Lecture 18.

1) Tensor product of algebras.
2) Exactness of additive functors.
Refs: [AM], Sections 2.9, 2.11

BONUS: Abelian categories.

1) A is comm've unital ring, B, C are comm've A-algebras
Last time: $\exists!$ comm've A-algebra str're on $B\otimes_A C$ s.t.
$(b_1\otimes c_1)(b_2\otimes c_2) = b_1b_2\otimes c_1c_2$.

Thm: $B\otimes_A C$ is the coproduct of B & C in the cat'y of comm've A-algebras.

i.e. the functors $\mathrm{Hom}_{A\text{-}Alg}(B\otimes_A C, \cdot)$, $\mathrm{Hom}_{A\text{-}Alg}(B,\cdot)\times \mathrm{Hom}_{A\text{-}Alg}(C,\cdot): A\text{-}CommAlg \longrightarrow Sets$ are isomorphic, equiv'y we have algebra homomorphisms $\iota^B: B\to B\otimes_A C$, $\iota^C: C\to B\otimes_A C$ s.t. $\forall$ alg. homom's $\varphi^B: B\to D$, $\varphi^C: C\to D$, where D is comm've A-alg'a $\exists!$ A-alg. homom. $\varphi: B\otimes_A C\to D$ making the following comm've

$$\begin{tikzcd}
B \arrow[dr, "\iota^B"] \arrow[ddr, bend right, "\varphi^B"'] & & C \arrow[dl, "\iota^C"'] \arrow[ddl, bend left, "\varphi^C"] \\
& B\otimes_A C \arrow[d, dashed, "\varphi"] & \\
& D &
\end{tikzcd}$$

Proof: Constr'n of ι^B, ι^C: $\iota^B(b) = b\otimes 1$, $\iota^C(c) = 1\otimes c$.

$\varphi(b\otimes c) = [b\otimes c = (b\otimes 1)(1\otimes c)] = \varphi(b\otimes 1)\varphi(1\otimes c) =$ [diagram is comm've] $= \varphi^B(b)\varphi^C(c)$, this what φ should satisfy.

The map $B\times C\to D$, $(b,c)\mapsto \varphi^B(b)\varphi^C(c)$ is A-bilinear, so

$\exists!$ A-linear map $\varphi: B\otimes_A C \to D$ w. $\varphi(b\otimes c) = \varphi^B(b)\varphi^C(c)$

What remains to check is: φ respects ring multipl'n, it's enough to do this on decomposable tensors:

$$\varphi(b_1\otimes c_1 \cdot b_2\otimes c_2) = \varphi(b_1b_2\otimes c_1c_2) = \varphi^B(b_1b_2)\varphi^C(c_1c_2) =$$
$$= \varphi^B(b_1)\varphi^B(b_2)\varphi^C(c_1)\varphi^C(c_2) = [D \text{ is comm've}] = (\varphi^B(b_1)\varphi^C(c_1))\cdot$$
$$(\varphi^B(b_2)\varphi^C(c_2)) = \varphi(b_1\otimes c_1)\varphi(b_2\otimes c_2)$$ $\square$

Example: $B = A[x_1,\dots,x_k]/(F_1,\dots F_{k'})$, $C = A[y_1,\dots y_\ell]/(G_1,\dots G_{\ell'})$.
Then $B\otimes_A C = A[x_1,\dots x_k, y_1,\dots y_\ell]/(\underbrace{F_1,\dots F_{k'}}_{\text{on } x_1\dots x_k}, \underbrace{G_1,\dots,G_{\ell'}}_{\text{on } y_1\dots y_\ell})$, denote the right hand side by D.

Will show isomorphism of functors:

$$\mathrm{Hom}_{A\text{-Alg}}(D,\cdot), \quad \mathrm{Hom}_{A\text{-Alg}}(B,\cdot)\times \mathrm{Hom}_{A\text{-Alg}}(C,\cdot)$$

$F_B := \mathrm{Hom}_{A\text{-Alg}}(B,\cdot)$, define another functor F_B': A-CommAlg $\to$ Sets:

F_B': comm've alg'a $E \mapsto \{(e_1,\dots,e_k)\in E^{\times k} \mid F_i(e_1,\dots e_k)=0,\ i=1,\dots k'\}$
for $\varphi: E\to E' \rightsquigarrow F_B'(\varphi): F_B'(E)\to F_B'(E')$, $(e_1,\dots e_k)\mapsto(\varphi(e_1),\dots,\varphi(e_k))$.
-well-defined map b/c $\varphi(e_1)\dots\varphi(e_k)$ satisfy rel'ns of $e_1,\dots e_k$.
Then $F_B \xRightarrow{\sim} F_B'$ (b/c a homom'm from B is uniquely determined by images of generators $x_1,\dots x_k$ & exists as long as the images satisfy relations, $F_1,\dots,F_{k'}$).
Similarly, we have $F_C \xRightarrow{\sim} F_C'$, $F_D \xRightarrow{\sim} F_D'$. From descriptions of B, C, D by generators & rel'ns see $F_D' \xRightarrow{\sim} F_B'\times F_C'$.
This completes example

2]

2) Exactness of additive functors – the main property of these functors Commut've Algebra cares about.

