DISCRETENESS CRITERION FOR SUBGROUPS OF PRODUCTS OF SL(2)

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Abstract. Let G be a finite product of $\mathrm{SL}(2, K_i)'s$ for local fields K_i of characteristic zero. We present a discreteness criterion for non-solvable subgroups of G containing an irreducible lattice of a maximal unipotent subgroup of G. In particular such a subgroup has to be arithmetic.

This extends a previous result of A. Selberg when G is a product of $\mathrm{SL}_2(\mathbb{R})'s$.

1. Introduction

Let $A = \prod_{i\in I} K_i$ be an algebra which is the product of finitely many local fields K_i of characteristic zero with $\mathrm{card}(I) \geq 2$ and $A^\times = \prod_i K_i^\times$ its multiplicative group. Let $G := \mathrm{SL}(2, A)$ be the group of matrices with coefficients in A and with determinant equal to $1 \in A$. This group is isomorphic to the product of the groups $\mathrm{SL}(2, K_i)$.

Recall that a lattice in a locally compact group is a discrete subgroup of finite covolume. It is well known that a lattice $\Omega \subset A$ must be cocompact. It will be said irreducible if $\Omega \subset A^\times \cup \{0\}$.

The following examples of irreducible lattices of A will play an important role in this paper : Let K be a number field and S be a finite set of places containing all the archimedean ones. Let $A = K_S$ be the ring which is the product of the completions K_v of K for all v in S, $\mathcal{O}_{K,S}$ be the ring of S-integers of K :

$$\mathcal{O}_{K,S} := \{k \in K \mid v(k) \geq 0 \text{ for all finite valuation } v \text{ of } K \text{ which is not in } S\}.$$

and $\sigma : K \to A$ be the diagonal embedding. We identify $\mathcal{O}_{K,S}$ with its image $\sigma(\mathcal{O}_{K,S})$. Then $\Omega := \mathcal{O}_{K,S}$ is an irreducible lattice in A [2].

By the arithmeticity theorem of Selberg when all K_i are equal to $\mathbb{R}$ [12], and of Margulis in general ([4], [5]), the groups $\mathrm{SL}(2, \mathcal{O}_{K,S})$ are the only irreducible non-uniform lattices in $\mathrm{SL}(2, A)$, up to conjugacy and commensurability. We note that $\mathrm{SL}(2, \mathcal{O}_{K,S})$ contains an irreducible lattice of the unipotent upper triangular subgroup of $\mathrm{SL}(2, A)$.

The following theorem says that, again up to conjugacy and commensurability, this property characterizes these arithmetic groups among all non-solvable discrete subgroups of $\mathrm{SL}(2, A)$.

It also gives a necessary and sufficient criterion for a subgroup of $\mathrm{SL}(2,A)$ containing an irreducible lattice of the unipotent upper triangular subgroup to be discrete.

There does exist a discreteness criterion for such subgroups in $\mathrm{SL}(2,\mathbb{R})$ (see [13]) but this is not known for $\mathrm{SL}(2,\mathbb{C})$.

The following theorem is due to Selberg in [13] when G is a product of $\mathrm{SL}(2,\mathbb{R})$'s.

Theorem 1.1. *Let K_i, $i \in I$, be a finite collection of local fields of characteristic zero for* $\mathrm{card}(I) \geq 2$, *and $A := \prod_{i\in I} K_i$. Let $\Omega \subset A$ be an irreducible lattice, and $g_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}$ be an element of* $\mathrm{SL}(2,A)$ *with $\gamma_0 \neq 0$. Let Γ be the subgroup of* $\mathrm{SL}(2,A)$ *generated by g_0 and Γ_Ω where*

$$\Gamma_\Omega := \{\tau_\omega = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \mid \omega \in \Omega\}.$$

Then the group Γ is discrete if and only if the following conditions are satisfied
(i) There exist $c \in A^\times$ and a number field $K \subset A$ such that
a) the projections $\sigma_i : K \to K_i$, $i \in I$, are inequivalent completions of K;
b) the set $S := \{\sigma_i \mid i \in I\}$ contains all the archimedean places of K;
c) $\Omega \subset c\mathcal{O}_{K,S}$.
(ii) For all $\omega \in \Omega$, the elements $\gamma_0^2\omega^2$, $(\alpha_0+\delta_0)\gamma_0\omega$ and $(\alpha_0+\delta_0)^2$ are in $\mathcal{O}_{K,S}$.
In this case Γ is commensurable to a conjugate of $\mathrm{SL}(2,\mathcal{O}_{K,S})$.

In particular, Γ is then a lattice in $\mathrm{SL}(2,A)$.

Notice that these conditions (i) and (ii) are invariant under the conjugation by an upper triangular matrix of $\mathrm{GL}(2,A)$.

