Lecture 16.

1) Tensor products of modules cont'd.
Refs: [AM], Section 2.7

BONUSES:
- Tensor products/noncomm've rings
- Another category of rings.

1.0) Recap.

M_1, M_2 A-modules (A is comm've & unital) $\rightsquigarrow$ another A-module $M_1 \otimes_A M_2$ w. a bilinear map $M_1 \times M_2 \to M_1 \otimes_A M_2$, $(m_1, m_2) \mapsto m_1 \otimes m_2$ w. the following univ. property:

$\forall$ A-bilinear map $\beta: M_1 \times M_2 \longrightarrow N$ $\exists!$ A-linear map $\tilde{\beta}: M_1 \otimes_A M_2 \to N$ s.t. $\beta(m_1, m_2) = \tilde{\beta}(m_1 \otimes m_2)$.

Thm: $M_1 \otimes_A M_2$ exists $\forall M_1, M_2$.

Step 1:
$$\begin{array}{ccc} A^{\oplus I} \otimes_A M & \xrightarrow{\sim} & M^{\oplus I} \\ \cup & & \cup \\ (a_i)_{i \in I} \otimes m & \longmapsto & (a_i m)_{i \in I} \end{array}$$

1.1) Existence of tensor products

Let M_1', M_2 be A-modules s.t. $M_1' \otimes_A M_2$ exists. Let $K_1 \subset M_1'$ be submodule $\rightsquigarrow M_1 := M_1'/K_1$ & $\pi_1: M_1' \twoheadrightarrow M_1$. Inside $M_1' \otimes_A M_2$ consider submodule $K := \operatorname{Span}_A(k_1 \otimes m_2 \mid k_1 \in K_1, m_2 \in M_2)$.
$\rightsquigarrow M_1' \otimes_A M_2$ & $\pi: M_1' \otimes_A M_2 \twoheadrightarrow M_1' \otimes_A M_2 / K$.

Proposition: $M_1' \otimes_A M_2 / K$ is the tensor product $M_1 \otimes_A M_2$ & for $m_1 = \pi_1(m_1') \in M_1$ & $m_2 \in M_2 \Rightarrow m_1 \otimes m_2 := \pi(m_1' \otimes m_2)$.

How this implies Thm: Pick M_1, M_2; know M_1 is a quotient of $A^{\oplus I} =: M_1'$ for some I; $M_1' \otimes_A M_2$ exists by Step 1 &

$M_1 \otimes_A M_2$ exists by Step 2.

Proof of Proposition: $m_1 \otimes m_2$ is well-defined (independent of choice of m_1') & gives a bilinear map $M_1 \times M_2 \to M_1' \otimes_A M_2 / K$. Now we only need to check univ'l property: $\forall$ bilinear $\beta: M_1 \times M_2 \to N$ $\exists!$ linear $\tilde{\beta}: M_1' \otimes_A M_2 / K \to N$ s.t. $\beta(m_1, m_2) = \tilde{\beta}(m_1 \otimes m_2)$.

Define $\beta': M_1' \times M_2 \to N$ by $\beta'(m_1', m_2) = \beta(\pi_1(m_1'), m_2)$ so β' is bilinear $\rightsquigarrow \exists!\ \tilde{\beta}': M_1' \otimes_A M_2 \to N$ s.t. $\tilde{\beta}'(m_1' \otimes m_2) = \beta'(m_1', m_2)$. Note that $\tilde{\beta}'(k_1 \otimes m_2) = \beta'(k_1, m_2) = \beta(0, m_2) = 0$ so $\tilde{\beta}'(K) = 0$. So $\exists!\ \tilde{\beta}: M_1' \otimes_A M_2 / K \to N$ s.t. $\tilde{\beta}' = \tilde{\beta} \circ \pi$. This is precisely the cond'n $\tilde{\beta}(m_1 \otimes m_2) = \beta(m_1, m_2)$ $\square$

1.2) Examples.

1) Tensor product of free modules: $A^{\oplus I} \otimes_A A^{\oplus J} = [\text{Step } 1] = (A^{\oplus J})^{\oplus I} \simeq A^{\oplus (I \times J)}$ w. basis $e_i \otimes e_j$ $(i \in I, j \in J)$.

