Lecture 13.

1) Objects representing functors.
2) Products & coproducts.

Refs: II.5 in Hilton, Stammbach "A course in Homological Algebra" GTM 4.

1.0) Recap: Considered Hom functors: $X \in Ob(\mathcal{C}) \rightsquigarrow F_X : \mathcal{C} \to Sets$
$F_X(Y) = Hom_{\mathcal{C}}(X,Y)$. We've produced a map

$$\begin{array}{ccc} Hom_{\mathcal{C}}(X',X) & \longrightarrow & Hom_{Fun}(F_X, F_{X'}) \\ \psi & & \psi \\ g & \longmapsto & \eta^g \end{array}$$

$$\begin{array}{ccc} \eta^g_Y: Hom_{\mathcal{C}}(X,Y) & \longrightarrow & Hom_{\mathcal{C}}(X',Y) \\ \psi & & \psi \\ \varphi & \longmapsto & \varphi \circ g \end{array}$$

Yoneda lemma: the map $g \mapsto \eta^g$ is a bijection.
Also: $\eta^g \circ \eta^{g'} = \eta^{g' \circ g}$.

Application: If $F = F_X$, $G = F_{X'}$, then can compute $Hom_{Fun}(F,G)$ (a set), $End_{Fun}(F)$, as a monoid: $\simeq End_{\mathcal{C}}(X)$ as a set, compositions are in opposite order (see example in the end of Lec 12)

Rem*: If $\mathcal{C}$ is a small category $\rightsquigarrow Fun(\mathcal{C}, Sets)$, then $X \mapsto F_X$ realizes $\mathcal{C}^{opp}$ as a $\underbrace{\text{full subcat'y}}_{\text{from Yoneda}}$ in $Fun(\mathcal{C}, Sets)$

1.1) Objects representing functors.
Definition: Let $F: \mathcal{C} \to Sets$. We say $X \in Ob(\mathcal{C})$ represents F if F is isomorphic to F_X.
Example (from Lec 12): Forgetful functor Groups $\to$ Sets

is represented by $\mathbb{Z}$.

A representing object may fail to exist.

Lemma: An object, X, representing $F: \mathcal{C} \to Sets$ is unique (up to isomorphism) if it exists.

Proof: Two representing objects X, X': $F_X \overset{\sim}{\Rightarrow} F \overset{\sim}{\Leftarrow} F_{X'}$, so $\eta: F_X \overset{\sim}{\Rightarrow} F_{X'} \underset{Yoneda}{\longleftrightarrow} X' \xrightarrow{g} X$ (w. $\eta = \eta^g$).
Since $g \mapsto \eta^g$ is compatible w. comp'ns, η is invertible $\Rightarrow$ g is invertible $\Rightarrow$ X, X' are isomorphic. $\square$

2) Products & coproducts.

2.1) Definition of product.

General constr'n: $F_1, F_2: \mathcal{C} \to Sets$. Can take direct products of sets $\rightsquigarrow F_1 \times F_2: \mathcal{C} \to Sets$

for $Y \in Ob(\mathcal{C}) \rightsquigarrow F_1 \times F_2(Y) = F_1(Y) \times F_2(Y)$

for $Y \xrightarrow{f} Y' \rightsquigarrow F_1 \times F_2(f) = F_1(f) \times F_2(f): F_1(Y) \times F_2(Y) \to F_1(Y') \times F_2(Y')$.

Definition: Let $X_1, X_2 \in Ob(\mathcal{C})$. By their product we mean an object, denoted $X_1 \times X_2$, representing the functor $F_{X_1}^{opp} \times F_{X_2}^{opp}$ (i.e. $Y \mapsto Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2)$ on objects).

Note: $X_1 \times X_2$ may fail to exist.

Exercise (on Lemma): $X_1 \times X_2$ is isom'c to $X_2 \times X_1$.