Note also that the product $\gamma_0\omega$ need not belong to K. For instance, let $A = \mathbb{R}\times\mathbb{R}$, $G = \mathrm{SL}(2,A)$, $\Gamma_0 = \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}]) \subset G$ and $g_0 = \begin{pmatrix} 0 & -\gamma_0^{-1} \\ \gamma_0 & 0 \end{pmatrix} \in G$ where the coefficient $\gamma_0 \in A$ is given by $\gamma_0 = (\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}})$. Since g_0 is of order 4 and normalizes Γ_0, the subgroup Γ of G generated by Γ_0 and g_0 is a two-fold extension of Γ_0. Hence Γ is discrete and so is the subgroup generated by Γ_Ω with $\Omega = \mathbb{Z}[\sqrt{2}]$ and g_0. But γ_0 does not belong to the field $K \simeq \mathbb{Q}[\sqrt{2}]$.

Here is a straightforward corollary of Theorem 1.1.

Corollary 1.2. *Let K_i, $i \in I$, be a finite collection of local fields of characteristic zero for* $\mathrm{card}(I) \geq 2$, *and $A := \prod_{i\in I} K_i$. Let Ω_1 and Ω_2 be two irreducible lattices of A, and Γ be the subgroup of* $\mathrm{SL}(2,A)$ *generated by all the matrices of the form*

$$\tau_{\omega_1} = \begin{pmatrix} 1 & \omega_1 \\ 0 & 1 \end{pmatrix} \text{ and } \tau^-_{\omega_2} = \begin{pmatrix} 1 & 0 \\ \omega_2 & 1 \end{pmatrix} \text{ with } \omega_1 \in \Omega_1 \text{ and } \omega_2 \in \Omega_2.$$

Then Γ is discrete if and only if there exists a number field $K \subset A$ such that the projections $\sigma_i : K \to K_i$ are inequivalent completions of K, the set

$S := \{\sigma_i \mid i \in I\}$ *contains all the archimedean places of* K*, and*

$$\Omega_1\Omega_2 \subset \mathcal{O}_{K,S}.$$

In this case Γ *is commensurable to a conjugate of* $\mathrm{SL}(2, \mathcal{O}_{K,S})$.

When all K_i are $\mathbb{R}$, Corollary 1.2 was shown by Selberg (see [13] or [1, Thm 1.4]).

Both Theorem 1.1 and Corollary 1.2 are not true when $\mathrm{card}(I) = 1$. For instance, for any $c > 2$, the subgroup Γ_c of $\mathrm{SL}(2, \mathbb{R})$ generated by $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has a fundamental domain $\{x + iy : y > 0, x^2 + y^2 \geq 1, -c/2 < x \leq c/2\}$ in the Poincare upper half plane. This implies that Γ_c is discrete of infinite covolume.

Similarly, for any $c > 2$, the subgroup Γ_c of $\mathrm{SL}(2, \mathbb{C})$ generated by $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & c\sqrt{-1} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ admits a fundamental domain $\{x + yi + rj : r > 0, x^2 + y^2 + r^2 \geq 1, -c/2 < |x| \leq c/2, -c/2 < |y| \leq c/2\}$ in the upper half space model of the hyperbolic 3 space $\mathbb{H}^3$ and hence is discrete of infinite covolume.

On the other hand, note that there is no lattice in a non-archimedean local field, considered as an additive group.

Analogs of Theorem 1.1 for higher rank simple groups have been proven by the second author, using deep ergodic properties of unipotent flows (see [6], [7], [8], [9], and [10]). This ergodic approach is not available in our case of products of SL_2's. Hence we adopt Selberg's number theoretic approach from [13]. We have also applied it recently to settle an analogous problem for $G = \mathrm{SL}(3, \mathbb{R})$ in [1].

The following proof follows the general strategy of Selberg's proof. But we replace tricky computations by a simple use of the Mahler compactness criterion and we avoid the use of a deep result on L-functions by replacing it with a result of Vaserstein in [14] related to the congruence subgroup problem for SL_2 so that the proof can be adapted to our more general situation.

2. Proof of the sufficiency of the criterion

Before beginning the proof, we first make some comments and introduce some further notations.

The algebra A is a product of fields K_i, $i \in I$, for a finite set I. Here K_i are local fields of characteristic zero, that is, they are either copies of $\mathbb{R}$, $\mathbb{C}$, or of a finite extension of $\mathbb{Q}_p$ for some prime p. We will sometimes write $\mathbb{Q}_\infty$ for $\mathbb{R}$ so that each field K_i is a finite extension of some $\mathbb{Q}_{p_i}$. We set A_p to be the product of all the K_i's extending $\mathbb{Q}_p$. Each K_i is endowed with the unique absolute value $|.|$ which extends the standard absolute value on $\mathbb{Q}_{p_i}$.

Examples of irreducible lattices $\Omega \subset A$ are

$$\mathbb{Z} \subset \mathbb{R} \ , \ \mathbb{Z}[\sqrt{-1}] \subset \mathbb{C} \ , \ \mathbb{Z}[\tfrac{1}{p}] \subset \mathbb{R} \times \mathbb{Q}_p \ , \ \mathbb{Z}[\sqrt{2}] \subset \mathbb{R} \times \mathbb{R},$$

where these injections are given by different ring morphisms into the factors.

The existence of a lattice Ω in A imposes a few restrictions on A: for all p, one must have $\dim_{\mathbb{Q}_p} A_p \leq \dim_{\mathbb{R}} A_\infty$. Each lattice Ω in A is then a torsion free $\mathbb{Z}$-module of rank[1] $\dim_{\mathbb{R}} A_\infty$. It is finitely generated if and only if all the K_i are archimedean.