2.1) Exact sequences:
Let $M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{k-1}} M_k$ be a sequence of A-modules & their homom'sm w. $\varphi_i \in Hom_A(M_i, M_{i+1})$, $i=0,..k-1$.

Defn: • Say this sequence is <u>exact</u> if $\operatorname{im}\varphi_{i-1} = \ker\varphi_i$ $\forall i=1,..k-1$.

• A <u>short exact sequence (SES)</u> is exact sequence of form:

$$0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \to 0$$

i.e.: φ_1 is injective, $\operatorname{im}\varphi_1 = \ker\varphi_2$ & φ_2 is surjective.

Example (of SES) if $N \subset M$ is an A-submodule, then have SES

$$0 \to N \underset{\text{inclusion}}{\longrightarrow} M \underset{\text{projection}}{\longrightarrow} M/N \to 0$$

In a way, every SES looks like this: φ_1 identifies M_1 w. submodule of M_2, φ_2 identifies M_3 w. $M_2/\operatorname{im}\varphi_1$.

2.2) Definition of exactness for functors.
A, B be two rings, $F: A\text{-Mod} \to B\text{-Mod}$ be an additive functor.

Definition (of left & right exact functors):

If $\forall$ SES $0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \to 0$

(i) the sequence $0 \to F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2)} F(M_3)$ is exact, then say F is <u>left exact</u>.

(ii) the sequence $F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3) \to 0$ is exact, then say F is <u>right exact</u>.

Rem: can define left/right exact functors $F: A\text{-Mod}^{opp} \to B\text{-Mod}$ e.g. in (i) require that

$$0 \to F(M_3) \xrightarrow{F(\varphi_2)} F(M_2) \xrightarrow{F(\varphi_1)} F(M_1)$$

is exact.

Def: For $F: A\text{-Mod} \to B\text{-Mod}$, $A\text{-Mod}^{opp} \to B\text{-Mod}$, exact = left & right exact, i.e. sends SES to SES.

2.3) Examples:

1) Let $S \subset A$ localizable subset $\rightsquigarrow$ local'n functor $\bullet_S: A\text{-Mod} \longrightarrow A_S\text{-Mod}$. We claim that it's exact.

Reminder (Lec 9, Section 2.2) : if $\varphi \in \mathrm{Hom}_A(M,N)$, then $\ker(\varphi_S) = (\ker\varphi)_S$, $\mathrm{im}(\varphi_S) = (\mathrm{im}\,\varphi)_S$.

For SES $\quad 0 \to M^1 \xrightarrow{\varphi^1} M^2 \xrightarrow{\varphi^2} M^3 \to 0$

φ^1 is inj've $\Rightarrow \varphi^1_S$ is injective; $\mathrm{im}\,\varphi^1 = \ker\varphi^2 \Rightarrow \mathrm{im}\,\varphi^1_S = \ker\varphi^2_S$ etc. This shows $\bullet_S$ is exact.

2) Let L be an A-module $\rightsquigarrow L\otimes_A \bullet: A\text{-Mod} \longrightarrow A\text{-Mod}$. We claim $L\otimes_A\bullet$ is right exact.

Indeed, consider SES $\quad 0 \to K_2 \xrightarrow{\iota_2} M_2' \xrightarrow{\pi_2} M_2 \to 0$ need to show

$$L\otimes_A K_2 \xrightarrow{id\otimes\iota_2} L\otimes_A M_2' \xrightarrow{id\otimes\pi_2} L\otimes_A M_2 \to 0$$

is exact.

By Lecture 16, Section 1.1: $id\otimes\pi_2$ is surj've, $\ker(id\otimes\pi_2) = \mathrm{Span}_A(l\otimes k_2 \mid l\in L, k_2\in K_2) = \mathrm{im}(id\otimes\iota_2)$; in that lecture we've fixed the 2nd factor of $\cdot \otimes_A \cdot$ but we know $\otimes_A$ is comm've so this doesn't matter).

