Lecture 11.

1) Categories, cont'd.
2) Functors

Bonus: homotopy category of topological spaces.

References: [R], Sections 1.1, 1.3, 2.2

1.1) Remarks:

1) Sometimes, objects in a category form a set (here we say our category is <u>small</u>). In general, they form a "class."

2) 1_X is uniquely determined. Moreover, if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, has a (2-sided) inverse g (i.e. $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X) \mid f\circ g = 1_Y, g\circ f = 1_X$) then g is unique, $f^{-1} := g$. In this case, f is called an <u>isomorphism</u>; we say X & Y are <u>isomorphic</u> (X & Y behave the same from the point of view of $\mathcal{C}$, e.g. $Z \in Ob(\mathcal{C}) \rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Z, X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(Z, Y) \\ \psi & \longmapsto & f\circ\psi \end{array} \quad \text{(inverse is } \psi' \mapsto f^{-1}\circ\psi').$$

Notation: $X \xrightarrow{f} Y$ means $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

1.2) Subcategories: $\mathcal{C}$ is a category.

Def'n: (i) By a <u>subcategory</u>, $\mathcal{C}'$, in $\mathcal{C}$ we mean:

(Data) • A subcollection, $Ob(\mathcal{C}')$, in $Ob(\mathcal{C})$.

• $\forall X, Y \in Ob(\mathcal{C}')$, a subset $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ s.t.

(Axioms) • If $f \in \mathrm{Hom}_{\mathcal{C}'}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}'}(Y,Z) \Rightarrow g\circ f \in \mathrm{Hom}_{\mathcal{C}'}(X,Z)$

• $1_X \in \mathrm{Hom}_{\mathcal{C}'}(X,X)$ $\forall X \in Ob(\mathcal{C}')$.

(ii) A subcategory $\mathcal{C}'$ in $\mathcal{C}$ is called <u>full</u> if $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

A subcategory $\mathcal{C}'$ has a natural category str're.

Examples: 1) A monoid M = category w. one object

A nonempty subcategory M' in M = a submonoid.

M' is full $\Leftrightarrow$ $M'=M$.

2) $\mathbb{Z}$-Mod (a.k.a. category of abelian groups) is a full subcategory in Groups

3) The category of commut've rings, CommRings is a full subcategory in Rings.

1.3) Constructions w. categories.

Definition: For a category, $\mathcal{C}$, its <u>opposite category</u>, $\mathcal{C}^{opp}$ by def'n, consists of • the same objects as $\mathcal{C}$,

• $Hom_{\mathcal{C}^{opp}}(X,Y) := Hom_{\mathcal{C}}(Y,X)$

• $g\circ^{opp} f := f\circ g$ $\quad (f\in Hom_{\mathcal{C}^{opp}}(X,Y)=Hom_{\mathcal{C}}(Y,X),$

$g\in Hom_{\mathcal{C}^{opp}}(Y,Z)=Hom_{\mathcal{C}}(Z,Y))$.

Definition: For categories $\mathcal{C}_1, \mathcal{C}_2$, their <u>product</u> $\mathcal{C}_1\times\mathcal{C}_2$ is defined by: • $Ob(\mathcal{C}_1\times\mathcal{C}_2)=Ob(\mathcal{C}_1)\times Ob(\mathcal{C}_2)$

• $Hom_{\mathcal{C}_1\times\mathcal{C}_2}((X_1,X_2),(Y_1,Y_2))=Hom_{\mathcal{C}_1}(X_1,Y_1)\times Hom_{\mathcal{C}_2}(X_2,Y_2)$

• composition is componentwise.

Rem*: for usual categories we care about (Groups, Rings, A-Mod), the opposite cat'y essentially has no independent meaning, except: $\mathcal{C}$ = CommRings, where $\mathcal{C}^{opp}$ is the category of affine schemes (Algebraic geometry)

2) Functors: functor is to a category what group homomorphism is to a group.

2.1) Definition: Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition: A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is

(Data) • an assignment $X \mapsto \mathcal{F}(X): Ob(\mathcal{C}) \to Ob(\mathcal{D})$.

• $\forall X, Y \in Ob(\mathcal{C})$, a map
$$\begin{array}{ccc} Hom_{\mathcal{C}}(X,Y) & \longrightarrow & Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \mathcal{F}(f) \end{array}$$

(Axioms) – compatibility between compositions & units

• $\forall f \in Hom_{\mathcal{C}}(X,Y), g \in Hom_{\mathcal{C}}(Y,Z) \Rightarrow \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$

equality in $Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Z))$.

• $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)} \ \forall X \in Ob(\mathcal{C})$

Remarks: • Have the identity functor $Id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$

• For functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{E}$ can take the composition $\mathcal{G} \circ \mathcal{F}: \mathcal{C} \to \mathcal{E}$ ($\mathcal{G} \circ \mathcal{F}(X) = \mathcal{G}(\mathcal{F}(X))$).