We now begin the proof of Theorem 1.1 and first check that the conditions (i) and (ii) are sufficient for the discreteness of Γ.

Using the invariance of these conditions under the conjugation by an upper triangular matrix of $\mathrm{GL}(2, A)$, we may assume that $\alpha_0 = 0$, $c = 1$ so that $\Omega \subset \mathcal{O}_{K,S}$. Since S contains all the archimedean places of K, the ring $\mathcal{O}_{K,S}$ is discrete in A. Let J be the ideal of $\mathcal{O}_{K,S}$ generated by Ω. The condition (ii) tells us that

$$\gamma_0^2 \in J^{-2} \ , \ \delta_0\gamma_0 \in J^{-1} \ , \ \delta_0^2 \in \mathcal{O}_{K,S},$$

where $J^{-1} \subset K$ is the $\mathcal{O}_{K,S}$-submodule of K given by

$$J^{-1} = \{a \in K \mid aj \in \mathcal{O}_{K,S} \text{ for all } j \text{ in } J\}$$

and J^{-2} is the square of J^{-1}. It is easy to check that the following set $\Gamma(J)$ is a subgroup of $\mathrm{SL}(2, A)$:

$$\Gamma(J) = \{g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, A) \mid \alpha \in \mathcal{O}_{K,S} \ , \ \beta \in J, \ \gamma \in J^{-1}, \ \delta \in \mathcal{O}_{K,S}\}.$$

Clearly $\Gamma(J)$ is discrete. By a direct computation, one checks that $\Gamma(J)$ contains Γ_Ω and g_0^2, and that $\Gamma(J)$ is normalized by g_0. Therefore the intersection $\Gamma \cap \Gamma(J)$ has index at most two in Γ. This proves that Γ is discrete too. □

3. Proof of the Necessity of the Criterion

For the rest of the paper, we check that the conditions (i) and (ii) in Theorem 1.1 are necessary for the discreteness of Γ.

Let $d = \mathrm{card}(I)$ so that $A = K_1 \times \cdots \times K_d$. We remark that the hypothesis $d \geq 2$ is used only in Lemma 3.5.

For any element g of $\mathrm{SL}(2, A)$ we denote by $\gamma_g \in A$ its lower-left coefficient. Note that this coefficient γ_g depends only on the Γ_Ω -double coset class $\Gamma_\Omega g \Gamma_\Omega$.

For $a = (a_1, \ldots, a_d)$ in A, we set $\|a\| = max|a_i|$.

[1] The rank of a torsion-free $\mathbb{Z}$-module M is defined to be the dimension of the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 3.1. *Assume that the subgroup* Γ *is discrete.*
a) For all $g \in \Gamma$ *with* $\gamma_g \neq 0$*, we have* $\gamma_g \in A^\times$*.*
b) For every $C > 0$*, the set of* Γ_Ω*-double coset classes*

$$\Gamma_\Omega \backslash \{g \in \Gamma \mid 0 < \|\gamma_g\| \leq C\} / \Gamma_\Omega$$

is finite. In particular, the set $\{\gamma_g \mid g \in \Gamma\}$ *is a discrete subset of* A*.*

Proof. *a*) Let g be an element of Γ whose lower-left coefficient $\gamma_g = (\gamma_1, \ldots, \gamma_d)$ is non-zero. We want to check that all the coordinates γ_i of γ_g are non-zero. Suppose that this is not the case. Then the product B of the factors K_i for which $\gamma_i \neq 0$ is not equal to A. Let T be the closure of the image of the projection of Ω onto B. Since Ω is irreducible, the subgroup T is a closed non-discrete subgroup of A and hence is uncountable. On the other hand, the commutator $[\tau_\omega, [\tau_\omega, g]]$ depends only on the coordinates ω_i for which $\gamma_i \neq 0$. Recall that τ_ω has been defined in Theorem 1.1. Since Γ is closed, it follows that Γ contains all the elements $h_t := [\tau_t, [\tau_t, g]]$ with t in T. The lower-left coefficient of h_t is equal to $t^3\gamma^4$. Since T is uncountable, this contradicts the discreteness of Γ.

b) Since Ω is a lattice in A, there exists a constant $C_\Omega > 0$ such that for all a in A, there exists $\omega \in \Omega$ with $\|a - \omega\| \leq C_\Omega$. For any $g \in \Gamma$ with $0 < \|\gamma_g\| \leq C$ γ_g is invertible by a), and hence there exist a and a' in A such that

$$\tau^-_{\gamma_g} = \tau_a g \tau_{a'}.$$

Therefore the Γ_Ω-double coset class of g has a representative

$$h = \tau_\omega g \tau_{\omega'} = \tau_{\omega - a} \tau^-_{\gamma_g} \tau_{\omega' - a'}$$

with ω, ω' in Ω, $\|\omega - a\| \leq C_\Omega$ and $\|\omega' - a'\| \leq C_\Omega$. Hence these representatives h are uniformly bounded. Since Γ is discrete, they belong to a finite set. □

The following lemma transforms our discreteness assumption on Γ into a property of the lattice Ω in A.

