Lecture 12.

1) Functor morphisms.
2) Yoneda lemma.

BONUS: category equivalences.

Refs: [R], Sections 1.4, 2.2

1) Functor morphism is to a functor $\approx$ a morphism is to an object.

1.1) Definition: Let $\mathcal{C}, \mathcal{D}$ be categories & $F, G: \mathcal{C} \to \mathcal{D}$ be functors.

Def'n: A <u>functor morphism</u> $\eta: F \Rightarrow G$ is

(Data) $\forall X \in Ob(\mathcal{C})$, a morphism $\eta_X \in Hom_{\mathcal{D}}(F(X), G(X))$ s.t.

(axiom) $f \in Hom_{\mathcal{C}}(X, Y) \rightsquigarrow F(f) \in Hom_{\mathcal{D}}(F(X), F(Y)), G(f) \in Hom_{\mathcal{D}}(G(X), G(Y))$. Want $\forall f$ the following is commutative:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}$$

Rem: Functor morphisms are also known as "<u>natural</u> transformations of functors". Natural $\approx$ uniform, independent of choices. This is what we'll see in examples.

1.2) Important example: Last time, for $X \in \mathcal{C}$ we defined a "Hom functor" $F_X: Y \in Ob(\mathcal{C}) \rightsquigarrow F_X(Y) := Hom_{\mathcal{C}}(X, Y)$ - a set, $f \in Hom_{\mathcal{C}}(Y_1, Y_2) \underset{map}{\rightsquigarrow} F_X(f): Hom_{\mathcal{C}}(X, Y_1) \longrightarrow Hom_{\mathcal{C}}(X, Y_2)$

$$\psi \longmapsto f \circ \psi$$

Now let $X' \in Ob(\mathcal{C})$ ($\rightsquigarrow F_{X'}$) & $g \in Hom_{\mathcal{C}}(X', X)$.
Goal: produce a functor morphism $\eta^g : F_X \Rightarrow F_{X'}$.
i.e. for each $Y \in Ob(\mathcal{C})$ need to define a map

$$\begin{array}{ccc} \eta^g_Y : Hom_{\mathcal{C}}(X,Y) & \longrightarrow & Hom_{\mathcal{C}}(X',Y) \\ \psi & \longmapsto & \psi \circ g \end{array} \qquad X' \xrightarrow{g} X \xrightarrow{\psi} Y$$

Now we need to check the axiom (comm've diagram)
$\forall f \in Hom_{\mathcal{C}}(Y_1, Y_2)$, $F_X(f) = f \circ ?$, $F_{X'}(f) = f \circ ?$

$$\left.\begin{array}{ccc} \psi \in Hom_{\mathcal{C}}(X, Y_1) & \xrightarrow[f\circ ?]{} & Hom_{\mathcal{C}}(X, Y_2) \\ \Big\downarrow \eta_{Y_1}(?) = ? \circ g & & \Big\downarrow \eta_{Y_2}(?) = ? \circ g \\ Hom_{\mathcal{C}}(X', Y_1) & \xrightarrow{f\circ ?} & Hom_{\mathcal{C}}(X', Y_2) \end{array}\right\} \text{is commutative}$$

$\downarrow \longrightarrow : \psi \mapsto \psi \circ g \mapsto f \circ (\psi \circ g)$
$\qquad\qquad\qquad\qquad\qquad \| \longleftarrow$ b/c composition in a category is associative.
$\longrightarrow \downarrow : \psi \mapsto f \circ \psi \mapsto (f \circ \psi) \circ g$

1.3) Remarks. (i) Have the identity morphism $id : F \Rightarrow F$.
(ii) Can take compositions of functor morphisms
$\tau : G \Rightarrow H$, $\eta : F \Rightarrow G \rightsquigarrow \tau \circ \eta : F \Rightarrow H$, $(\tau \circ \eta)_X = \tau_X \circ \eta_X$

Very importantly, in example above:
$$\eta^{g'} \circ \eta^g = \eta^{g \circ g'}$$ (note that the order of g, g' is reversed b/c we compose on the right).