• A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is the same thing as a functor $\mathcal{C}^{opp} \to \mathcal{D}^{opp}$.

2.2) Examples:

0) Let M, M' be monoids (= categories w. one object).
A functor $M \to M'$ = monoid homomorphism.

1) Let $\mathcal{C}'$ be a subcategory in $\mathcal{C}$. Then have <u>inclusion functor</u> $\mathcal{C}' \hookrightarrow \mathcal{C}$ taking objects/morphisms in $\mathcal{C}'$ & sending them to the same objects/morphisms now in $\mathcal{C}$; axioms are clear.

2) Forgetful functors: forget part of a structure

2a) For: Groups $\to$ Sets;

On objects: For$(G) = G$ viewed as a set.

On morphisms: For$(f) = \underbrace{f}$ ← map of sets.

Axioms: clear.

2b) Rings $\to$ Sets

2c) For a comm've ring A, For: A-Alg $\to$ A-Mod (forget multipl'n)

2d) Let $B \xrightarrow{\varphi} A$ comm've ring homom'm $\rightsquigarrow$ For: A-Mod $\to$ B-Mod (just remember how elements of $\varphi(B)$ act on a module).

3) The most important example of functors from a general category $\mathcal{C}$ to Sets. Will depend on $X \in Ob(\mathcal{C})$, $\mathcal{F}_X$.

On objects: $\mathcal{F}_X(Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)$, a set.

On morphisms:

$$Y_1 \xrightarrow{f} Y_2 \rightsquigarrow \text{map } \mathcal{F}_X(f): \mathrm{Hom}_{\mathcal{C}}(X, Y_1) \to \mathrm{Hom}_{\mathcal{C}}(X, Y_2)$$
$$\psi \longmapsto f \circ \psi$$

Check axioms: composition: $\mathcal{F}_X(g \circ f) = \mathcal{F}_X(g) \circ \mathcal{F}_X(f)$ for $Y_1 \xrightarrow{f} Y_2 \xrightarrow{g} Y_3$

$$\mathcal{F}_X(g) \circ \mathcal{F}_X(f): \mathrm{Hom}_{\mathcal{C}}(X, Y_1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y_3) \ni g \circ (f \circ \psi)$$

$\psi \mapsto f \circ \psi \in \mathrm{Hom}_{\mathcal{C}}(X, Y_2) \ni \psi' \mapsto g \circ \psi'$

$$g \circ (f \circ \psi) = (g \circ f) \circ \psi = \mathcal{F}_X(g \circ f)(\psi)$$

So $\mathcal{F}_X(g\circ f) = \mathcal{F}_X(g)\circ \mathcal{F}_X(f)$.

Unit axiom: $\mathcal{F}_X(1_Y) =$ identity map $Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{C}}(X,Y)$.
- exercise.

3^{opp}) We can apply this construction to $\mathcal{C}^{opp} \rightsquigarrow$
$F_X^{opp}: Y \mapsto Hom_{\mathcal{C}^{opp}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$
$f \in Hom_{\mathcal{C}^{opp}}(Y_1, Y_2) = Hom_{\mathcal{C}}(Y_2, Y_1) \rightsquigarrow$

$$\begin{array}{cccl} F_X^{opp}(f): Hom_{\mathcal{C}}(Y_1, X) & \longrightarrow & Hom_{\mathcal{C}}(Y_2, X) & \text{- map of sets} \\ \psi & \longmapsto & \psi\circ f & \end{array}$$

We can view F_X^{opp} as a functor $\mathcal{C} \longrightarrow Sets^{opp}$
(a traditional name: contravariant functor $\mathcal{C} \to Sets$)

4) Algebra constructions as functors:
4a) A is a ring. Want to define a functor Free: Sets $\to$ A-Mod
I, set, $\rightsquigarrow$ Free$(I) := A^{\oplus I}$
$f: I \to J \rightsquigarrow$ Free$(f): A^{\oplus I} \to A^{\oplus J}$ - the unique map sending the basis element e_i $(i\in I)$ to $e_{f(i)} \in A^{\oplus J}$.
Checking axioms of functor: exercise.

4b) Abelinization functor Ab: Groups $\to \mathbb{Z}$-Mod
Recall (from Galois th'y): in a group G have group commutator
$(g,h) = ghg^{-1}h^{-1}$ $((g,h)=1 \Leftrightarrow gh=hg)$

The derived subgr'p $(G,G) :=$ subgroup generated by (g,h) $\forall g,h \in G$, it's normal.

Then $Ab(G) = G/(G,G)$ is abelian.