Lemma 3.2. *Assume that the subgroup* Γ *is discrete.*
Then the set $\Omega^2 := \{\omega_1\omega_2 \mid \omega_1 \in \Omega, \omega_2 \in \Omega\}$ *is a discrete subset of* A*.*

Proof. Choose an element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ with $\gamma \neq 0$. For instance $g = g_0$. We just look at the simplest elements of Γ whose lower-left coefficients involve the product of two elements of Ω. More precisely, we consider elements of Γ of the form

$$h = h(\omega_1, \omega_2) = g \tau_{\omega_1} g^{-1} \tau_{\omega_2} g$$

where ω_1 and ω_2 vary in Ω. Setting $n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ so that $\tau_{\omega_i} = I + \omega_i n$, one computes

$$h = g + \omega_1 gn + \omega_2 ng + \omega_1\omega_2 gng^{-1}ng \ ,$$

that is,

$$h = \begin{pmatrix} \alpha + \omega_2\gamma - \omega_1\omega_2\alpha\gamma^2 & \beta + \omega_1\alpha + \omega_2\delta - \omega_1\omega_2\alpha\delta\gamma \\ \gamma - \omega_1\omega_2\gamma^3 & \delta + \omega_1\gamma - \omega_1\omega_2\delta\gamma^2 \end{pmatrix} \tag{3.1}$$

By Lemma 3.1.b and the above computation of the lower-left coefficients $\gamma_h = \gamma - \omega_1\omega_2\gamma^3$ of the elements h, the set Ω^2 is discrete. □

To exploit the conclusion of Lemma 3.2, we will need the following general proposition. It is thanks to this proposition that the number field K we are looking for is identified.

Proposition 3.3. *Let A be a finite product of local fields K_i of characteristic zero. Let Ω be an irreducible lattice of A such that the set $\Omega^2 := \{\omega_1\omega_2 \mid \omega_1 \in \Omega, \omega_2 \in \Omega\}$ is discrete.*

Let $\Omega_{\mathbb{Q}} \subset A$ be the $\mathbb{Q}$-vector space generated by Ω,

$$K := \{a \in A \mid a\Omega_{\mathbb{Q}} \subset \Omega_{\mathbb{Q}}\}, \tag{3.2}$$

and $L_i \subset K_i$ be the closure of the image of K by the projection $\sigma_i : A \to K_i$. Then the following hold:
a) K is a number field and the completions $\sigma_i : K \to L_i$ are inequivalent.
b) The set $S = \{\sigma_i\}$ of places of K contains all the archimedean ones.
c) The dimension $t := \dim_K(\Omega_{\mathbb{Q}})$ is equal to 1 or 2.
d) The lattice Ω is commensurable to a free $\mathcal{O}_{K,S}$-submodule of rank t in A.

The main tools in the proof of Proposition 3.3 are the Minkowski lemma and the Mahler compactness criterion (cf. [3], [11]). But before beginning the proof we introduce some notations we will need.

Recall that A_∞ denotes the product of the archimedean factors in A and that d is the total number of factors in A.

Here is the Minkowski lemma [3]:

If $B \subset A$ is compact and $\mathrm{vol}(B) \geq \mathrm{covol}_A(\Omega)$, *then* $\Omega \cap (B - B) \neq \{0\}$.

We normalize the Haar measure on each K_i so that the ball

$$\{a \in K_i \mid \; |a| \leq 1\}$$

has volume 1 and we choose the Haar measures on A and A_∞ to be the product of the Haar measures on the K_i's.

Let A_U be the open subgroup of A given by

$$A_U := \{a \in A \mid |a_i| \leq 1 \text{ for all } i \text{ with } K_i \text{ non-archimedean}\}.$$

We will denote by Ω_U the lattice of A_U which is

$$\Omega_U = \Omega \cap A_U$$

and by Ω_∞ the lattice of A_∞ which is the image of Ω_U under the projection $A \to A_\infty$. One has the equalities

$$\Omega_{\mathbb{Q}} = (\Omega_U)_{\mathbb{Q}}. \tag{3.3}$$

We remark that one has the equality

$$\mathrm{covol}_A(\Omega) = \mathrm{covol}_{A_U}(\Omega_U). \tag{3.4}$$

To see this, first note that the countable group A/A_U does not contain finite index subgroups; if M is a finite index subgroup of A containing A_U, then for each i with K_i non-archimedean, M must contain $\pi_i^{-m_i}$ for some sequence $m_i \to \infty$ for the uniformizer π_i of K_i. It follows $K_i \subset M$ and hence $M = A$.

Since the image of Ω in A/A_U must be of finite index as Ω being of finite covolume in A, it follows that the projection from Ω to A/A_U is surjective and this implies (3.4).

Moreover one has the equality

$$\mathrm{covol}_{A_U}(\Omega_U) = \mathrm{covol}_{A_\infty}(\Omega_\infty). \tag{3.5}$$

This follows from the injectivity of the projection from Ω_U to Ω_∞ and the normalization of the measures.

