathcal{F}^s$ is the singular hull of $\mathcal{F}$. By Proposition 4.1 and Lemma 4.4, the sub-building $X_{\mathcal{F}^s}$ is equal to the building

X_L associated to some σ-invariant Levi-subgroup $\mathbf{L} \subsetneq \mathbf{G}$ such that every σ-apt in X_L is also an σ-apt in X_G. We apply our induction hypothesis to the reductive subgroup $\mathbf{L}$ to conclude.

Note that the distance on the building X_G^{ess} is unique only up to some scalar factors, one for each quasi-simple isotropic normal k-subgroup of $\mathbf{G}$. The constant C_G will depend on this normalization of those scalar factors. The same is true for $\mathbf{L}$.

Each σ-invariant Levi subgroup $\mathbf{L}$ gives rise to some constant C_L. This constant C_L depends only on the H-conjugacy class of $\mathbf{L}$.

To obtain a finite upper bound for these C_L, we need to know that *there are only finitely many H-conjugacy classes of σ-invariant Levi subgroup of* $\mathbf{G}$. Since a k-split torus contains only finitely many singular k-split sub-tori, this assertion is a direct consequence of the finiteness of the number of H-conjugacy classes of maximal σ-invariant k-split tori of $\mathbf{G}$ (Proposition 3.3). □

4.5. **Orbits of (k,σ)-split tori in X_G.** The following corollary is the link between Theorem 1.1 and Theorem 4.6.

As in the introduction, we choose maximal (k,σ)-split tori $\mathbf{A}_1, \dots \mathbf{A}_n$, which are representatives of H-conjugacy classes of all maximal (k,σ)-split tori of $\mathbf{G}$ (cf. Proposition 3.2). Let A be the union of the groups of k-points A_i, $1 \le i \le n$.

Corollary 4.8. *Let k be a non-archimedean local field with* $\mathrm{char}(k) \neq 2$*, $\mathbf{G}$ a connected reductive k-group, σ a k-involution of $\mathbf{G}$ and $\mathbf{H}$ an open k-subgroup of $\mathbf{G}^\sigma$.*

For any fixed $x_0 \in X_G$, there exists a constant $C > 0$ such that for all $x \in X_G$, there exist $h \in H$ and $1 \le i \le n$ with

$$d(hx, A_i x_0) \le C.$$

In other words, *the union of the H-orbits meeting Ax_0 is quasi-dense in X_G.*

Proof By Theorem 4.6, the set of σ-apartments are C_0-dense in X_G for some $C_0 > 0$. Hence $d(x, x_1) < C_0$ for some x_1 contained in a σ-apartment $\mathcal{F}$. Then by Lemma 4.4, x_1 is contained the sub-building $X_{Z(S)}$ of the centralizer of a maximal (k,σ)-split torus $\mathbf{S}$ of $\mathbf{G}$. By Proposition 3.2 (c), $\mathbf{S}$ is conjugate to one of $\mathbf{A}_i$ by an element of H and hence for some $h \in H$, $hx_1 \in X_{Z(A_i)}$.

Fix a σ-apartment $\mathcal{F}_i$ associated to A_i as in Lemma 4.4 for each $1 \le i \le n$ and let e_i denote the Hausdorff distance between $\mathcal{F}_i$ and $A_i x_0$. By Lemma 4.4, the group $H \cap Z(A_i)$ acts co-compactly on the quotient building $X_{Z(A_i)}/V(A_i)$. In other words, there is $d_i > 0$ such that any point in $X_{Z(A_i)}$ is within distance d_i from any σ-apartment in $X_{Z(A_i)}$ parallel to $\mathcal{F}_i$, up to translation by $H \cap Z(A_i)$.

Therefore

$$d(h'hx_1, \mathcal{F}_i) < d_i \quad \text{for some } h' \in H \cap Z(A_i).$$

Hence

$$d(h'hx, A_i x_0) \le d(x, x_1) + d(h'hx_1, \mathcal{F}_i) + d(\mathcal{F}_i, A_i x_0) \le C_0 + d_i + e_i.$$

It remains to put $C := C_0 + \max_{1\le i\le n}(d_i + e_i)$.