Notice: $\forall$ group homom. $f: G \to H \Rightarrow f((G,G)) \subset (H,H)$

$\rightsquigarrow$ well-defined homom'm $Ab(f): Ab(G) \to Ab(H)$:

$$Ab(f)(g(G,G)) := f(g)(H,H).$$

Need to check: $Ab(f_2 \circ f_1) = Ab(f_2) \circ Ab(f_1)$ follows from

$$Ab(1_G) = 1_{Ab(G)} \text{ — manifest.}$$

4c) Localization of modules is a functor: $S \subset A$ localizable $\rightsquigarrow \bullet_S: A\text{-Mod} \longrightarrow A_S\text{-Mod}$. (see previous lecture)

BONUS: homotopy category of topological spaces.

B1) Equivalence on morphisms.

Let $\mathcal{C}$ be a category. Suppose that $\forall X,Y \in Ob(\mathcal{C})$, the set $Hom_{\mathcal{C}}(X,Y)$ is endowed with an equivalence relation $\sim$ s.t.

(1) If $g,g' \in Hom_{\mathcal{C}}(Y,Z)$ are equivalent & $f \in Hom_{\mathcal{C}}(X,Y)$, then $g \circ f \sim g' \circ f$.

(2) If $f,f' \in Hom_{\mathcal{C}}(X,Y)$ are equivalent and $g \in Hom_{\mathcal{C}}(Y,Z)$, then $g \circ f \sim g \circ f'$.

We write $[f]$ for the equivalence class of f.

Given such an equivalence relation, we can form a new category to be denoted by $\mathcal{C}/\sim$ as follows:

- $Ob(\mathcal{C}/\sim) := Ob(\mathcal{C})$

· $Hom_{\mathcal{C}/\sim}(X,Y) := Hom_{\mathcal{C}}(X,Y)/\sim$ - the set of equivalence classes

· $[g]\circ[f] = [g\circ f]$ - well-defined precisely b/c of (1) & (2)

We note that there is a natural functor $\pi: \mathcal{C} \to \mathcal{C}/\sim$ given by $X \mapsto X$, $f \mapsto [f]$.

Example: Let M be a monoid. Note that the equivalence class of $1 \in M$ is a submonoid, say M_0, moreover, (1) & (2) imply that $mM_0 = M_0 m$ $\forall$ $m \in M$. Such submonoids are called normal (for groups we recover the usual condition). And if $M_0 = [1]$ is normal, then (1) and (2) hold - an exercise. For a normal submonoid M_0 we can M/M_0 with a natural monoid structure - just as we do for groups. The category $\mathcal{C}/\sim$ corresponds to the quotient monoid M/M_0 and the functor π is just the natural epimorphism $M \twoheadrightarrow M/M_0$.

Rem*: $\mathcal{C}/\sim$ looks like a quotient category. But in situations where the term "quotient" is used and that are closer to quotients of abelian groups (Serre quotients of abelian categories) the construction is different - and more difficult.

B2) Homotopy category of topological spaces.

Let's recall the usual category of topological spaces. Let X be a set. One can define the notion of <u>topology</u> on X: we declare some subsets of X to be "open", these are supposed to satisfy certain axioms. A set w. topology is called a topological space. A map $f: X \to Y$ of topological spaces is called continuous if $U \subset Y$ is open $\Rightarrow f^{-1}(U) \subset X$ is open.

We define the category Top of topological spaces w.

$Ob(Top)$ = topological spaces.

$Hom_{Top}(X, Y)$:= continuous maps $X \to Y$

Composition = composition of maps.

One issue: this category is hard to understand - hard to study topological spaces up to homeomorphisms.

Now we introduce our equivalence relation of $Hom_{Top}(X, Y)$

Definition: Continuous maps $f_0, f_1 : X \to Y$ are called <u>homotopic</u> if $\exists$ a continuous map $F: X \times [0,1] \to Y$ s.t. $f_0(x) = F(x,0)$ & $f_1(x) = F(x,1)$.

Informally, f_0, f_1 are homotopic if one can continuously deform f_0 to f_1. It turns out that being homotopic is an equivalence relation satisfying (1) & (2) from B1. The corresponding category $Top/\sim$ is known as the <u>homotopy category of topol'l spaces</u>. Note that in this category morphisms are <u>not</u> maps!

Here is why we care about the homotopy category. Isomorphic here means homotopic (X is homotopic to Y if $\exists$ $X \xrightarrow{f} Y$, $Y \xrightarrow{g} X$ s.t. fg is homotopic to 1_Y & gf is homotopic to 1_X) and this is easier to understand then being homeomorphic. Second, the classical invarients such as homology and homotopy groups only depend on homotopy type. A more educated way to state this: these invariants are functors from the homotopy category of topological spaces to Groups (true as stated for homology, for homotopy it's more subtle, this requires fixing a point in X and hence need to work w. an auxiliary category of "pointed" topological spaces -up to homotopy).