We will apply the Mahler compactness criterion to the set of lattices in A_∞ [3]:

A set $\mathcal{F}$ of lattices in A_∞ is relatively compact if and only if the lattices in $\mathcal{F}$ have uniformly bounded covolume and intersect trivially a fixed neighborhood of 0 in A_∞.

Proof of Proposition 3.3. a) Notice first that K is a $\mathbb{Q}$-algebra whose dimension over $\mathbb{Q}$ is at most $\dim_{\mathbb{Q}} \Omega_{\mathbb{Q}}$. Since Ω is irreducible (that is, $\Omega \subset A^\times \cup \{0\}$) and the zero-divisors of K are non-zero elements of A one of whose coordinates is zero, the definition of K immediately implies that K contains no zero divisors and hence K is a field. Fix $i \neq j$. Recalling that we have chosen the normalized absolute values on K_i's, in order to show that the completions given by σ_i and σ_j are inequivalent, it is enough to find an element

$$a = (a_1, \ldots, a_d) \in K \ \text{ such that } \ |a_i| < |a_j|. \tag{3.6}$$

We set $\varepsilon_i := \dim_{\mathbb{Q}_{p_i}} K_i$ and $D = (\mathrm{covol}_A \Omega)^{1/d'}$ with $d' = \sum_i \varepsilon_i$. Applying the Minkowski Lemma, one can find a sequence ω_n of non-zero elements in Ω whose coordinates are bounded by

$$|\omega_{n,i}| \leq 2n^{-1/\varepsilon_i} D\ ,\ \ |\omega_{n,j}| \leq 2n^{1/\varepsilon_j} D \text{ and } |\omega_{n,k}| \leq 2D \text{ for } k \neq i,\ k \neq j. \tag{3.7}$$

Since $\lim_{n\to\infty} \omega_{n,i} = 0$ and Ω is irreducible (and hence none of $\omega_{n,j}$ is zero), one must have $\lim_{n\to\infty} |\omega_{n,j}| = \infty$. We introduce the lattices $\Omega_n := \omega_n \Omega$. The covolume of these lattices Ω_n are uniformly bounded. Moreover, since Ω^2 is discrete, the union $\cup_{n\geq 1} \Omega_n$ is a discrete subset of A. Therefore there exists a neighborhood of 0 in

A meeting trivially all the lattices Ω_n. The Mahler compactness criterion tells us that, by passing to a subsequence, the lattices Ω_n^∞ converge to some lattice in A_∞. Hence, by passing further to a subsequence, the lattices $(\Omega_n)_U$ of A_U converge to some lattice of A_U. Using again the discreteness of the union $\cup_{n\geq 1}\Omega_n$, since the lattices $(\Omega_n)_U$ are finitely generated, one deduces that, starting from some $n_0 \geq 1$, the sequence $(\Omega_n)_U$ is constant. That is

$$(\omega_n\Omega)_U = (\omega_{n_0}\Omega)_U \; . \tag{3.8}$$

Using the equality (3.3), the sequence of $\mathbb{Q}$-vector spaces $(\omega_n\Omega)_\mathbb{Q}$ is also constant for $n \geq n_0$. Hence the element $a = \omega_n\omega_{n_0}^{-1}$ belongs to K and, by (3.7), satisfies $|a_i| < |a_j|$ for n large enough.

b) Since the ring of integers $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module, one can find a non-zero ω_0 in Ω such that $\mathcal{O}_K\omega_0 \subset \Omega$. Hence $\mathcal{O}_K$ is discrete in A and S has to contain all the archimedean completions of K.

c) Let d_1 and d_2 be the number of real and complex factors in A respectively. Since Ω is a lattice in A, one has $\dim_\mathbb{Q}\Omega_\mathbb{Q} = d_1 + 2d_2$. Since by a), K has $d_1 + d_2$ inequivalent archimedean completions, one has the lower bound $[K : \mathbb{Q}] \geq d_1 + d_2$. Since $\Omega_\mathbb{Q}$ is a K vector space, it follows that $\dim_K \Omega_\mathbb{Q} = 1$ or 2. This second case is possible only if $d_1 = 0$, $[K : \mathbb{Q}] = d_2$ and K is totally real.

d) Let us for instance assume $t = 2$. The following proof can easily be adapted to the case $t = 1$. One can find non-zero c, c' in Ω such that $\Omega \subset Kc \oplus Kc'$.

We first prove that one can choose non-zero c, c' in Ω so that

$$\mathcal{O}_{K,S}\, c \oplus \mathcal{O}_{K,S}\, c' \subset \Omega. \tag{3.9}$$

Since Ω_U is a free $\mathbb{Z}$-module of rank $2d$, one can choose non-zero c, c' in Ω so that

$$\mathcal{O}_K\, c \oplus \mathcal{O}_K\, c' \subset \Omega_U. \tag{3.10}$$

Fix a finite place σ_i and an archimedean place σ_j. We introduce again for these i and j the distinct elements $\omega_n \in \Omega$ as in the proof of a). Hence there exists $n_0 \geq 1$ such that one has the equality (3.8) for infinitely many $n \geq n_0$.