Alternative definition (via universal property): Let $X_1, X_2 \in Ob(\mathcal{C})$

By the <u>product</u> of X_1 & X_2 we mean (X, π_1, π_2) w. $X \in Ob(\mathcal{C})$, $\pi_i \in Hom_{\mathcal{C}}(X, X_i)$ $(i=1,2)$ s.t. $\forall$ $Y \in Ob(\mathcal{C})$, $\psi_i \in Hom_{\mathcal{C}}(Y, X_i)$ $\exists!$ $\psi \in Hom_{\mathcal{C}}(Y, X)$ s.t. the following is comm'ive:

$$\begin{tikzcd}
 & Y \arrow[d, dashed, "\psi"] \arrow[dl, bend right, "\psi_1"'] \arrow[dr, bend left, "\psi_2"] & \\
X_1 & X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2
\end{tikzcd}$$

Lemma: The two definitions are equivalent.

Proof: An object constructed in Altern. def'n satisfies Definition:
Let (X, π_1, π_2) be as above. We need a functor isomorphism
$\eta: Hom_{\mathcal{C}}(\bullet, X) \xRightarrow{\sim} Hom_{\mathcal{C}}(\cdot, X_1) \times Hom_{\mathcal{C}}(\cdot, X_2)$

- Construct $\eta_Y : Hom_{\mathcal{C}}(Y, X) \xrightarrow{\sim} Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2)$

$$\psi \longmapsto (\pi_1 \circ \psi, \pi_2 \circ \psi)$$

η_Y is a bijection b/c of the universal property.

- Check η is a functor morphism, i.e. $\forall$ $Y' \xrightarrow{f} Y$ the following diagram commutes:

$$\begin{tikzcd}
Hom_{\mathcal{C}}(Y, X) \arrow[r, "\eta_Y : \psi \mapsto (\pi_1 \circ \psi, \pi_2 \circ \psi)"] \arrow[d, "\psi \mapsto \psi \circ f"] & Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2) \arrow[d, "(\psi_1, \psi_2) \mapsto (\psi_1 \circ f, \psi_2 \circ f)"] \\
Hom_{\mathcal{C}}(Y', X) \arrow[r, "\psi' \mapsto (\pi_1 \circ \psi', \pi_2 \circ \psi')"] & Hom_{\mathcal{C}}(Y', X_1) \times Hom_{\mathcal{C}}(Y', X_2)
\end{tikzcd}$$

— exercise.

Now: Definition $\rightsquigarrow$ Alternative definition:

Note $F_1 \times F_2$ admits a functor morphism $\pi^i : F_1 \times F_2 \to F_i$

π^i_Y is the projection $F_1(Y) \times F_2(Y) \to F_i(Y)$ (exercise). Now we have

$\eta : Hom_{\mathcal{C}}(\cdot, X_1 \times X_2) \longrightarrow Hom_{\mathcal{C}}(\cdot, X_1) \times Hom_{\mathcal{C}}(\cdot, X_2) \rightsquigarrow$

$\pi^i \circ \eta : Hom_{\mathcal{C}}(\cdot, X_1 \times X_2) \longrightarrow Hom_{\mathcal{C}}(\cdot, X_i)$. By Yoneda,

$\pi^i \circ \eta = \eta^{\pi_i}$ for the unique $\pi_i \in Hom_{\mathcal{C}}(X_1 \times X_2, X_i)$ $(i=1,2)$:

$(\pi^i \circ \eta)_Y(\psi) = \pi_i \circ \psi$. $(\forall\, \psi \in Hom_{\mathcal{C}}(Y, X_1 \times X_2))$. Since η is a functor isomorphism $\Rightarrow$ $\eta_Y = ((\pi^1 \circ \eta)_Y, (\pi^2 \circ \eta)_Y)$ is a bijection

$$\begin{array}{ccc} Hom_{\mathcal{C}}(Y, X_1 \times X_2) & \xrightarrow{\sim} & Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2). \\ \psi \in & & \in \\ \psi & \longmapsto & (\pi_1 \circ \psi, \pi_2 \circ \psi) \ (= \eta_Y(\psi)) \end{array}$$

