Lecture 15.

1) Additive functors.
2) Tensor products of modules.
Reference: [AM], Section 2.7

BONUS: Additive categories.

1.1) Definition: Let A, B be comm've unital rings $\rightsquigarrow$ categories of modules A-Mod, B-Mod. Hom sets in these cat's are abelian groups

Definition: a functor $F: A\text{-Mod} \to B\text{-Mod}$ is <u>additive</u> if $\forall M, N \in Ob(A\text{-Mod}) \Rightarrow Hom_A(M,N) \longrightarrow Hom_B(F(M), F(N))$, $\psi \longmapsto F(\psi)$ is a group homomorphism.

Similarly, can talk about additive functors $A\text{-Mod}^{opp} \to B\text{-Mod}$.

1.2) Examples of additive functors.

1) Fix $M \in Ob(A\text{-Mod}) \rightsquigarrow F_M = Hom_A(M, \cdot): A\text{-Mod} \longrightarrow Sets$

Can promote F_M to be a functor, $\tilde{F}_M = Hom_A(M,\cdot): A\text{-Mod} \longrightarrow A\text{-Mod}$ $\rightsquigarrow F_M = For \circ \tilde{F}_M$

To see $\tilde{F}_M$ makes sense we need to know that:

1) $\forall$ A-module $N \Rightarrow \tilde{F}_M(N) = Hom_A(M,N)$ is an A-module (we know this already)

2) $\forall \psi \in Hom_A(N,N') \Rightarrow \tilde{F}_M(\psi): Hom_A(M,N) \longrightarrow Hom_A(M,N')$, $\varphi \longmapsto \psi\circ\varphi$ is A-linear.

1) & 2) $\Rightarrow$ $\widetilde{F}_M$ is a functor A-Mod $\longrightarrow$ A-Mod (functor axioms for $\widetilde{F}_M \Leftarrow$ the axioms for F_M).

To see that $\widetilde{F}_M$ is additive need to show

3) The map $\psi \mapsto \widetilde{F}_M(\psi)$ is a group homom'm

$$\mathrm{Hom}_A(N,N') \longrightarrow \mathrm{Hom}_A(\widetilde{F}_M(N), \widetilde{F}_M(N')).$$

2) & 3) follow from:

Very Important Exercise: The composition map

$$\mathrm{Hom}_A(M,N) \times \mathrm{Hom}_A(N,N') \longrightarrow \mathrm{Hom}_A(M,N'), \ (\varphi,\psi) \mapsto \psi\circ\varphi$$

is A-bilinear (i.e. if we fix one of the arguments, this map is A-linear in the other).

2) $\Leftrightarrow$ comp'n map is A-linear in φ; 3) $\Leftrightarrow$ additive in ψ.

1^{opp}) Fix $N \in Ob(A\text{-Mod})$, $\widetilde{F}_N^{opp} := \mathrm{Hom}_A(\cdot, N): A\text{-Mod}^{opp} \longrightarrow A\text{-Mod}$.

Exercise: $\widetilde{F}_N^{opp}$ is an additive functor.

2) General'n of 1). Let $\zeta: A \to B$ be a ring homom'm. Let M be a B-module (hence also an A-module thx to ζ). Then $\mathrm{Hom}_A(M,\bullet)$ is an additive functor A-Mod $\to$ B-Mod

In order to see this we need:

1) Define a B-module str're on $\mathrm{Hom}_A(M,N)$ $\forall$ A-module N.

2) $\forall\, \psi \in \mathrm{Hom}_A(N,N')$, the map $\psi\circ ?: \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,N')$ is B-linear.

1: $\varphi \in Hom_A(M,N)$, $[b\varphi](m) := \varphi(bm), \forall m \in M$.

Exercise: • Show that $b\varphi$ is indeed A-linear map $M \to N$;
• Show that $(b,\varphi) \mapsto b\varphi$ defines a B-module structure on A-module $Hom_A(M,N)$.
• Show property 2).

2^{opp}) In assumptions of 2), for $N \in B\text{-Mod}$, $Hom_A(\cdot, N)$ is an additive functor $A\text{-Mod}^{opp} \to B\text{-Mod}$.
Exercise: work out details.