(iii)* If $\mathcal{C}$ is small ($Ob(\mathcal{C})$ is a set), then the collection of functor morphisms $F \Rightarrow G$, denoted $Hom_{Fun}(F, G)$, is a set $\rightsquigarrow$ category of functors $\mathrm{Fun}(\mathcal{C},\mathcal{D})$: objects are functors, morphisms = morphisms of functors.

(iv): (i) & (ii) allow to talk about functor <u>isomorphisms</u>; $\eta: F \Rightarrow G$ is an isom'm $\Leftrightarrow$ $\forall\, Y \in Ob(\mathcal{C})$ have that $\eta_Y \in \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Y))$ is an isom'm:
$\eta^{-1}: G \Rightarrow F$ is given $(\eta^{-1})_Y = (\eta_Y)^{-1}$

2) Yoneda lemma: gives a powerful tool to compute the set of morphisms between certain functors.

Theorem: $\mathcal{C}$ is a category, $X \in Ob(\mathcal{C}) \rightsquigarrow F_X: \mathcal{C} \longrightarrow Sets$, & F - another functor. Then we have bijection of sets,

$$\mathrm{Hom}_{Fun}(F_X, F) \xrightarrow{\sim} F(X).$$

Proof: Step 1: Construct a map

$$\begin{array}{ccc} F(X) & \longrightarrow & \mathrm{Hom}_{Fun}(F_X, F) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & \eta^a \end{array}$$

Need to define $\forall\, Y \in Ob(\mathcal{C}) \rightsquigarrow$ map of sets

$$\eta^a_Y : F_X(Y) = \underset{\underset{\varphi}{\rotatebox{90}{$\in$}}}{\mathrm{Hom}_{\mathcal{C}}(X, Y)} \longrightarrow F(Y)$$

$(a, \varphi) \rightsquigarrow$ el't of $F(Y)$, $\varphi \rightsquigarrow$ map of sets $F(\varphi): F(X) \to F(Y)$
$\eta^a_Y(\varphi) := [F(\varphi)](a) \in F(Y)$

Now: need to check η^a_Y's satisfy the axiom of functor morphism: $\forall\, f \in \mathrm{Hom}_{\mathcal{C}}(Y_1, Y_2)$ we need to check that the following diagram is commutative

$$\begin{array}{ccc} \psi \in \mathrm{Hom}_{\mathcal{C}}(X, Y_1) & \xrightarrow{f \circ ?} & \mathrm{Hom}_{\mathcal{C}}(X, Y_2) \\ \downarrow \eta^a_{Y_1} & & \downarrow \eta^a_{Y_2} \\ F(Y_1) & \xrightarrow{F(f)} & F(Y_2) \end{array}$$

$\longrightarrow\downarrow$: $\psi \mapsto \eta^a_{Y_2}(f \circ \psi) = [F(f \circ \psi)](a)$

$\qquad\qquad\qquad\qquad \parallel \longleftarrow$ b/c $F(f \circ \psi) = F(f) \circ F(\psi)$

$\downarrow\longrightarrow$: $\psi \mapsto [F(\psi)](a) \mapsto [F(f) \circ F(\psi)](a)$

Concl'n: $\eta^a \in \mathrm{Hom}_{Fun}(F_X, F)$.

Step 2: Construct a map $\mathrm{Hom}_{Fun}(F_X, F) \longrightarrow F(X)$

$\qquad\qquad\qquad\qquad \eta \longmapsto a_\eta$

$\eta \rightsquigarrow$ map $\eta_X : \mathrm{Hom}_{\mathcal{C}}(X, X) \longrightarrow F(X)$

$\qquad\qquad\qquad 1_X \longmapsto a_\eta$

$a_\eta = \eta_X(1_X)$.

Step 3: $a_{\eta^a} = a$: $a_{\eta^a} = \eta^a_X(1_X) = [F(1_X)](a) = [F(1_X) = 1_{F(X)}]$
$= 1_{F(X)}(a) = a$.