In particular, choosing an integer $N_0 \geq 1$ such that

$$N_0\Omega_U \subset \omega_{n_0}^{-1}(\omega_{n_0}\Omega)_U.$$

and setting $a = N_0^{-1}\omega_n\omega_{n_0}^{-1}$, one gets

$$\Omega_U \subset \tfrac{1}{N_0}\, \omega_{n_0}^{-1}(\omega_n\Omega)_U \subset a\Omega.$$

Hence, using (3.10), one has

$$a^{-1}(\mathcal{O}_K\, c \oplus \mathcal{O}_K\, c') \subset \Omega. \tag{3.11}$$

We decompose the fractional ideal $\mathcal{O}_K\, a$ of K as a ratio N/D of two relatively prime ideals of $\mathcal{O}_K$. From inclusions (3.10) and (3.11), using the fact that N and D are relatively prime, one gets

$$(N^{-1}\, c \oplus N^{-1}\, c') \subset \Omega.$$

Set $\pi_i \subset \mathcal{O}_K$ to be the prime ideal of $\mathcal{O}_K$ associated to the place σ_i. The assumption (3.7) on ω_n implies that N is divisible by a power $\pi_i^{m_n}$ of π_i with $\lim_{n\to\infty} m_n = \infty$. Hence

$$(\pi_i^{-m_n}\, c \oplus \pi_i^{-m_n}\, c') \subset \Omega.$$

Since this is true for all finite place σ_i in S this proves the inclusion (3.9).

Let $t_i := \dim_{L_i} K_i$. We prove now that $t_i = 2$. The closure Ω_i of the image of $\Omega_{\mathbb{Q}}$ in K_i is an L_i-vector subspace of K_i. Since Ω is a lattice in A, Ω_i must be equal to K_i. Since $\Omega_{\mathbb{Q}}$ is a K-vector space of dimension 2, $\dim_{L_i} \Omega_i \le 2$. Hence $t_i \le 2$.

Let B be the algebra which is the product of all L_i's. The inclusion (3.9) proves that $\mathcal{O}_{K,S}c \oplus \mathcal{O}_{K,S}c'$ is discrete in A. Since $\mathcal{O}_{K,S}$ is cocompact in B, it follows that the intersection $Bc \cap Bc'$ is zero. In particular, one has $A = Bc \oplus Bc'$, and $\mathcal{O}_{K,S}c \oplus \mathcal{O}_{K,S}c'$ is a lattice in A. Since Ω is also a lattice in A, the inclusion (3.9) proves that these two lattices are commensurable, finishing the proof of d).

We note that this argument also proves that, for every i, one has $t_i = t$. □

Note that the second case $\dim_K \Omega_{\mathbb{Q}} = 2$ of Proposition 3.3 may occur. For instance, for any lattice Ω in $\mathbb{C}$, the set Ω^2 is discrete. More generally:

Remark *Let K be a totally real number field of degree d_0 seen as a subfield of the ring $\mathbb{C}^{d_0}$ through the d_0 distinct embeddings $\sigma_i : K \to \mathbb{R} \subset \mathbb{C}$. Let a be an element of $\mathbb{C}^{d_0}$ all of whose coordinates are non-real and Ω be the lattice $\Omega = \mathcal{O}_K \oplus \mathcal{O}_K a$. Then the square Ω^2 is a discrete subset of $\mathbb{C}^{d_0}$.*

Since we will not use this remark, we leave the proof to the reader.

Corollary 3.4. *Let K be a number field, S a finite set of places of K containing all the archimedean ones. Let $A = K_S$ be the algebra which is the product of the completions K_v for v in S. We identify K with its image in A by the diagonal embedding. Let Ω be a lattice of A which is included in K. Assume that the set $\Omega^2 = \{\omega_1\omega_2 \mid \omega_1 \in \Omega\ ,\ \omega_2 \in \Omega\}$ is discrete in A.*

Then Ω is commensurable to $\mathcal{O}_{K,S}$.

The hypothesis "Ω^2 discrete" can not be omitted. Here is an example. Let $K = \mathbb{Q}[i]$, $S = \{\infty, v\}$ where v is the completion for the prime ideal $(2+i)$ so that $K_v = \mathbb{Q}_5$. Then $\Omega := \mathbb{Z}[\frac{1}{5}] \oplus \mathbb{Z}i$ is a lattice in $\mathbb{C} \times \mathbb{Q}_5$ included in $\mathbb{Q}[i]$ but Ω is not commensurable to $\mathcal{O}_{K,S} = \mathbb{Z}[\frac{1}{2+i}]$.

Proof. This is a direct consequence of Proposition 3.3.d since the assumption $\Omega \subset K$ implies that $\Omega_{\mathbb{Q}} = K$. □

The following lemma tells us that the number field K constructed in Proposition 3.3 is the one we are looking for.

We pick again an element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of Γ with $\gamma \neq 0$. For instance $g = g_0$.

Lemma 3.5. *Assume that $d \geq 2$ and that the subgroup Γ is discrete. Let $K \subset A$ be the number field given by* (3.2) *and c be a non-zero element of Ω. Then*
a) One has $\Omega \subset cK$.
b) The elements $c^2\gamma^2$, $(\alpha + \delta)^2$ and $(\alpha + \delta)c\gamma$ are in K.