□

Proof of Theorem 1.1 It is now very easy to conclude. Fixing a point $x_0 \in X_G$, let $C > 0$ be a constant given by Corollary 4.8 and K the compact subset of G given by

$$K := \{k \in G \mid d(x_0, kx_0) \le C\} .$$

For any $g \in G$, we apply Corollary 4.8 to $x = g^{-1}x_0$ to obtain elements $h \in H$, $a \in \cup_{i=1}^n A_i$ such that

$$d(hx, a^{-1}x_0) \le C ,$$

or, equivalently,

$$d(x_0, gh^{-1}a^{-1}x_0) \le C .$$

In other words, we have $g = kah$ for some $k \in K$. □

5. Examples

In order to put our polar decomposition in perspective, we now recall some well-known examples.

5.1. **The real case.** We first discuss the archimedean case $k = \mathbb{R}$ and recall the proof of the following basic fact:

Fact 5.1. *Every non-degenerate quadratic form q on the Euclidean space $\mathbb{R}^n$ can be put into diagonal form by an orthogonal base change.*

Proof Choose a point e_1 on the Euclidean unit sphere in $\mathbb{R}^n$ where q achieves the maximum. Note that the Euclidean orthogonal of e_1 is also orthogonal for q. Hence we can use an induction argument. □

Remark Fact 5.1 is a special case of the polar decomposition $G = KAH$ for a real symmetric space G/H with $G = \mathrm{GL}(n, \mathbb{R})$, $K = \mathrm{O}(n)$, A the subgroup of diagonal matrices and $H = \mathrm{O}(p, n-p)$ for any $0 \le p \le n$: Write $q = q_0 \circ g^{-1}$ with $q_0 = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$ and $g \in G$. If we write $g = kah \in KAH$, then the quadratic form $q \circ k = q_0 \circ a^{-1}$ is diagonal.

We now recall more precisely the "real" polar decomposition and its proof. Let $\mathbf{G}$ be a connected reductive $\mathbb{R}$-group, σ an $\mathbb{R}$-involution of $\mathbf{G}$ and $\mathbf{H}$ an open $\mathbb{R}$-subgroup of $\mathbf{G}^\sigma$. Recall that an $\mathbb{R}$-involution θ of $\mathbf{G}$ is called *Cartan* if the group $\mathbf{K} = \mathbf{G}^\theta$ of θ-fixed points is a maximal $\mathbb{R}$-anisotropic subgroup of $\mathbf{G}$.

The following facts are well-known (see [Sc]):

- There exists a *Cartan involution* θ of $\mathbf{G}$ which commutes with σ, i.e., $\sigma\theta = \theta\sigma$.

- A maximal $(\mathbb{R}, \sigma)$-split torus $\mathbf{A}$ of the group $\mathbf{H}' := \mathbf{G}^{\sigma\theta}$ of $\sigma\theta$-fixed points is a maximal $\mathbb{R}$-split torus of $\mathbf{H}'$ as well as a maximal $(\mathbb{R}, \sigma)$-split torus of $\mathbf{G}$. Moreover, all maximal $(\mathbb{R}, \sigma)$-split tori of $\mathbf{G}$ are H-conjugate.

Letting $\mathbf{K} = \mathbf{G}^\theta$ and $\mathbf{A}$ a maximal $(\mathbb{R}, \sigma)$-split torus of the group $\mathbf{H}'$, the polar decomposition for G/H is given as follows:

Proposition 5.2. (see [Sc]**)** *We have $G = KAH$ with $K = \mathbf{K}_\mathbb{R}$, $A = \mathbf{A}_\mathbb{R}$ and $H = \mathbf{H}_\mathbb{R}$.*

We recall here a sketch of proof to emphasize both analogies and differences with our proof in the non-archimedean case.

