Lecture 17.

1) Tensor-Hom adjunction.
2) Tensor product of algebras.

Ref: [AM], Sections 2.8, 2.9, 2.11.

Goal: investigate interactions between:
- tensor products (last week)
- representing objects/adjoint functors (2 weeks ago)

1) Tensor-Hom adjunction.

A is a comm've ring, A-module $L \rightsquigarrow$ functors $L\otimes_A \bullet, \operatorname{Hom}_A(L,\bullet)$: $A\text{-Mod} \to A\text{-Mod}$.

Preliminary Thm: $L\otimes_A \bullet$ is left adjoint to $\operatorname{Hom}_A(L,\bullet)$.

We'll need a more general version of Thm involving different rings.

1.1) Tensor products of modules over different rings.

Let $\varsigma: A\to B$, a homomorphism of comm've rings. Let L^B be B-module (& hence also A-module), M^A be A-module $\rightsquigarrow$ A-module $L^B\otimes_A M^A$.

Lemma: $\exists!$ B-module str're on $L^B\otimes_A M^A$ that extends A-module str're & satisfies $b(l\otimes m) = bl\otimes m$.

(compare w. B-module str're on $\operatorname{Hom}_A(L^B, M^A)$ - Lecture 15).

Proof: For $b\in B$, $\varphi_b: L\to L$, $\varphi_b(l) = bl$, A-linear map $\rightsquigarrow$ $bx := (\varphi_b \otimes id_{M^A})(x)$ for $x\in L^B\otimes_A M^A \rightsquigarrow$ map

$$B\times (L^B\otimes_A M^A) \longrightarrow L^B\otimes_A M^A.$$

We claim that

this is a B-module str're. Need to check assoc'y & distributivity in the B-argument, it's enough to do this on generators of $L^B \otimes_A M^A$ (b/c $x \mapsto bx$ is A-linear in x by const'n). We'll check assoc'y: $b_1 \cdot (b_2 \cdot l \otimes m)) = (b_1 b_2 l) \otimes m = (b_1 b_2) \cdot (l \otimes m)$
So we have B-module str're w. required properties.
Uniqueness follows from cond'n $b(l \otimes m) = bl \otimes m$ b/c $l \otimes m$ span $L^B \otimes_A M^A$. $\square$

Exer: For $\varphi \in Hom_A(M_1^A, M_2^A)$, $id_{L^B} \otimes \varphi : L^B \otimes_A M_1^A \to L^B \otimes_A M_2^B$ is B-linear.

1.2) Main result. Fix $L^B \rightsquigarrow$ functor $L^B \otimes_A \bullet : A\text{-Mod} \to B\text{-Mod}$
Also have a functor $Hom_B(L^B, \bullet): B\text{-Mod} \to A\text{-Mod}$, the composition
$$B\text{-Mod} \xrightarrow[Hom_B(L_B, \cdot)]{} B\text{-Mod} \xrightarrow[For]{} A\text{-Mod}.$$

Theorem: $L^B \otimes_A \bullet : A\text{-Mod} \to B\text{-Mod}$ is left adjoint to $Hom_B(L^B, \bullet): B\text{-Mod} \to A\text{-Mod}$.

Proof: We need: for A-module M^A, B-module N^B, produce a set bijection $\eta_{M,N}: Hom_B(L^B \otimes_A M^A, N^B) \xrightarrow{\sim} Hom_A(M^A, Hom_B(L^B, N^B))$ which is "natural" (a.k.a. functorial) in M^A & N^B.
We'll establish "natural" bijections of two sets above w. $Bilin_{B,A}(L^B \times M^A, N^B) := \{\beta: L^B \times M^A \to N^B \mid B\text{-linear in } L\text{-argument}, A\text{-linear in the } M\text{-argument}\}$

- Bijection $\eta^1_{M,N}$:

$$\begin{array}{ccc} Hom_B(L^B \otimes_A M^A, N^B) & \overset{\sim}{\dashrightarrow} & Bilin_{B,A}(L^B \times M^A, N^B) \\ \cap & & \cap \\ Hom_A(L^B \otimes_A M^A, N^B) & \xrightarrow{\sim} & Bilin_A(L^B \times M^A, N^B) \\ \tilde{\beta} \in & \longleftrightarrow & \ni \beta \end{array}$$

w. $\tilde{\beta}(\ell\otimes m)=\beta(\ell,m)$. We need to show that $\tilde{\beta}$ is B-linear $\Leftrightarrow$ β is B-linear in the L-argument; enough to check on generators $\ell\otimes m$:

$$\overbrace{\tilde{\beta}(b(\ell\otimes m))=b\tilde{\beta}(\ell\otimes m)}^{\tilde{\beta}\text{ is } B\text{-linear}} \Leftrightarrow \tilde{\beta}((b\ell)\otimes m)=b\tilde{\beta}(\ell\otimes m) \Leftrightarrow \underbrace{\beta(b\ell,m)=b\beta(\ell,m)}_{\beta\text{ is } B\text{-linear in } L\text{-argument.}}.$$

Now need check functoriality for $\eta^1_{M,N}$ – 2 comm've diagrams, e.g. $\forall\ \psi\in \mathrm{Hom}_B(N^B, N'^B)$ we have comm've diagram

$$\begin{tikzcd}
\mathrm{Hom}_B(L^B\otimes_A M^A, N^B) \arrow[r, "\eta^1_{M,N}", "\sim"'] \arrow[d, "\psi\circ ?"] & \mathrm{Bilin}_{B,A}(L^B\times M^A, N^B) \arrow[d, "\psi\circ ?"] \\
\mathrm{Hom}_B(L^B\otimes_A M^A, N'^B) \arrow[r, "\eta^1_{M,N'}", "\sim"'] & \mathrm{Bilin}_{B,A}(L^B\times M^A, N'^B)
\end{tikzcd}$$

– comm've by constr'n (exercise).