2) $A = \mathbb{F}[x,y]$ ($\mathbb{F}$ is field), ideal $I = (x,y)$, want $I \otimes_A I$.
Need to present I as a quotient of free module:
$A^{\oplus 2} \twoheadrightarrow I$, $(a,b) \mapsto (ax+by)$.
Compute the kernel $K_1 = \{(a,b) \mid ax = -by\} = [A \text{ is UFD} \Rightarrow a = gy, b = -gx, g \in A] = \mathrm{Span}_A((y,-x))$.

Step 2: $I \otimes_A I = A^{\oplus 2} \otimes_A I / \mathrm{Span}_A (g(y,-x) \otimes m = (y,-x) \otimes gm)$

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$= [\text{Step 1}] = I^{\oplus 2} / \mathrm{Span}_A\left((ym, -xm) \mid m \in I\right).$

Now want to present $I \otimes_A I$ by generators & rel'ns.
$I = A^{\oplus 2} / \mathrm{Span}_A((y,-x))$ (the 1st generator of $I \leftrightarrow \xi_1 \in A^{\oplus 2}$, the 2nd gen'r $\leftrightarrow \xi_2 \in A^{\oplus 2}$)

$$I \otimes_A I = \frac{\left(A^{\oplus 4} / \mathrm{Span}_A((y,-x,0,0),\ (0,0,y,-x))\right)}{\{\underbrace{(ym, -xm)}_{\text{elements of this form for gen'rs } m \text{ of } I}\mid m \in I\}}.$$

elements of this form for gen'rs m of I,

$$= A^{\oplus 4} / \mathrm{Span}_A\left((y,-x,0,0), (0,0,y,-x), (y,0,-x,0), (0,y,0,-x)\right).$$

1.3) Generators of tensor product.

NOT EVERY element of $M_1 \otimes_A M_2$ has the form $m_1 \otimes m_2$ (we'll call such elements <u>decomposable tensors</u>)

Exercise: $A = \mathbb{F}$ (a field), $M_1 = \mathbb{F}^{\oplus k}$, $M_2 = \mathbb{F}^{\oplus \ell}$ $\rightsquigarrow$
$M_1 \otimes_{\mathbb{F}} M_2 = \{\sum_{i=1}^{\ell} \sum_{j=1}^{\ell} a_{ij}\, e_i \otimes e_j\} \cong \{k \times \ell\text{-matrices}\}$
Show that $\{\text{elements of the form } m_1 \otimes m_2\} \xrightarrow{\sim} \{\text{rk } 1 \text{ matrices}\}$

However, it's still true that $m_1 \otimes m_2$ span (generate) A-module $M_1 \otimes_A M_2$.

Lemma: If $M_k = \mathrm{Span}_A(m_k^i \mid i \in I_k)$ $k = 1,2$, then
$M_1 \otimes_A M_2 = \mathrm{Span}_A(m_1^i \otimes m_2^j \mid i \in I_1, j \in I_2)$.

Proof: $N := M_1 \otimes_A M_2 / \mathrm{Span}_A(m_1^i \otimes m_2^j) \rightsquigarrow$ linear map $\tilde{\beta}: M_1 \otimes_A M_2 \twoheadrightarrow N \rightsquigarrow$ bilinear map $\beta: M_1 \times M_2 \to N$

$\underbrace{\beta(m_1^i, m_2^j) = \tilde{\beta}(m_1^i \otimes m_2^j) = 0}$. Since m_1^i span M_1, m_2^j span M_2

$\Rightarrow \beta = 0 \overset{\text{[univ. property of tens. product]}}{\Longrightarrow} \tilde{\beta} = 0 \Rightarrow N = \{0\}$ $\square$

1.4) Tensor products of linear maps & functoriality.

M_1, M_1', M_2, M_2' A-modules & A-linear maps $\varphi_i: M_i' \to M_i$

Goal: define A-linear map $\varphi_1 \otimes \varphi_2 : M_1' \otimes_A M_2' \to M_1 \otimes_A M_2$.