3) With the same assumptions, the forgetful functor $B\text{-Mod} \longrightarrow A\text{-Mod}$ is additive.

4) Let $S \subset A$ be localizable subset $\rightsquigarrow$ the localization functor $\bullet_S : A\text{-Mod} \longrightarrow A_S\text{-Mod}$ is additive ($\Leftarrow$ part 1 of Prob 5 in HW3).

2) Tensor products of modules - source of more additive functors.

2.1) Bilinear maps: Let A be a comm've ring, M_1, M_2, N be A-modules $\rightsquigarrow$ set $Bilin_A(M_1 \times M_2, N) = \{A\text{-bilinear maps } M_1 \times M_2 \to N\}$.

Digression: why should we care about bilinear maps - b/c they are everywhere!

· Linear algebra: for an $\mathbb{F}$-vector space V can talk about bilinear forms := bilinear maps $V\times V \to \mathbb{F}$, fundamentally important in linear algebra & beyond.

· if M is an A-module $\Rightarrow$ mult'n map $A\times M \to M$ is A-bilinear.

· the composition map from Very Important Exercise is bilinear

· etc.

Observation: $F_{M_1,M_2} := Bilin_A(M_1\times M_2, \bullet)$ is actually a functor $A\text{-Mod} \longrightarrow Sets$:

To $\psi \in Hom_A(N,N')$ we assign

$$F_{M_1,M_2}(\psi): Bilin_A(M_1\times M_2, N) \longrightarrow Bilin_A(M_1\times M_2, N')$$
$$\beta \longmapsto \psi\circ\beta$$

Exercise: Show $\psi\circ\beta$ is A-bilinear & F_{M_1,M_2} is indeed a functor $A\text{-Mod} \to Sets$.

2.2) Definition of tensor product:

Definition: By the <u>tensor product</u> $M_1 \otimes_A M_2$ we mean a representing object for $Bilin_A(M_1\times M_2, \bullet)$ i.e. want a functor isomorphism $Hom_A(M_1\otimes_A M_2, \bullet) \xrightarrow{\sim} Bilin_A(M_1\times M_2, \bullet)$

Equivalently (compare to products in Lec 13) can define tensor products via universal property:

tensor product of M_1 & M_2 is an A-module $M_1\otimes_A M_2$ w. a bilinear map $M_1\times M_2 \longrightarrow M_1\otimes_A M_2$, $(m_1,m_2)\longmapsto m_1\otimes m_2$ w. the following universal property:

$\forall$ A-module N & A-bilinear map $\beta: M_1\times M_2 \to N$ $\exists!$
A-linear map $\tilde\beta: M_1\otimes_A M_2 \to N$ s.t. $\beta(m_1,m_2)=\tilde\beta(m_1\otimes m_2)$

$$\begin{tikzcd}
M_1\times M_2 \arrow[d, "{(m_1,m_2)\mapsto m_1\otimes m_2}"'] \arrow[dr, "\beta"] & \\
M_1\otimes_A M_2 \arrow[r, dashed, "\tilde\beta"] & N
\end{tikzcd}$$

Rem: Under isomorphism $\mathrm{Bilin}_A(M_1\times M_2, M_1\otimes_A M_2) \xrightarrow{\sim}$
$\mathrm{Hom}_A(M_1\otimes_A M_2, M_1\otimes_A M_2)$, the map $(m_1,m_2)\mapsto m_1\otimes m_2$ corresponds
to the identity on the r.h.s.

2.3) Existence of tensor products.

Being representing object, $M_1\otimes_A M_2$ is unique up to an iso.
More precisely, if $M_1\otimes_A M_2$ w. $(m_1,m_2)\mapsto m_1\otimes m_2$, $M_1\otimes_A' M_2$
w. $(m_1,m_2)\mapsto m_1\otimes' m_2$ be two tensor products, then $\exists!$
A-linear map $M_1\otimes_A M_2 \to M_1\otimes_A' M_2$ s.t. $m_1\otimes m_2\mapsto m_1\otimes' m_2$
$\forall m_i\in M_i$ & is an isomorphism.
But existence is not guaranteed.