Proof. We first prove that

$$(\omega_1\omega_2\gamma^2 - 1)^{-1}\omega_1 \in \Omega_{\mathbb{Q}} \text{ for all } \omega_1,\, \omega_2 \text{ in } \Omega \smallsetminus 0. \tag{3.12}$$

Since $d \geq 2$, according to the Dirichlet S-unit theorem, there exists an element u of infinite order in $\mathcal{O}_{K,S}^{\times}$. Since, by Proposition 3.3.e, Ω is commensurable to a finitely generated $\mathcal{O}_{K,S}$-submodule, replacing u by a suitable power, one can assume $u\Omega = \Omega$.

Fix any non-zero elements ω_1, ω_2 in Ω. For any two integers n' and n'', one introduce the elements h' and h'' of Γ

$$h' = g\tau_{\omega_1'}g^{-1}\tau_{\omega_2'}g \text{ and } h'' = g\tau_{\omega_1''}g^{-1}\tau_{\omega_2''}g$$

with

$$\omega_1' = u^{n'}\omega_1 \ , \ \omega_2' = u^{-n'}\omega_2 \ , \ \omega_1'' = u^{n''}\omega_1 \text{ and } \omega_2'' = u^{-n''}\omega_2\,.$$

The coefficients of these matrices

$$h' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \text{ and } h'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$$

have been computed in (3.1). Since one has

$$\omega_1\omega_2 = \omega_1'\omega_2' = \omega_1''\omega_2'',$$

one has $\gamma' = \gamma''$. Hence according to Lemma 3.1, one can choose n', n'' distinct such that h' and h'' are in the same Γ_Ω-double coset class. As a consequence, there exists a non-zero element $\omega_0 \in \Omega$ such that

$$\delta'' = \delta' + \gamma'\omega_0\,.$$

Using the computation in (3.1), this can be rewritten as

$$\delta + \omega_1''\gamma - \omega_1\omega_2\gamma^2\delta = \delta + \omega_1'\gamma - \omega_1\omega_2\gamma^2\delta + (\gamma - \omega_1\omega_2\gamma^3)\omega_0,$$

that is

$$(u^{n''} - u^{n'})\omega_1 = (1 - \omega_1\omega_2\gamma^2)\omega_0.$$

It follows that $(\omega_1\omega_2\gamma^2 - 1)$ is invertible. Since $(u^{n''} - u^{n'})^{-1}$ is in K and ω_0 is in Ω, $(\omega_1\omega_2\gamma^2 - 1)^{-1}\omega_1$ belongs to $\Omega_{\mathbb{Q}}$. This proves the statement (3.12).

a) According to Proposition 3.3, one has $\dim_K(\Omega_{\mathbb{Q}}) = 1$ or 2. We show now that this second case is excluded. If not, one would have $\Omega_{\mathbb{Q}} = Kc \oplus Kc'$ for another non-zero element c' in Ω. Using the invariance by conjugation, one may assume that c' is the identity element $c' = 1$ of $A^\times$. Choosing $\omega_1 = 1$ and $\omega_2 = \pm 1$ in equation (3.12), one deduces that $(\gamma^2 \pm 1)$ are invertible and that the inverses $(\gamma^2 \pm 1)^{-1}$ are in $K \oplus Kc$. Hence there exists a non-zero triple (m_1, m_2, m_3) in K^3 such that

$$m_1(\gamma^2 - 1)^{-1} + m_2(\gamma^2 + 1)^{-1} = m_3.$$

This proves that γ^2 is algebraic over K of degree 1 or 2. We now divide the proof into two cases:

First case: $\gamma^2 \notin K$. In this case, since $(\gamma^2 - 1)^{-1}$ is in $K \oplus Kc$, the element c would be algebraic of degree 2 over K and $K \oplus Kc$ would be field. This would contradict the definition (3.2) of K.

Second case: $\gamma^2 \in K$. In this case, one can choose $\omega_1 = 1$ and $\omega_2 = c$ in equation (3.12) and one gets that $(\gamma^2 c - 1)^{-1} \in K \oplus Kc$. Hence c would be algebraic of degree 2 over K which gives the same contradiction as in the first case.

Hence one has $\dim_K(\Omega_{\mathbb{Q}}) = 1$ and $\Omega \subset Kc$

b) The equation (3.12) with $\omega_1 = \omega_2 = c$ gives

$$c^2\gamma^2 \in K. \tag{3.13}$$

Applying (3.13) to the lower-left coefficient $(\alpha + \delta)\gamma$ of g^2, one obtains $c^2(\alpha + \delta)^2\gamma^2 \in K$ and hence

$$(\alpha + \delta)^2 \in K. \tag{3.14}$$

Applying (3.14) to each element $\tau_\omega g \in \Gamma$, for all $\omega \in \Omega$, one gets that the square of the trace of the matrices $\tau_\omega g = \begin{pmatrix} \alpha + \gamma\omega & \beta + \delta\omega \\ \gamma & \delta \end{pmatrix}$ is also in K. That is,

$$(\alpha + \delta)^2 + 2(\alpha + \delta)\gamma\omega + \gamma^2\omega^2 \in K.$$

Hence one has

$$(\alpha + \delta)c\gamma \in K. \tag{3.15}$$

This ends the proof of Lemma 3.5. □

To prove Theorem 1.1, it remains to check the following lemma.