Similarly, have comm've diagram for $\varphi\in \mathrm{Hom}_A(M'^A, M^A)$

- Bijection $\eta^2_{M,N}: \mathrm{Bilin}_{B,A}(L^B\times M^A, N^B) \xrightarrow{\sim} \mathrm{Hom}_A(\underset{\psi}{M^A}, \underset{\psi}{\mathrm{Hom}_B(L^B,N^B)})$

$$\beta \longmapsto [m\mapsto \beta_m]$$

$\beta_m(\ell)=\beta(\ell,m)$.

Exercise: $\eta^2_{M,N}$ is a functorial bijection. $\square$

1.3) Special cases & variations.

- $L^B := B$, then $L^B\otimes_A \bullet = B\otimes_A \bullet$ & $\mathrm{Hom}_B(L^B,N^B)=\mathrm{Hom}_B(B,N^B)$ $= N^B$ so the Hom-functor $=$ For $: B\text{-Mod}\to A\text{-Mod}$.

Corollary: $B\otimes_A\bullet : A\text{-Mod}\to B\text{-Mod}$ (known as induction, base change, extension of scalars) is left adjoint to For: B-Mod $\to$ A-Mod.

E.g. $\mathbb{C}\otimes_{\mathbb{R}}\bullet$ is complexification (often used in Linear algebra)

- Variation of Thm: let L^B be a B-module $\rightsquigarrow$ $\mathrm{Hom}_A(L^B,\bullet)$: A-Mod $\to$ B-Mod (see Section 2.1 in Lec 15), $L^B\otimes_B\bullet$: B-Mod $\to$ A-Mod

Thm: $L^B\otimes_B\bullet$: B-Mod $\to$ A-Mod is left adjoint of $\mathrm{Hom}_A(L^B,\bullet)$: A-Mod $\to$ B-Mod.

Sketch of proof: Need to identify $\mathrm{Hom}_A(L^B\otimes_B M^B, N^A)$ & $\mathrm{Hom}_B(M^B, \mathrm{Hom}_A(L^B, N^A))$ w. the set:

$$\{\beta\in \mathrm{Bilin}_A(L^B\times M^B, N^A)\mid \beta(bl,m)=\beta(l,bm)\ \forall b\in B, l\in L^B, m\in M^B\}$$

Exercise: to establish a bijection of $\mathrm{Hom}_A(L^B\otimes_B M^B, N^A)$ w. this set of bilinear maps show that we have a well-defined A-linear map $L^B\otimes_A M^B\longrightarrow L^B\otimes_B M^B$, $l\otimes m\mapsto l\otimes m$, it's surjective & $\ker=\mathrm{Span}_A(bl\otimes m - l\otimes bm)$. $\square$

Corollary: The functor $\mathrm{Hom}_A(B,\bullet)$: A-Mod $\to$ B-Mod (coinduction) is right adjoint to For: B-Mod $\to$ A-Mod.

2) Tensor product of algebras.

A comm've unital ring, B, C are A-algebras that are comm've (& unital) $\rightsquigarrow$ $B\otimes_A C$, A-module.

Proposition: $\exists!$ A-algebra str're on $B\otimes_A C$ s.t.

$(b_1\otimes c_1)\cdot(b_2\otimes c_2) = b_1b_2\otimes c_1c_2$ & this alg. str're is comm've & unital.

Proof: Uniqueness will follow b/c $b_1\otimes c_1$ span $B\otimes_A C$.

Need to show existence. The product map $B\times B\to B$ is A-bilinear $\rightsquigarrow$ $\mu_B : B\otimes_A B\to B$, $b_1\otimes b_2\mapsto b_1b_2$. Similarly, have $\mu_C: C\otimes_A C\to C$ $\rightsquigarrow$

$$\begin{tikzcd}
 & \mu_B\otimes\mu_C : B\otimes_A B\otimes_A C\otimes_A C \arrow[r] & B\otimes_A C \\
x\otimes y \in & (B\otimes_A C)\otimes_A(B\otimes_A C) \arrow[u, "\wr"] \arrow[ur] & \\
(x,y) \arrow[u, mapsto] \in & (B\otimes_A C)\times(B\otimes_A C) \arrow[u] \arrow[uur] &
\end{tikzcd}$$

$(b_1\otimes c_1)\otimes(b_2\otimes c_2)\mapsto(b_1b_2)\otimes(c_1c_2)$

our multiplication map

So we've shown existence. The properties (associativity, comm'y & unit) - distributivity follows from bilinearity) can be checked on decomposable tensors, e.g.

comm'y: $(b_1\otimes c_1)(b_2\otimes c_2) = b_1b_2\otimes c_1c_2 = [b_1b_2 = b_2b_1,\ c_1c_2 = c_2c_1\ ] =$
$= b_2b_1\otimes c_2c_1 = (b_2\otimes c_2)(b_1\otimes c_1)$

$1\otimes 1$ is a unit. $\square$

In next lecture, we'll see $B\otimes_A C$ for B, C comm've, is the coproduct of B & C in A-CommAlg.