Consider: $\begin{bmatrix} M_1' \times M_2' & \longrightarrow & M_1 \otimes_A M_2 \\ \Psi & & \Psi \\ (m_1', m_2') & \longmapsto & \varphi_1(m_1') \otimes \varphi_2(m_2') \end{bmatrix}$ gives rise to

Exercise: this map is bilinear.

A-linear map $M_1' \otimes_A M_2' \longrightarrow M_1 \otimes_A M_2$ denoted by $\varphi_1 \otimes \varphi_2$

$\varphi_1 \otimes \varphi_2 (m_1' \otimes m_2') = \varphi_1(m_1') \otimes \varphi_2(m_2')$

Properties of tensor products of maps:

- $id_{M_1} \otimes id_{M_2} = id_{M_1 \otimes_A M_2}$

- Compositions: $M_1'' \xrightarrow{\varphi_1'} M_1' \xrightarrow{\varphi_1} M_1$, $M_2'' \xrightarrow{\varphi_2'} M_2' \xrightarrow{\varphi_2} M_2$

$(\varphi_1\varphi_1') \otimes (\varphi_2\varphi_2') = (\varphi_1 \otimes \varphi_2)(\varphi_1' \otimes \varphi_2')$ b/c they coincide on generators $m_1'' \otimes m_2''$ of $M_1'' \otimes_A M_2''$.

So: we have tensor product functor

$$A\text{-Mod} \times A\text{-Mod} \longrightarrow A\text{-Mod}$$

Important exercise: Prove that $(\varphi_1, \varphi_2) \mapsto \varphi_1 \otimes \varphi_2$:
$\mathrm{Hom}_A(M_1', M_1) \times \mathrm{Hom}_A(M_2', M_2) \longrightarrow \mathrm{Hom}_A(M_1' \otimes_A M_2', M_1 \otimes_A M_2)$
is A-bilinear (hint: check on generators of $M_1' \otimes_A M_2'$)

We will mostly be interested in situation when fix one of the modules: $M_1 := L \rightsquigarrow$ functor $L \otimes_A \bullet : A\text{-Mod} \to A\text{-Mod}$

$M \mapsto L \otimes_A M$

$\varphi: M' \to M \mapsto id_L \otimes \varphi : L \otimes_A M' \longrightarrow L \otimes_A M.$

This is an additive functor!

1.5) "Algebra properties" of tensor products.

Theorem: Let M_1, M_2, M_3 be A-modules. Then:

1) There is a unique isomorphism $(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A (M_2 \otimes_A M_3)$ s.t. $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$. (i.e. tensor product is associative).
2) $\exists!$ isom'm $M_1 \otimes_A M_2 \xrightarrow{\sim} M_2 \otimes_A M_1$ w. $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$.
3) $\exists!$ isom'm $M_1 \otimes_A (M_2 \oplus M_3) \xrightarrow{\sim} M_1 \otimes_A M_2 \oplus M_1 \otimes_A M_3$ w. $m_1 \otimes (m_2, m_3) \mapsto (m_1 \otimes m_2, m_1 \otimes m_3)$
4) $\exists!$ unique isom'm $A \otimes_A M \xrightarrow{\sim} M$ s.t. $a \otimes m \mapsto am$.

Proof: (1) We want A-linear map

$$\tilde{\beta}: (M_1 \otimes_A M_2) \otimes_A M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3), (m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$$

i.e. want a bilinear map $\beta:(M_1\otimes_A M_2)\times M_3 \longrightarrow M_1\otimes_A(M_2\otimes_A M_3)$

$(m_1\otimes m_2, m_3)\mapsto m_1\otimes(m_2\otimes m_3)$

Fix $m_3 \rightsquigarrow$ a linear map $M_2\longrightarrow M_2\otimes_A M_3$, $m_2\mapsto m_2\otimes m_3$. Define $\beta_{m_3}: M_1\otimes_A M_2 \longrightarrow M_1\otimes_A(M_2\otimes_A M_3)$ to be the tensor product of id_{M_1} & $[m_2\mapsto m_2\otimes m_3]$ so $\beta_{m_3}(m_1\otimes m_2)=m_1\otimes(m_2\otimes m_3)$

Note that β_{m_3} depends <u>linearly</u> on m_3

$\rightsquigarrow$ bilinear map $\beta:(M_1\otimes_A M_2)\times M_3 \longrightarrow M_1\otimes_A(M_2\otimes_A M_3)$,

$\beta(x,m_3):=\beta_{m_3}(x) \rightsquigarrow \tilde{\beta}$ as needed.