Lemma 3.6. *Assume that $d \geq 2$ and that the subgroup Γ is discrete. Let $K \subset A$ be the number field given by* (3.2) *and ω be a non-zero element of Ω. Then the elements $\omega^2\gamma^2$, $(\alpha + \delta)^2$ and $(\alpha + \delta)\omega\gamma$ are in $\mathcal{O}_{K,S}$.*

This is the most delicate and the longest part in Selberg's proof for a product of $\mathrm{SL}(2, \mathbb{R})$'s. Selberg uses subtle properties of L-functions associated to the number field K. We will base this part of our argument on the following result of Vaserstein in [14].

Proposition 3.7. (Vaserstein) *Let K be a number field, S a finite set of places of K containing all the archimedean ones, and N a positive integer. Set Γ_N to be the subgroup of* $\mathrm{SL}(2,\mathcal{O}_{K,S})$ *generated by all the elements* $\tau_\omega := \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ *and* $\tau_\omega^- := \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$ *with $\omega \in N\mathcal{O}_{K,S}$.*

Assume that $\mathrm{card}(S) \geq 2$. *Then Γ_N is of finite index in* $\mathrm{SL}(2,\mathcal{O}_{K,S})$.

We now end the proof of Theorem 1.1.

Proof of Lemma 3.6. According to Lemma 3.5.a and Corollary 3.4, after conjugation by a diagonal matrix, one may assume that Ω is commensurable to the ring of S-integers $\mathcal{O}_{K,S}$ of K.

After conjugating by an upper triangular unipotent matrix, we may also assume that the upper-left coefficient α of the matrix g is $\alpha = 0$. One has then

$$g = \begin{pmatrix} 0 & -\gamma^{-1} \\ \gamma & \delta \end{pmatrix}.$$

But then, for all $\omega \in \Omega$, one computes

$$g\tau_\omega g^{-1} = \tau^-_{\gamma^2\omega} \tag{3.16}$$

Since, by Lemma 3.5.b, γ^2 is in K, one can find a positive integer N such that $N\mathcal{O}_{K,S}$ is included in both Ω and $\gamma^{-2}\Omega$.

According to (3.16), the group Γ contains the group Γ_N which is of finite index in $\mathrm{SL}(2,\mathcal{O}_{K,S})$.

Since Γ is discrete and Γ_N is a lattice in $\mathrm{SL}(2,A)$, the subgroup Γ_N is of finite index in Γ. This implies that Γ is a lattice of $\mathrm{SL}(2,A)$ and is commensurable to $\mathrm{SL}(2,\mathcal{O}_{K,S})$.

We say that an algebraic element x over K is an S-integer, resp. an S-unit, if it is an S'-integer, resp. an S'-unit, in the field $K' = K[x]$ for the set S' of places of K' lying over S. In particular the eigenvalues of all the elements of Γ are S-units, and the traces of the elements of Γ are S-integers in some extension of K. Hence $\delta = \mathrm{trace}(g)$ is an S-integer and, for all $\omega \in \Omega$, $\gamma\omega = \mathrm{trace}(\tau_\omega g - g)$ is an S-integer. Their squares δ^2, $\gamma^2\omega^2$ and their product $\delta\gamma\omega$ are S-integers too. Since, by Lemma 3.5.b, they belong to K, one gets that δ^2, $\gamma^2\omega^2$ and $\delta\gamma\omega$ are elements of $\mathcal{O}_{K,S}$. □

We have proven the statement (i) and (ii) of Theorem 1.1 and also that there exists some element h in $\mathrm{GL}(2,A)$ such that $h\Gamma h^{-1}$ is commensurable to $\mathrm{SL}(2,\mathcal{O}_{K,S})$. Since one has the equality $\mathrm{GL}(2,A) = \mathrm{GL}(2,K)\,\mathrm{SL}(2,A)\,Z(A)$

where $Z(A)$ is the center of $\mathrm{GL}(2, A)$, one can choose h in $\mathrm{SL}(2, A)$. This ends the proof of Theorem 1.1. □

Proof of Corollary 1.2. The sufficiency of the conditions are proven as in section 2. Note that the condition $\Omega_1\Omega_2 \subset \mathcal{O}_{K,S}$ implies that there exists $c \in A^\times$ such that $\Omega_1 \subset c\,\mathcal{O}_{K,S}$. One then shows that a conjugate of Γ is included in one of the discrete groups $\Gamma(J)$.

The necessity of the conditions is a direct consequence of Theorem 1.1. This theorem applied to each matrix $g = \tau^-_{\omega_2}$ with $\omega_2 \in \Omega_2$ tells us that for each $\omega_1 \in \Omega_1$ the elements $\omega_1^2\omega_2^2$ and $2\omega_1\omega_2$ are in $\mathcal{O}_{K,S}$. Hence $\omega_1\omega_2$ is in $\mathcal{O}_{K,S}$ too. □

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