Step 4: $\eta^{a_\eta} = \eta \Leftrightarrow \forall\, Y \in Ob(\mathcal{C})$ have $\eta^{a_\eta}_Y = \eta_Y$ equality of maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow F(Y)$

$\psi \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$

$$\eta_Y^{a_\eta}(\psi) = [F(\psi)](a_\eta) = F(\psi)(\eta_X(1_X))$$

Will use comm've diagram:

$$\begin{tikzcd}
1_X \in Hom_{\mathcal{C}}(X,X) \arrow[r, "\psi\circ ?"] \arrow[d, "\eta_X"] & Hom_{\mathcal{C}}(X,Y) \arrow[d, "\eta_Y"] \\
F(X) \arrow[r, "F(\psi)"] & F(Y)
\end{tikzcd}$$

$\downarrow \longrightarrow$ $\quad 1_X \mapsto F(\psi)(\eta_X(1_X))$

$\longrightarrow \downarrow$ $\quad 1_X \mapsto \psi\circ 1_X = \psi \mapsto \eta_Y(\psi)$

So $\eta_Y^{a_\eta}(\psi) = F(\psi)(\eta_X(1_X)) = \eta_Y(\psi) \Rightarrow \eta^{a_\eta} = \eta.$ $\square$

Corollary (Yoneda lemma): (modified Oct 9) For $F = F_{X'}$, the map $Hom_{\mathcal{C}}(X',X) \longrightarrow Hom_{Fun}(F_X, F_{X'})$ from Example 1.2 is the same as in the theorem, hence is a bijection.

Proof: Need to show $\forall g \in Hom_{\mathcal{C}}(X',X)$ the construction of η^g in the proof is the same as in the example:

$\eta_Y^g : Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{C}}(X',Y)$ gives $\psi \mapsto \psi\circ g$

$$\eta_Y^g(\psi) = F_{X'}(\psi)(g) = [F_{X'}(\psi) = \psi\circ ?] = \psi\circ g$$ $\square$

2.2) Example:

Q: Compute $End_{Fun}(F,F)$, where $F: Groups \to Sets$

is the forgetful functor.

Solution: we know how to compute $End_{Fun}(F_X, F_X) \xrightarrow{\sim} End_{Groups}(X)$, an isomorphism of monoids. We are looking at $X \in Groups$ w.

$$Hom_{Groups}(X, G) \xrightarrow[\eta_G]{\sim} G \quad \text{(as sets)}$$

$X := \mathbb{Z}$: $\quad \eta_G : Hom_{Groups}(\mathbb{Z}, G) \xrightarrow{\sim} G$

$$\varphi \longmapsto \varphi(1)$$

η_G is a bijection.

Check η is a functor isomorphism; iso is b/c η_G is a bijection

Morphism: $\forall\, \tau : G \to H$, group homom'm

$$\begin{tikzcd}
Hom_{Groups}(\mathbb{Z}, G) \arrow[r, "\tau \circ ?"] \arrow[d, "?(1)"] & Hom_{Groups}(\mathbb{Z}, H) \arrow[d, "?(1)"] \\
G \arrow[r, "\tau"] & H
\end{tikzcd}$$

– commutes (exercise)

Now we need to compute $End_{Groups}(\mathbb{Z}) \overset{\text{Yoneda}}{=} F_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Z}$ w. composition = multiplication:

(added on Oct 9) note that under identification $End_{Groups}(\mathbb{Z}) \simeq \mathbb{Z}$ an element $n \in \mathbb{Z}$ goes to the endomorphism $z \mapsto nz : \mathbb{Z} \to \mathbb{Z}$, the claim about composition = multiplication follows from here

Note that for $\eta \in End_{Fun}(F)$ corresponding to $n \in \mathbb{Z}$ have $\eta_G(g) = [\eta(1) = g \Rightarrow \eta(n) = g^n] = g^n$.

