Lecture 14.

1) Adjoint functors | BONUS: adjunction unit & counit.

Reference: [R], Section 4.1, Hilton-Stammbach, Section 2.7.

1.1) Definition (of adj't functors).

Let $\mathcal{C}, \mathcal{D}$ be cat's, $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ be functors.

Def'n: F is <u>left adjoint</u> to G if

$\forall X \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D})$ $\exists$ bijection $\eta_{X,Y}: Hom_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim}$ $Hom_{\mathcal{C}}(X, G(Y))$ s.t.

(1) $\forall$ $X, X' \in Ob(\mathcal{C})$, $Y \in Ob(\mathcal{D})$, $X' \xrightarrow{\varphi} X$ the following is comm've:

$\rightsquigarrow F(X') \xrightarrow{F(\varphi)} F(X)$

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F(X), Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "? \circ F(\varphi)"] & Hom_{\mathcal{C}}(X, G(Y)) \arrow[d, "? \circ \varphi"] \\
Hom_{\mathcal{D}}(F(X'), Y) \arrow[r, "\eta_{X',Y}"] & Hom_{\mathcal{C}}(X', G(Y))
\end{tikzcd}$$

(2) $\forall$ $Y, Y' \in Ob(\mathcal{D})$, $Y \xrightarrow{\psi} Y'$; $X \in Ob(\mathcal{C})$, the following is comm've

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F(X), Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "\psi \circ ?"] & Hom_{\mathcal{C}}(X, G(Y)) \arrow[d, "G(\psi) \circ ?"] \\
Hom_{\mathcal{D}}(F(X), Y') \arrow[r, "\eta_{X,Y'}"] & Hom_{\mathcal{C}}(X, G(Y'))
\end{tikzcd}$$

Remarks: 1) Fix X & consider composition of functors

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{F_X = Hom_{\mathcal{C}}(X,\bullet)} Sets$$

If F is left adj't to G, then $F(X)$ represents this comp'n i.e. $Hom_{\mathcal{D}}(F(X),\bullet)$ is isom'c to this comp'n via $\eta_{X,?}$, which is a functor isomorphism by diagram (2).

2^*) Can view $Hom_{\mathcal{C}}(\bullet,?)$ as a functor $\mathcal{C}^{opp}\times\mathcal{C} \longrightarrow Sets$
Similarly for $\mathcal{D} \rightsquigarrow$ compositions $\mathcal{C}^{opp}\times\mathcal{D} \longrightarrow Sets$

$$Hom_{\mathcal{D}}(F(\bullet),?),\ Hom_{\mathcal{C}}(\bullet, G(?))$$

Diagrams (1) & (2) combine to show that [F is left adj't to G] $\Longleftrightarrow$ the two functors above are isomorphic (via $\eta_{\bullet,?}$)

3) Often we get interesting adjoint functors F/G starting from (sometimes boring) functors - which is why we care about adjoint functors in this course.

4) G is called <u>right adjoint</u> to F.

1.2) Examples.

Ex 1: Let G be $For: A\text{-}Mod \to Sets$ (A is comm've unital ring), $F := Free: Sets \longrightarrow A\text{-}Mod$

$$I \longmapsto A^{\oplus I}$$

Claim: F is left adj't to G ($Maps := Hom_{Sets}$)

• construct $\eta_{I,M}: \mathrm{Hom}_A(A^{\oplus I}, M) \xrightarrow{\sim} M^{\times I} = \mathrm{Maps}(I, M)$

$$\tau \overset{\eta_{I,M}}{\longmapsto} [i \mapsto \tau(e_i)]$$

• check comm'ive diagram (1): $\forall$ maps $\varphi: I' \to I$:

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\eta_{I,M}", "\sim"'] \arrow[d, "? \circ \mathrm{Free}(\varphi)"] & \mathrm{Maps}(I, M) \arrow[d, "? \circ \varphi"] \\
\mathrm{Hom}_A(A^{\oplus I'}, M) \arrow[r, "\eta_{I',M}", "\sim"'] & \mathrm{Maps}(I', M)
\end{tikzcd}$$

where $\mathrm{Free}(\varphi)$ is unique $[\mathrm{Free}(\varphi)](e_i) := e_{\varphi(i)}$

$\downarrow \longrightarrow$: $\tau \mapsto [\text{unique } \tau': A^{\oplus I'} \to M \text{ s.t. } \tau'(e'_j) := \tau(e_{\varphi(j)})]$

$$\downarrow$$

$$[j \mapsto \tau(e_{\varphi(j)})]$$

$\longrightarrow \downarrow$: $\tau \mapsto [i \mapsto \tau(e_i)] \mapsto [j \mapsto \tau(e_{\varphi(j)})]$