Sketch of proof As in the proof of Fact 5.1, the proof uses a minimizing argument. Let $g \in G$. Fix a base point $x_0 = K/K$ in the Riemannian symmetric space $X_G = G/K$ and and let $x := g^{-1}x_0$. Let $y = h^{-1}x_0$, with $h \in H$, be the nearest point to x on the totally geodesic sub-manifold $X_H := H/(H \cap K)$. The geodesic from x_0 to hx is orthogonal to X_H. Hence the point hx is in the symmetric subspace $X_{H'} = H'/(H' \cap K)$ "orthogonal at x_0" to X_H. Therefore we have $x_0 = h'hx$ with $h' \in H'$, or, in other words, $g = kh'h$ with $k \in K$. Using the Cartan decomposition $H' = (H \cap K)A(H \cap K)$ for the subgroup H', we then obtain $g \in KAH$, as desired. $\square$

5.2. **Quadratic forms.** We now discuss the diagonalization of quadratic forms in the non-archimedean case. The following well-known fact is analogous to Fact 5.1:

Fact 5.3. *Let k be a non-archimedean local field of residual characteristic not 2.*

Every non-degenerate quadratic form on k^n can be put into diagonal form by a base change preserving the sup norm.

Recall that the stabilizer of the sup norm $\|x\| = \sup_{1 \le i \le n} |x_i|$ is the maximal compact subgroup $K_0 = \mathrm{GL}(n, \mathcal{O}) \subset \mathrm{GL}(n, k)$ where $\mathcal{O}$ is the valuation ring of k.

Proof Choose a point e_1 on the unit ball $\mathcal{O}^n$ where $|q|$ is maximum. Note that e_1 is primitive, i.e., $ke_1 \cap \mathcal{O}^n = \mathcal{O}e_1$. Hence e_1 can be completed to a basis, say $(e_1, \ldots, e_n)$, of $\mathcal{O}^n$. Write $q(\sum x_i e_i) = \sum a_{ij} x_i x_j$ where the matrix $b = (a_{ij})$ is symmetric. One can assume $a_{11} = 1$. By the maximality of $|q(e_1)| = a_{11} = 1$, we have $|q(e_i)| = |a_{i,i}| \le 1$ for each i. It follows from $|q(e_1 + e_i)| \le 1$ that $|2a_{1,i}| \le 1$ for each i. Since the residual characteristic is not 2, we have $|a_{1,i}| \le 1$. Therefore the n-tuple $(e_1, e_2', \ldots, e_n')$ with $e_i' := e_i - a_{1,i}e_1$ is also a basis of $\mathcal{O}^n$ and $e_2', \ldots e_n'$ are orthogonal to e_1. We then use an induction argument to conclude. $\square$

We may translate Fact 5.3 in terms of our polar decomposition. Let q_0 be a non-degenerate diagonal quadratic form on k^n, $\mathbf{G} := \mathbf{GL}(n)$, $K_0 := \mathrm{GL}(n, \mathcal{O})$, $\mathbf{A}_0$ the k-group of diagonal matrices, $\mathbf{H} := \mathbf{O}(q_0)$, so that $\mathbf{X} := \mathbf{G}/\mathbf{H}$ is the space of non-degenerate quadratic forms and $X := G/H$ is the space of quadratic forms G-equivalent to q_0. The subset of diagonal quadratic forms is $X_{\mathrm{diag}} := X \cap \mathbf{A}_0 q_0$. Fact 5.3 gives the equality

$$X = K_0 X_{\mathrm{diag}} \quad \text{or} \quad G = K_0(\mathbf{A}_0\mathbf{H})_k \ .$$

The set $(\mathbf{A}_0\mathbf{H})_k$ is a finite union of double classes $A_0 g_i H$, where the set of indices i is given by $(k^*/k^{*2})^n$. Letting $\mathbf{A}_i := g_i^{-1}\mathbf{A}_0 g_i$, A the union of A_i's and K the union of $K_0 g_i$'s. we obtain the polar decomposition $G = KAH$ as in Theorem 1.1. Note here that K is a compact subset.

Remark If the residual characteristic is 2 but the characteristic of k is not 2, the same argument shows that there exists a compact subset K_0 of $\mathrm{GL}(n, k)$ such that *every non-degenerate quadratic form on k^n can be put into diagonal form by a base change in K_0* i.e., $G = K_0(\mathbf{A}_0\mathbf{H})_k$. One can easily find an example where K_0 cannot be chosen to be a subgroup.