3) $\mathrm{Hom}_A(L, \cdot): A\text{-Mod} \longrightarrow A\text{-Mod}$ is left exact. Need to show: for SES $0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \to 0$, the following is exact

$$0 \longrightarrow \mathrm{Hom}_A(L, M_1) \xrightarrow{\varphi_1 \circ ?} \mathrm{Hom}_A(L, M_2) \xrightarrow{\varphi_2 \circ ?} \mathrm{Hom}_A(L, M_3)$$

φ_1 is inj've $\Rightarrow$ $\varphi_1 \circ ?$ is inj've (exercise)

Let's check $\ker[\varphi_2 \circ ?] = \mathrm{im}[\varphi_1 \circ ?]$, i.e. the following are equivalent (for $\psi_2 \in \mathrm{Hom}_A(L, M_2)$)

(a) $\varphi_2 \circ \psi_2 = 0$ (i.e. $\psi_2 \in \ker$)

(b) $\psi_2 = \varphi_1 \circ \psi_1$ for $\psi_1 \in \mathrm{Hom}_A(L, M_1)$ (i.e. $\psi_2 \in \mathrm{im}$).

(b) $\Rightarrow$ (a) b/c $\varphi_2 \circ \varphi_1 = 0$

(a) $\Rightarrow$ (b): $\varphi_2 \circ \psi_2 = 0 \Leftrightarrow \mathrm{im}\, \psi_2 \subset \ker \varphi_2 = \mathrm{im}\, \varphi_1$ & $\varphi_1 : M_1 \xrightarrow{\sim} \mathrm{im}\, \varphi_1$ so we can view ψ_2 as an A-linear map $L \longrightarrow M_1$, this is ψ_1 in (b). $\square$ of example.

Remark: $\cdot$ $\mathrm{Hom}_A(L, \cdot)$ may fail to be exact: $A = \mathbb{Z}$, $L = \mathbb{Z}/2\mathbb{Z}$, consider SES

$$0 \to \mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \{0\}$ but $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq \{0\}$.

$\cdot$ Similarly, $L \otimes_A \cdot$ may fail to be exact (see Prob 6 in HW 4).

Another example: $\mathrm{Hom}_A(\cdot, L): A\text{-Mod}^{opp} \longrightarrow A\text{-Mod}$ is left exact.

(the proof wasn't discussed in the lecture).

We need to show that, for an exact sequence

$$0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0$$

the sequence

$$0 \to \mathrm{Hom}(M_3, L) \xrightarrow{?\circ\varphi_2} \mathrm{Hom}(M_2, L) \xrightarrow{?\circ\varphi_1} \mathrm{Hom}(M_1, L)$$

is exact.

- $\psi_3 \mapsto \psi_3 \circ \varphi_2$ is injective. Indeed, since φ_2 is surjective, $\psi_3 \circ \varphi_2 = 0 \Longrightarrow \psi_3 = 0$.
- $\ker[?\circ\varphi_1] = \mathrm{im}[?\circ\varphi_2]$: as in example of $\mathrm{Hom}_A(L, \cdot)$, we get $\supset$

To show $\subset$: $\psi_2 \circ \varphi_1 = 0 \Longleftrightarrow \ker\psi_2 \supset \mathrm{im}\,\varphi_1$ so $\exists!\ \underline{\psi}_2$ s.t. the following diagram is commutative:

$$\begin{tikzcd}
M_2 \arrow[d] \arrow[dr, "\psi_2"] & \\
M_2/\mathrm{im}\,\varphi_1 \arrow[r, dashed, "\underline{\psi}_2"'] & L
\end{tikzcd}$$

But φ_2 identifies $M_2/\mathrm{im}\,\varphi_1$ with M_3, so that the projection $M_2 \twoheadrightarrow M_2/\mathrm{im}\,\varphi_1$ becomes $\varphi_2: M_2 \twoheadrightarrow M_3$. Take $\psi_3: M_3 \longrightarrow L$ corresponding to $\underline{\psi}_2$ so that $\psi_3 = \psi_3 \circ \varphi_2$.

BONUS: Abelian categories.

Additive functors make sense between additive categories, Lec 15.

Question for today: what additional structures/conditions do we need to impose in order to be able to talk about exact sequences? It turns out that no additional structures are needed but we need to impose additional conditions.

Exact sequences are about kernels, images and their coincidence. One can define them easily when we talk about modules but in the generality of additive categories, objects are not sets & morphisms are not maps, so we need to explain what we mean by kernels.

As usual, a recipe to define the kernels (and cokernels = quotients by images) are to look at their universal properties in the usual setting of abelian groups.