Check (2): for $\psi \in \mathrm{Hom}_A(M, M')$, the following is comm'ive:

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\sim"'] \arrow[d, "\psi \circ ?"] & \mathrm{Maps}(I, M) \arrow[d, "\psi \circ ?"] \\
\mathrm{Hom}_A(A^{\oplus I}, M') \arrow[r, "\sim"'] & \mathrm{Maps}(I, M')
\end{tikzcd}$$

where now ψ is viewed as map of sets

$\overset{\longrightarrow}{\ \ \downarrow}$: $\tau \mapsto [i \mapsto \tau(e_i)] \longmapsto [i \mapsto \varphi(\tau(e_i))]$

$\downarrow\!\longrightarrow$: $\tau \mapsto \varphi \circ \tau \longmapsto$

Adjointness is established

Ex 2: G is the inclusion functor $\mathbb{Z}$-Mod $\hookrightarrow$ Groups,
$F := Ab$: Groups $\longrightarrow \mathbb{Z}$-Mod, $G \mapsto G/(G,G)$.
Claim: F is left adjoint to G.
$\pi_G :=$ natural epimorphism $G \twoheadrightarrow G/(G,G)$.

· Construct bijection $\eta_{G,M} : \mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M) \longrightarrow \mathrm{Hom}_{\mathrm{Groups}}(G,M)$
$$\tau \longmapsto \tau \circ \pi_G$$

Exercise: $\eta_{G,M}$ is a bijection.

· Check comm've diagram (1): $\forall\ \varphi : G' \to G$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M) \arrow[r, "?\circ\pi_G"] \arrow[d, "?\circ Ab(\varphi)"] & \mathrm{Hom}_{\mathrm{Groups}}(G,M) \arrow[d, "?\circ\varphi"] \\
\mathrm{Hom}_{\mathbb{Z}}(G'/(G',G'), M) \arrow[r, "?\circ\pi_{G'}"] & \mathrm{Hom}_{\mathrm{Groups}}(G',M)
\end{tikzcd}$$

where $Ab(\varphi)(g'(G',G')) = \varphi(g')(G,G) \quad (g' \in G')$

$\rightarrow\downarrow \; = \; ?\circ\underbrace{(\pi_G\circ\varphi)}$

$\qquad\qquad\qquad\;\; \parallel \quad \longleftarrow$ follows from constr'n of $Ab(\varphi)$

$\downarrow\rightarrow \; = \; ?\circ\overbrace{(Ab(\varphi)\circ\pi_{G'})}$

Check comm've diagram (2): $\varphi: M\to M'$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M) \arrow[r, "?\circ\pi_G"] \arrow[d, "\varphi\circ ?"] & \mathrm{Hom}_{Groups}(G, M) \arrow[d, "\varphi\circ ?"] \\
\mathrm{Hom}_{\mathbb{Z}}(G/(G,G), M') \arrow[r, "?\circ\pi_G"] & \mathrm{Hom}_{Groups}(G, M')
\end{tikzcd}$$

Both $\rightarrow\downarrow$ & $\downarrow\rightarrow$ give $\varphi\circ ?\circ\pi_G$, manifestly the same!

Adjointness is checked.

1.3) Uniqueness.

Proposition: Let $G:\mathcal{D}\to\mathcal{C}$ be a functor. Claim: if its left adjoint exists, then it's unique up to a functor isom'm.

Proof: Suppose $F^1, F^2: \mathcal{C}\to\mathcal{D}$ are both left adj't to $G:\mathcal{D}\to\mathcal{C}$ $\rightsquigarrow$ $\eta^i_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F^i(X), Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ that make (1) & (2) comm've $\rightsquigarrow$

$$\eta_{X,Y} := (\eta^2_{X,Y})^{-1}\circ\eta^1_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F^1(X), Y)\longrightarrow\mathrm{Hom}_{\mathcal{D}}(F^2(X), Y)$$

which makes analogs of (1) & (2) comm've:

(1) $\forall\ X' \xrightarrow{\varphi} X$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F^1(X), Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "? \circ F^1(\varphi)"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y) \arrow[d, "? \circ F^2(\varphi)"] \\
\mathrm{Hom}_{\mathcal{D}}(F^1(X'), Y) \arrow[r, "\eta_{X',Y}"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X'), Y)
\end{tikzcd}$$

(2) $\forall\ Y \xrightarrow{\psi} Y'$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F^1(X), Y) \arrow[r, "\eta_{X,Y}"] \arrow[d, "\psi \circ ?"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y) \arrow[d, "\psi \circ ?"] \\
\mathrm{Hom}_{\mathcal{D}}(F^1(X), Y') \arrow[r, "\eta_{X,Y'}"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y')
\end{tikzcd}$$