But $\tilde{\beta}$ is an isom'm: have $M_1\otimes_A(M_2\otimes_A M_3)\longrightarrow(M_1\otimes_A M_2)\otimes M_3$

$m_1\otimes(m_2\otimes m_3)\mapsto(m_1\otimes m_2)\otimes m_3$. It's inverse of $\tilde{\beta}$ b/c this is so on generators. $\square$ of (1).

(2) - commutativity - is an exercise & (4) - unit - follows from our construction.

Added on Oct 27: proof of (3) - distributivity.

$\pi_i: M_2\oplus M_3 \longrightarrow M_i$, $i=2,3$; $\iota_i: M_i\hookrightarrow M_2\oplus M_3$, inclusion

$\rightsquigarrow id_{M_1}\otimes\pi_i: M_1\otimes_A(M_2\oplus M_3)\longrightarrow M_1\otimes_A M_i$, $id_{M_1}\otimes\iota_i$

$$(id_{M_1}\otimes\pi_2, id_{M_1}\otimes\pi_3): M_1\otimes_A(M_2\oplus M_3) \rightleftarrows M_1\otimes_A M_2\oplus M_1\otimes_A M_3 : (id_{M_1}\otimes\iota_2, id_{M_1}\otimes\iota_3)$$

$$id_{M_1}\otimes\iota_2(x)+id_{M_1}\otimes\iota_3(y) \longleftarrow\!\shortmid (x,y)$$

Important (in particular, for HW): check that these maps are mutually inverse.

BONUS 1: Tensor products over noncommutative rings.

Let A be a comm've unital ring & R be an A-algebra (associative & unital but perhaps non-commutative). Recall that it makes sense to talk about left & right R-modules & also about bimodules. Also (compare to Bonus of Lec 3) for two left R-modules M_1, M_2, the Hom set $\operatorname{Hom}_R(M_1, M_2)$ is only an A-module, not an R-module.

As for tensor products, we can tensor left R-modules w. right R-modules. Namely, let M be a left R-module & N be a right R-module. For an A-module L consider the set $\operatorname{Bilin}_R(N\times M, L)$ consisting of all A-bilinear maps $\varphi: N\times M \to L$ s.t. in addition $\varphi(nr, m) = \varphi(n, rm)$ $\forall\, r\in R, n\in N, m\in M$.

Definition: $N\otimes_R M \in Ob(A\text{-Mod})$ represents the functor $\operatorname{Bilin}_R(N\times M, \bullet): A\text{-Mod} \longrightarrow Sets$.

Important exercise: If R is comm've, then this definition gives the same as the definition in Lec 15.

To construct $N\otimes_R M$ we can use the same construction as we did in the lecture. Alternatively, $N\otimes_R M$ is the quotient of $N\otimes_A M$ by the A-submodule $\operatorname{Span}_A(nr\otimes m - n\otimes rm \mid n\in N, m\in M, r\in R)$.

Now suppose we have 2 more A-algebras, S and T. Let N be an S-R-bimodule & M be an R-T-bimodule.

Important exercise: $\exists!$ S-T-bimodule str're on $N\otimes_R M$ s.t.

$s(n\otimes m) = sn\otimes m$, $(n\otimes m)t = n\otimes mt$.

BONUS 2: Another category of rings.

We define a category $\mathcal{C}$, where the objects are (associative unital) rings. For ring R & S define $Hom_{\mathcal{C}}(R,S)$ to be the set of isomorphism classes of finitely generated S-R-bimodules. For an S-R-bimodule M & T-S-bimodule N define set $N\circ M := N\otimes_S M$, this again finitely generated. The unit in $Hom_{\mathcal{C}}(R,R)$ is R itself. Associativity of composition follows from associativity of tensor product.