Let N, M be abelian groups & $\varphi: M \to N$ be a homomorphism. Let K be the kernel of φ and $\iota: K \hookrightarrow M$ be the inclusion. Then we have the following:

(*) $\forall$ $L \in Ob(\mathbb{Z}\text{-Mod})$ & $\psi: L \to M$ a homom'm s.t. $\varphi \circ \psi = 0$ $\exists!$ $\psi': L \to K$ making the following diagram comm've

$$\begin{tikzcd} L \arrow[d, dashed, "\psi'"'] \arrow[dr, "\psi"] & \\ K \arrow[r, "\iota"] & M \end{tikzcd}$$

Definition (of kernel in an additive category) Let $\mathcal{C}$ be an additive category, $M, N \in Ob(\mathcal{C})$, $\varphi \in Hom_{\mathcal{C}}(M, N)$. By the <u>kernel</u> of φ we mean a pair (K, ι) w $K \in Ob(\mathcal{C})$, $\iota \in Hom_{\mathcal{C}}(K, M)$ s.t.

- $\varphi \circ \iota = 0$
- (K, ι) has a universal property that is a direct generali-

zation of $(*)$

Definition (of cokernel in an additive category) The cokernel in $\mathcal{C}$ = the kernel in $\mathcal{C}^{opp}$. I.e. in the notation of the previous definition, we get a pair (C, π) w. $C \in Ob(\mathcal{C})$, $\pi \in Hom_{\mathcal{C}}(N, C)$ s.t.

- $\pi \circ \varphi = 0$
- and the universal property: $\forall \psi \in Hom_{\mathcal{C}}(N, L)$ s.t. $\psi \circ \varphi = 0$ $\exists! \psi' \in Hom_{\mathcal{C}}(C, L)$ s.t.

$$\begin{tikzcd} N \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ C \arrow[r, dashed, "\psi'"] & L \end{tikzcd}$$ is comm've

Exercise: In the category of abelian groups, the cokernel of $\varphi: M \to N$ is $N/\mathrm{im}\,\varphi$ w. the projection $\pi: N \twoheadrightarrow N/\mathrm{im}\,\varphi$.

Definition: We say that $\varphi \in Hom_{\mathcal{C}}(M, N)$ is a <u>monomorphism</u> if $(0,0)$ is its kernel and is an <u>epimorphism</u> is $(0,0)$ is its cokernel.

For example, in A-Mod, monomorphism = injective & epimorphism = surjective. Note that a monomorphism in $\mathcal{C}$ = epimorphism in $\mathcal{C}^{opp}$.

Exercise: • The following 2 conditions are equivalent

(a) $\varphi: M \to N$ is a monomorphism

(b) $\varphi \circ ?: Hom_A(L, M) \longrightarrow Hom_A(L, N)$ is inj've $\forall L \in Ob(\mathcal{C})$

• Similarly, φ is an epimorphism $\Leftrightarrow$ $? \circ \varphi: Hom_A(N, L) \hookrightarrow Hom_A(M, L)$ $\forall L \in Ob(\mathcal{C})$.

• In particular, for any kernel (K, ι) we have that ι is a monomorphism & for any cokernel (C, π), π is an epimorphism.

Definition: We say that an additive category $\mathcal{C}$ is <u>abelian</u> if the following conditions hold:

(K) every morphism in $\mathcal{C}$ has a kernel

(C) every morphism in $\mathcal{C}$ has a cokernel

(M) for every monomorphism $\iota \in Hom_{\mathcal{C}}(K,M)$ $\exists$ N & $\varphi \in Hom_{\mathcal{C}}(M,N)$ s.t. (K,ι) is the kernel of φ.

(E) for every epimorphism $\pi \in Hom_{\mathcal{C}}(N,C)$ $\exists$ M & $\varphi \in Hom_{\mathcal{C}}(M,N)$ s.t. (C,π) is the cokernel of φ.

Example: A-Mod & A-Mod^{opp} are abelian categories.

Non-example: The category of free A-modules is not abelian if A is not a field. This is because every (not necessarily free) A-module is the cokernel (in the usual sense) of a linear map between free modules.

Example: A full subcategory of A-Mod (where A is an associative ring) that is closed under taking sub- & quotient modules is abelian. Added on 11/4: in particular, for A Noetherian, the category of fin. generated A-modules is abelian.

In an abelian category it makes sense to speak about subobjects of M (a pair of $K \in Ob(\mathcal{C})$ & a monomorphism $\iota \in Hom_{\mathcal{C}}(K,M)$) quotient objects etc. Axioms (M) & (E) ensure that these objects behave in a way we expect them to. In particular, it does make sense to talk about exact sequences.

Premium exer: in abelian category, isomorphism $\Leftrightarrow$ monomorphism & epimorphism