Fix X, look at (2): it tells us that $\eta_{X,?}$ is a functor morphism (and hence isomorphism - b/c each $\eta_{X,Y}$ is bijection) between $\mathrm{Hom}_{\mathcal{D}}(F^1(X), \bullet)$ & $\mathrm{Hom}_{\mathcal{D}}(F^2(X), \bullet)$. By Yoneda Lemma, have the unique isomorphism $\tau_X \in \mathrm{Hom}_{\mathcal{C}}(F^2(X), F^1(X))$ s.t. $\eta_{X,?} = - \circ \tau_X$.

Claim: (1) now tells us that τ is a functor morphism (hence, an isomorphism). Indeed, in (1):

$\longrightarrow \downarrow$: $? \circ \underbrace{\left(\tau_X \circ F^2(\varphi)\right)}$

$\qquad\qquad \| \qquad$ — b/c (1) is comm've

$\downarrow \longrightarrow$: $? \circ \overbrace{\left(F^1(\varphi) \circ \tau_{X'}\right)}$

i.e.

$$\begin{tikzcd}
F^2(X') \arrow[r, "\tau_{X'}"] \arrow[d, "F^2(\varphi)"] & F^1(X') \arrow[d, "F^1(\varphi)"] \\
F^2(X) \arrow[r, "\tau_X"] & F^1(X)
\end{tikzcd}$$

is commutative

So τ is indeed a functor (iso)morphism $\square$

1.4) Analogy between adjoint functors & adjoint linear maps.

Linear map story	Functor story
Fin. dim. vector space V	Category $\mathcal{C}$
Dual space V^*	Category $\mathcal{C}^{opp}$
Map $V^* \times V \to \mathbb{F}$ (pairing)	Functor $\mathcal{C}^{opp} \times \mathcal{C} \longrightarrow Sets$ $Hom_{\mathcal{C}}(\bullet, ?)$
Cond'n for $A^*: W^* \to V^*$ being adj't to $A: V \to W$:	Condition for functors being adjoint:

$\langle A^* \beta, v \rangle = \langle \beta, Av \rangle$ $\forall \beta \in W^*, v \in V.$	Isom'm of functors $\mathcal{C}^{opp} \times \mathcal{D} \to Sets$ $Hom_{\mathcal{D}}(F(\cdot), ?) \xrightarrow{\sim} Hom_{\mathcal{C}}(\cdot, G(?))$

Note: this is just an analogy:

BONUS: adjunction unit & counit.
Let $F: \mathcal{C} \to \mathcal{D}$ be left adjoint to $G: \mathcal{D} \to \mathcal{C}$. We claim that this gives rise to functor morphisms: the adjunction unit $\varepsilon: Id_{\mathcal{C}} \Longrightarrow GF$ & counit $\eta: FG \Longrightarrow Id_{\mathcal{D}}$.

We construct ε and leave η as an exercise.

Consider $X_1, X_2 \in Ob(\mathcal{C})$. Then we have the bijection
$$\eta_{X_1, F(X_2)}: Hom_{\mathcal{D}}(F(X_1), F(X_2)) \xrightarrow{\sim} Hom_{\mathcal{C}}(X_1, GF(X_2))$$

Note that F gives rise to a map $Hom_{\mathcal{C}}(X_1, X_2) \to Hom_{\mathcal{D}}(F(X_1), F(X_2))$
Composing this map w. the bijection $\eta_{X_1, F(X_2)}$ we get
$$\varepsilon_{X_1, X_2}: Hom_{\mathcal{C}}(X_1, X_2) \longrightarrow Hom_{\mathcal{C}}(X_1, GF(X_2)).$$
Now we can argue as in the proof of Proposition 1.3 to see that $\exists! \ \varepsilon: Id_{\mathcal{C}} \Rightarrow GF$ s.t. $\varepsilon_{X_1, X_2}(\varphi) = \varepsilon_{X_2} \circ \varphi$.

A natural question to ask is: for two functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ & functor morphisms $\varepsilon: Id_{\mathcal{C}} \Rightarrow GF$, $\eta: FG \Rightarrow Id_{\mathcal{D}}$ when is F left adjoint to G (& ε, η unit & counit).

Very Premium Exercise: TFAE

a) F is left adjoint to G w. unit ε & counit η

6) The composed morphisms $F \Rightarrow FGF \Rightarrow F$, $G \Rightarrow GFG \Rightarrow G$ induced by ε, η (cf. Problem 8 in HW3) are the identity endomorphisms (of F & G).