BONUS: Category equivalences.

Our question here: when are two categories the "same"? Turns out, functor isomorphisms play an important role in answering this question.

Before we address this we should discuss an easier question: when are two sets the same? Well, they are literally the same if they consist of the same elements. But this definition is quite useless: sets arising from different constructions won't be the same in this sense. Of course, we use isomorphic instead of being literally the same.

Now back to categories. Again, being the same is useless. How about being isomorphic? Turns out, this is not useful as well. Let's see why. Let $\mathcal{C}, \mathcal{D}$ be categories. We say that $\mathcal{C}, \mathcal{D}$ are isomorphic if there are functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ such that $FG = Id_{\mathcal{D}}$, $GF = Id_{\mathcal{C}}$. The issue is: two functors obtained by different constructions are never the same (compare to sets). The solution: replace "equal" w. "isomorphic" (as functors).

Definition: • We say F, G as above are <u>quasi-inverse</u> if $FG \xRightarrow{\sim} Id_{\mathcal{D}}$, $GF \xRightarrow{\sim} Id_{\mathcal{C}}$ (isomorphic).

• We say $\mathcal{C}, \mathcal{D}$ are <u>equivalent</u> if there are quasi-inverse functors (called <u>equivalences</u>) $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$.

Now we are going to state a general result (not to be used in HW: ☹).

Definitions: A functor $F: \mathcal{C} \to \mathcal{D}$ is called

- <u>fully faithful</u> if $\forall X, X' \in Ob(\mathcal{C}) \Rightarrow$
$f \mapsto F(f)$ is a bijection $Hom_{\mathcal{C}}(X,X') \xrightarrow{\sim} Hom_{\mathcal{D}}(F(X),F(X'))$
- <u>essentially surjective</u> if $\forall Y \in Ob(\mathcal{D})\ \exists\ X \in Ob(\mathcal{C})$ such that $F(X)$ is isomorphic to Y.

Thm: A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence $\Leftrightarrow$ F is fully faithful & essentially surjective.

We won't prove this, but we will give an example - that illustrates how the proof works in general.

Example: Consider the category $\mathcal{D} = \mathbb{F}\text{-}Vect_{fd}$ of finite dimensional vector spaces over $\mathbb{F}$ and its full subcategory $\mathcal{C}$ w. objects $\mathbb{F}^n$ $(n \geq 0)$. We claim that the inclusion functor $F: \mathcal{C} \hookrightarrow \mathcal{D}$ is an equivalence. It's fully faithful by def'n and the claim that it's essentially surjective.

Now we produce a quasi-inverse functor, G. In each $V \in Ob(\mathcal{D})$ we fix a basis, which leads to an isomorphism $\eta_V: V \xrightarrow{\sim} \mathbb{F}^n$. We define $G(V)$ as $\mathbb{F}^n$. For a linear map $f: U \to V$ (w. $\dim U = m$, $\dim V = n$) we set $G(f) := \eta_V^{-1} \circ f \circ \eta_U$.

Exercise: Check G is a functor.

Now we are going to simplify our life a bit an assume that $\eta_{\mathbb{F}^n}: \mathbb{F}^n \xrightarrow{\sim} \mathbb{F}^n$ is the identity.

Exercise: $GF: \mathcal{C} \longrightarrow \mathcal{C}$ is the identity functor (not just isomorphic to it).

Now we produce a functor isomorphism $\eta: \mathrm{Id}_{\mathcal{D}} \xRightarrow{\sim} FG$

So we need to have $\eta_V: V \to \mathbb{F}^{\dim V}$ and this is the isomorphism from above.

Exercise: prove that η is indeed a functor morphism

Then η is an isomorphism of functors. So F is indeed a category equivalence.

A different exercise: prove that the duality functor $\bullet^*$ is an equivalence $\mathbb{F}\text{-}\mathrm{Vect}_{fd} \longrightarrow \mathbb{F}\text{-}\mathrm{Vect}_{fd}^{opp}$.