Theorem: $M_1\otimes_A M_2$ exists $\forall M_1, M_2\in Ob(A\text{-Mod})$.
Construction/proof is in 2 steps.
Step 1: Assume $M_1\simeq A^{\oplus I}$ for some set I.

Lemma: $A^{\oplus I} \otimes_A M$ exists & is identified w. $M^{\oplus I}$ w. $((a_i)_{i\in I}) \otimes m \longleftrightarrow (a_i m)_{i\in I}$

Proof: Observe that the map $A^{\oplus I} \times M \longrightarrow M^{\oplus I}$, $((a_i), m) \mapsto (a_i m)$ is bilinear. We need to show universal property: $\forall$ bilinear map $\beta: A^{\oplus I} \times M \longrightarrow N$ $\exists!$ linear map $\tilde{\beta}: M^{\oplus I} \to N$ s.t. $\beta((a_i), m) = \tilde{\beta}((a_i m))$.

Define, for $i \in I$, a map $\beta_i: M \to N$, $\beta_i(m) := \beta(e_i, m)$, linear

Recall that (Prob 7 in HW1):

$$\begin{array}{ccc} \mathrm{Hom}_A(M^{\oplus I}, N) & \xrightarrow{\sim} & \mathrm{Hom}_A(M,N)^{\times I} \\ \Downarrow & & \Downarrow \\ \tilde{\beta} & \longleftarrow & (\beta_i)_{i\in I} \end{array}$$

definition of $\tilde{\beta}$ $\rightsquigarrow$

Check: $\tilde{\beta}((a_i m)) = \beta((a_i), m))$ $(*)$

$$\tilde{\beta}((a_i m)) \underset{\text{by def'n of } \tilde{\beta}}{=} \sum_{i\in I} \beta_i(a_i m) \underset{\beta_i \text{ is linear}}{=} \sum a_i \beta_i(m) \underset{\text{by def'n of } \beta_i}{=} \sum a_i \beta(e_i, m) =$$

$= [\beta$ is linear in 1st argument$] = \beta((a_i), m)$.

Exercise: $\tilde{\beta}$ is a unique linear map $M^{\oplus I} \to N$ satisfying $(*)$. $\square$

BONUS: additive categories.

In our definition of additive functors we need to consider categories A-Mod, A-Mod^{opp} separately. This is awkward. The concept of an "additive category" includes these examples & much more. And we can talk about additive functors between additive categories.

Definition: An additive category $\mathcal{C}$ is

(Data) · a category

· together w. abelian group structure on $Hom_{\mathcal{C}}(X,Y)$ $\forall\, X,Y \in Ob(\mathcal{C})$

These data have to satisfy the following axioms:

· (added on Oct 30) $\exists\, 0 \in Ob(\mathcal{C})$ w. $Hom_{\mathcal{C}}(X,0) = Hom_{\mathcal{C}}(0,X) = \{0\}$.

· $\forall\, X,Y \in Ob(\mathcal{C})$, $\exists$ a product $X \times Y \in Ob(\mathcal{C})$.

· the composition map $Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \to Hom_{\mathcal{C}}(X,Z)$ is bi-additive (a.k.a. $\mathbb{Z}$-bilinear), $\forall\, X,Y,Z \in Ob(\mathcal{C})$.

Recall that in $\mathbb{Z}$-Mod, the product of two objects (in fact, of any finite collection) coincides w. their coproduct. This property carries over to arbitrary additive categories. The (co)product $X \times Y$ is usually called the direct sum and is denoted by $X \oplus Y$.

Examples (of additive categories):

1) A-Mod (for a ring A, not necessarily comm've).

2) A-Mod^{opp}

3) A full subcategory in an additive category is additive iff it's closed under taking finite direct sums. For example, in A-Mod we can consider the full subcategories consisting of free (or of projective) objects. They are closed under direct sums hence additive.

4*) In various parts of Geometry / Topology people consider

categories of "sheaves". These categories are additive.
5*) Various constructions in Homological Algebra produce more complicated additive categories from A-Mod: homotopy categories of complexes, derived categories etc.