Lecture 10

1) Localization of modules, con'd.

2) Categories.

Refs: 1) – same as in Lec 9; 2) – [R], Section 1.1.

1.1) Submodules in M_S

M is an A-module, for A-submodule $N\subset M \Rightarrow N_S$ is an A_S-submodule in M_S (Section 2.2 of Lec 9)

Proposition: We have mutually inverse bijections:

$$N' \in \{A_S\text{-submodules } N'\subset M_S\} \ni N_S \longleftarrow\!\!\shortmid\; N$$
$$\downarrow$$
$$\iota^{-1}(N') \in \{A\text{-submodules } N\subset M \mid sm\in N \text{ for } s\in S, m\in M \Rightarrow m\in N\} \ni N$$

Here, as before, $\iota: M\to M_S$, $m\mapsto \frac{m}{1}$.

Proof: Step 1: $\iota^{-1}(N')$ is a submodule w. add'l condition.

N' is A_S-submodule hence also A-submodule in M_S & ι is A-linear. Therefore, $\iota^{-1}(N')$ is A-submodule.

Check the cond'n: $sm\in \iota^{-1}(N') \Leftrightarrow \iota(sm)\in N' \Leftrightarrow \frac{s}{1}\iota(m)\in N'$ $\Leftrightarrow \iota(m)\in N' \Leftrightarrow m\in \iota^{-1}(N')$

$\frac{s}{1}$ is invertible in A_S.

Step 2: $\iota^{-1}(N_S)=N$: $\iota^{-1}(N_S)=\{m\in M \mid \iota(m)\in N_S \Leftrightarrow$ $\frac{m}{1}=\frac{n}{s}$ for $n\in N, s\in S \Leftrightarrow \exists\, u\in S \mid usm=un\in N \Leftrightarrow$ [choice of N] $\Leftrightarrow m\in N\} = N$.

Step 3: $(\iota^{-1}(N'))_S = N'$: $(\iota^{-1}(N'))_S=\{\frac{n}{s} \mid \frac{n}{1}\in N' \Leftrightarrow \frac{n}{s}\in N'\}$ $=N'$

b/c $\frac{s}{1}$ is invertible $\square$

2]

Corollary: If M is Noetherian A-module, then M_S is Noetherian A_S-module. In particular, if A is Noeth'n ring, then so is A_S.

Proof: By Prop'n $\forall$ submodule of M_S has the form N_S if N is generated by $m_1, \dots m_k$, then N_S is gen'd by $\frac{m_1}{1}, \dots \frac{m_k}{1}$:
$m = \sum a_i m_i \Rightarrow \frac{m}{s} = \sum \frac{a_i}{s} \frac{m_i}{1}$ $\square$

Rem: Can also prove that AC (resp. DC) condition for M implies that for M_S. This is because bijections in Prop'n are strictly compatible w. inclusions, i.e. $\subsetneq \Leftrightarrow \subsetneq$.

2.2) Local rings.

Recall (Lec 9) if $\mathfrak{p} \subset A$ is prime ideal, then $S := A \setminus \mathfrak{p}$ is localizable; $A_\mathfrak{p} := A_{A\setminus\mathfrak{p}} \rightsquigarrow$ ideal $\mathfrak{p}_\mathfrak{p} \subset A_\mathfrak{p}$.

Proposition: $\mathfrak{p}_\mathfrak{p}$ is the unique maximal ideal of $A_\mathfrak{p}$.

Proof: Pick $I' \subsetneq A_S$. Need to show $I' \subseteq \mathfrak{p}_\mathfrak{p} \Leftarrow \iota^{-1}(I') \subseteq \mathfrak{p}$

from Prop'n in Sect. 1.1

Equiv. (by Prop from 1.1) need to show every ideal $I \subset A$ that satisfies $sa \in I \Rightarrow a \in I$ is contained in $\mathfrak{p}$.

Assume contrary: $I \not\subset \mathfrak{p} \Leftrightarrow S \cap I \neq \emptyset$. Pick $s \in S \cap I$, $a = 1$, so $sa \in I$ but $a \notin I$. Contr'n w. choice of I. $\square$

Definition: A comm've unital ring B is <u>local</u> if it has a unique maximal ideal.

Ex: $A_\mathfrak{p}$ is local.

Rem: Local rings are : important &
have especially nice properties.

2) Categories.

2.1) Definition: definitions below will have a familiar strive: have data & axioms (compare: a group consists of
data: a set, G, and a map $G\times G\to G$, subject to
axioms: associative, has a unit, inverse.)

Definition: A <u>category</u>, $\mathcal{C}$, consists of

(Data): • a "collection" of objects, $Ob(\mathcal{C})$.
• $\forall X, Y\in Ob(\mathcal{C}) \rightsquigarrow$ a <u>set</u> of morphisms, $Hom_{\mathcal{C}}(X,Y)$
• $\forall X,Y,Z\in Ob(\mathcal{C})$, a map (of sets) called <u>composition</u>
$$Hom_{\mathcal{C}}(X,Y)\times Hom_{\mathcal{C}}(Y,Z)\longrightarrow Hom_{\mathcal{C}}(X,Z),\ (f,g)\mapsto g\circ f$$
($\circ$ is often omitted)

These satisfy:

(Axioms): i) composition is associative:
$(f\circ g)\circ h = f\circ(g\circ h)$ for $f\in Hom_{\mathcal{C}}(W,X)$, $g\in Hom_{\mathcal{C}}(X,Y)$, $h\in Hom_{\mathcal{C}}(Y,Z)$.

ii) Units: $\forall X\in Ob(\mathcal{C})\ \exists\ 1_X\in Hom_{\mathcal{C}}(X,X)$ s.t.
• $f\circ 1_X = f\ \ \forall f\in Hom_{\mathcal{C}}(X,Y)$
• $1_X\circ g = g\ \ \forall g\in Hom_{\mathcal{C}}(Z,X)$.

2.2) Examples

1) Category of sets, Sets: objects = sets, morphisms = maps of sets, composition = compositions of maps. Axioms: classical

(unit $1_X = id_X$).

2) Sets w. additional str're: objects = sets w. addi'l str're, morphisms = maps compatible w. this str're, composition = = comp'n of maps. This includes

a) Category of groups: objects are groups, morphisms = homomorphisms of groups. (Groups)

b) Category of rings (Rings)

c) For a ring A, have categories of A-modules, A-Mod, & A-algebras (A-Alg), in the latter morphisms = A-linear homomorphisms of rings.

3a) Let Γ be an oriented graph w. vertices V & edges E.
$\rightsquigarrow$ category $\mathcal{C}(\Gamma)$:

- Objects = vertices
- Morphisms = paths in the graph:

$$X \bullet \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_k} Y$$

this includes empty paths, one for every vertex.

- Composition: concatenation of paths.

Axioms: associativity is manifest, 1_X = empty path in X.

3b) A <u>monoid</u> is a set w. binary operation that is assoc've & has unit (unlike for groups, we don't require inverses).

Note: $\forall X \in Ob(\mathcal{C}) \Rightarrow Hom_{\mathcal{C}}(X,X)$ is a monoid w.r.t. $\circ$

Every monoid, M, gives a category w. one object, X, $(Hom_{\mathcal{C}}(X,X), \circ) := M$.