5.3. **The group case.** The following is a well-known corollary of Proposition 2.2.

Corollary 5.4. *Let k be a non-archimedean local field, $\mathbf{G}$ a reductive k-group and $\mathbf{A}$ a maximal k-split torus of $\mathbf{G}$. Then there exists a compact subset K of G such that $G = KAK$.*

Proof We keep the same notation as in Proposition 2.2. Let $Z_c = K_x \cap Z$ be the maximal compact subgroup of Z. The quotient Z/Z_c is an abelian group of rank r and the image of T in Z/Z_c is a subgroup of finite index. □

This corollary is also a special case of our polar decomposition (Theorem 1.1) for the group $\mathbf{G} \times \mathbf{G}$ with the involution given by $\sigma(g_1, g_2) = (g_2, g_1)$ for which $\mathbf{H}$ is the diagonal embedding of $\mathbf{G}$ into $\mathbf{G} \times \mathbf{G}$. In this example there exists only one H-conjugacy class of maximal (k, σ)-split tori of $\mathbf{G} \times \mathbf{G}$.

Remark Note that even in this corollary one can not always choose the compact subset K to be a subgroup of G because the map from T to Z/Z_c is not surjective in general.

5.4. **All σ-apartments are useful.** The following lemma with Lemma 4.4 (a) shows that *all the representatives* $\mathbf{A}_i$ of H-conjugacy classes of maximal (k, σ)-split tori of $\mathbf{G}$ *are useful in our polar decomposition* (Theorem 1.1 or 4.6).

Lemma 5.5. *Let k be a non-archimedean local field with* $\mathrm{char}(k) \neq 2$, $\mathbf{G}$ *a connected reductive k-group, σ a k-involution of $\mathbf{G}$ and $\mathbf{H}$ an open k-subgroup of $\mathbf{G}^\sigma$.*

Let $\mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be maximal σ-apartments (see definition 4.2) of X_G such that

$$\sup_{x \in \mathcal{F}} d(x, \cup_{1 \leq i \leq \ell} H.\mathcal{F}_i) < \infty .$$

Then $\mathcal{F}$ is parallel to one of the translates $h\mathcal{F}_i$ with $h \in H$ and $1 \leq i \leq \ell$.

Proof Since there are only finitely many directions of singular flats in $\mathcal{F}$, we can choose a σ-invariant geodesic $t \to c_t$ in $\mathcal{F}$ with the same singular hull as $\mathcal{F}$ and satisfying $\sigma(c_t) = c_{-t}$ for all t. By hypothesis we can find $C_0 > 0$, $1 \leq i \leq \ell$ and sequences $t_n \to \infty$, $h_n \in H$ such that

$$d(c_{t_n}, h_n\mathcal{F}_i) \leq C_0 .$$

Applying σ, we have

$$d(c_{-t_n}, h_n\mathcal{F}_i) \leq C_0 \ .$$

Using the fact that in a CAT(0)-space, the distance function to a convex subset is a convex function, we deduce that for all $s \in [-t_n, t_n]$

$$d(c_s, h_n\mathcal{F}_i) \leq C_0$$

and that

$$d(c_0, h_n x_i) \leq C_0$$

where x_i denotes the unique point of $\mathcal{F}_i$ which is σ-invariant. The sequence h_n remains in a compact subset of G and hence converges to an element $h \in H$, after passing to a sub-sequence. We then have that for all $s \in \mathbb{R}$,

$$d(c_s, h\mathcal{F}_i) \leq C_0 \ .$$

Hence the geodesic $\mathcal{G} := \{c_t \ , \ t \in \mathbb{R}\}$ is parallel to a geodesic in $h\mathcal{F}_i$ (see 2.3.3 in [KL]). But then $h\mathcal{F}_i$ is included in the sub-building $X_{\mathcal{G}}$. Since $\mathcal{G}$ and $\mathcal{F}$ have the same singular hull, we have $h\mathcal{F}_i \subset X_{\mathcal{G}} = X_{\mathcal{F}}$. Therefore Lemma 4.4 implies that $h\mathcal{F}_i$ is parallel to $\mathcal{F}$. $\square$

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