So we recover the universal property. $\square$

2.2) Coproduct := product in $\mathcal{C}^{opp}$. Notation: $X_1 * X_2$

Universal property: have $X_i \xrightarrow{\iota_i} X_1 * X_2$ s.t. $\forall\, Y \in Ob(\mathcal{C})$, $X_i \xrightarrow{\varphi_i} Y$, $i=1,2$ $\exists!\ X_1 * X_2 \xrightarrow{\varphi} Y$ s.t. the following is comm've:

$$\begin{tikzcd} & Y & \\ & X \arrow[u, dashed, "\varphi"] & \\ X_1 \arrow[ur, "\iota_1"] \arrow[uur, bend left, "\varphi_1"] & & X_2 \arrow[ul, "\iota_2"'] \arrow[uul, bend right, "\varphi_2"'] \end{tikzcd}$$

Equiv.: $X_1 * X_2$ represents $Hom_{\mathcal{C}}(X_1, \cdot) \times Hom_{\mathcal{C}}(X_2, \cdot)$.

2.3) Examples: Products in Sets, Groups, Rings, A-Mod etc: product = direct product.

Coproducts:

In Sets, coproduct = disjoint union.

In A-Mod, coproduct = direct sum (see Prob 7 in HW1)

In general alg'c cat's can talk about objects defined by generators & relations:

Let X_1 be given by generators a_i, $(i\in I)$, w. relations F_α, $\alpha\in A$; let X_2 have generators b_j, $(j\in J)$ w. relations G_β, $\beta\in B$

Then $X_1 * X_2$ is generated by the union: a_i $(i\in I)$, b_j $(j\in J)$ w. relations F_α (on a_i's) & G_β (on b_j's) - also the union

2.4) Remarks.

- can talk about (co)products indexed by an arbitrary set, I (rather than $\{1,2\}$), for example, direct sums & products in A-Mod.
- products may fail to exist: Let $\mathcal{C}$ be the full subcat. in $\mathbb{F}$-Vect w. objects being odd-dimensional vector spaces For any two objects in $\mathcal{C}$, there's no product (in $\mathcal{C}$).

2.5) Initial/terminal objects in cat's.

Definition: Let $\mathcal{C}$ be a category. Say $X\in Ob(\mathcal{C})$ is:

- <u>initial</u> if $\forall\, Y\in Ob(\mathcal{C}) \Rightarrow Hom_{\mathcal{C}}(X,Y)$ consists of 1 element.
- <u>terminal</u> — · — · — · — · $Hom_{\mathcal{C}}(Y,X)$ — · — · — · — · —

Initial in $\mathcal{C}$ = terminal in $\mathcal{C}^{opp}$.

Examples: • In A-Mod, $\{0\}$ is both initial & terminal

• In Rings: initial object: $\mathbb{Z}$
terminal object: $\{0\}$.

Exercise: For $X_1, X_2 \in Ob(\mathcal{C}) \rightsquigarrow$ new cat'y $\mathcal{C}(X_1, X_2)$

Ob: triples $\{(Y, \psi_1, \psi_2) \mid Y \in Ob(\mathcal{C}), \psi_i \in Hom_{\mathcal{C}}(Y, X_i), i=1,2\}$

$Hom((Y, \psi_1, \psi_2), (Y', \psi_1', \psi_2')) := \{\varphi \in Hom_{\mathcal{C}}(Y, Y') \mid \psi_i = \psi_i' \circ \varphi, i=1,2\}$

Composition of morphisms is restricted from $\mathcal{C}$.

Persuade yourself that $\mathcal{C}(X_1, X_2)$ is a category & $(X_1 \times X_2, \pi_1, \pi_2)$ is a terminal object (if exists)