Lecture 19.

1) Properties of left/right exact functors.
2) Localization vs tensor product functors.
3) Projective & flat modules.

Bonus: injective modules.

[AM], Sections 2.9 & intro to 3; [E], A.3.2, 6.1, 6.3.

1) **Lemma:** Let $F: A\text{-Mod} \to B\text{-Mod}$ be left exact additive functor. Then

(a) F sends injections to injections.

(b) F sends every exact sequence $0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3$ to an exact sequence $0 \to F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3)$

(c) F is exact $\Leftrightarrow$ F sends surjections to surjections.

Proof: (a) $N \overset{\varphi_1}{\hookrightarrow} M$ can be included into SES

$0 \to N \xrightarrow{\varphi_1} M \longrightarrow M' \longrightarrow 0$, $M' := M/\operatorname{im}\varphi_1$.

$$\Downarrow F$$

$0 \to F(N) \xrightarrow{F(\varphi_1)} F(M) \longrightarrow F(M')$ - exact $\Rightarrow F(\varphi_1)$ is inj've.

(b): $M_3' := \operatorname{im}\varphi_2 \subset M_3$; $\varphi_2' := \varphi_2$ viewed as a map to its image
$\iota: M_3' \hookrightarrow M_3$ - incl'n, so $\varphi_2 = \iota \circ \varphi_2'$.

$0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2'} M_3' \to 0$ is exact $\Rightarrow$

$$0 \to F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2')} F(M_3') \qquad (*)$$

is exact. Further, ι is inj've $\Rightarrow$ [by (a)] $F(\iota)$ is inj've;
F is a functor $\Rightarrow$ $F(\varphi_2) = \underbrace{F(\iota)}_{\text{inj've}} \circ F(\varphi_2') \Rightarrow \ker F(\varphi_2) = \ker F(\varphi_2')$

$+ (*)$

$0 \to F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3)$ is exact.

$\square$

(c) is exercise. $\square$

Rem: There are direct analogs of this lemma for <u>all other</u> types of partial exactness. E.g. left exact functor $F: A\text{-Mod}^{opp} \longrightarrow B\text{-Mod}$ sends $\forall$ exact sequence
$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ to exact sequence
$0 \to F(M_3) \longrightarrow F(M_2) \longrightarrow F(M_1)$ (exercise).

2) Localization functors vs tensor products

A comm've unital ring, $S \subset A$ localizable subset $\rightsquigarrow$ A-algebra A_S & local'n functor $\bullet_S: A\text{-Mod} \longrightarrow A_S\text{-Mod}$.

Theorem: The functors $\bullet_S$, $A_S \otimes_A \bullet : A\text{-Mod} \longrightarrow A_S\text{-Mod}$ are isomorphic.

Proof: Step 1: construct $\eta: A_S \otimes_A \bullet \Rightarrow \bullet_S$
$\eta_M: A_S \otimes_A M \longrightarrow M_S$, consider $A_S \times M \longrightarrow M_S$,
A_S-module

the map $A_S \times M \longrightarrow M_S$, $(\frac{a}{s}, m) \longmapsto \frac{a}{s}\frac{m}{1} \in M_S$, is A-bilinear $\rightsquigarrow$
A-linear map $\eta_M: A_S \otimes_A M \to M_S$, $\frac{a}{s} \otimes m \mapsto \frac{a}{s}\frac{m}{1}$.

Exercise: η_M are A_S-linear & constitute a functor morphism.

Step 2: prove $\eta_{A^{\oplus I}}: A_S \otimes_A A^{\oplus I} \xrightarrow{\sim} (A^{\oplus I})_S$ $\forall$ set I.
Recall:

$$\begin{tikzcd}
A_S \otimes_A A^{\oplus I} \arrow[rr, "\eta"] \arrow[dr, "\sim"] & & (A^{\oplus I})_S \arrow[dl, "\sim"'] \\
& (A_S)^{\oplus I} &
\end{tikzcd}$$

From def'n of η this diagram is comm've: for $|I|=1$ - follows directly from the const'n; in general all maps in this diagram are componentwise.

Step 3: here from Step 2 & exactness we deduce that η_M is isom'm $\forall$ M. Have exact sequence:

$$A^{\oplus J} \longrightarrow A^{\oplus I} \longrightarrow M \longrightarrow 0$$

Apply functors $A_S \otimes_A \bullet$, $\bullet_S$ to this exact sequence to get a diagram

$$\begin{tikzcd}
A_S \otimes_A A^{\oplus J} \arrow[r] \arrow[d, "\wr", "\eta_{A^{\oplus J}}"'] & A_S \otimes_A A^{\oplus I} \arrow[r] \arrow[d, "\wr", "\eta_{A^{\oplus I}}"'] & A_S \otimes_A M \arrow[r] \arrow[d, "\eta_M"] & 0 \\
(A^{\oplus J})_S \arrow[r] & (A^{\oplus I})_S \arrow[r] & M_S \arrow[r] & 0
\end{tikzcd}$$

This diagram is comm've b/c η is a functor morphism
Rows are exact b/c $A_S \otimes_A \bullet$, $\bullet_S$ are right exact. (analogous to Lemma in Section 1).

The claim that η_M is isom'm follows from the next Lemma.

Lemma: Suppose we have a comm've diagram w. exact rows:

$$\begin{tikzcd} M_1 \arrow[r, "\tau_1"] \arrow[d, two heads, "\varphi_1"] & M_2 \arrow[r, "\tau_2"] \arrow[d, "\varphi_2", "\wr"'] & M_3 \arrow[r] \arrow[d, "\varphi_3"] & 0 \\ N_1 \arrow[r, "\psi_1"] & N_2 \arrow[r, "\psi_2"] & N_3 \arrow[r] & 0 \end{tikzcd}$$

Assume φ_2 is isom'm & φ_1 is surjective. Then φ_3 is an isom'm.

Proof of lemma: φ_1 is surj've & left square is comm've $\Rightarrow$ $\varphi_2 : \operatorname{im} \tau_1 \xrightarrow{\sim} \operatorname{im} \psi_1$.

Rows are exact: $M_3 \simeq M_2 / \operatorname{im} \tau_1$ (via τ_2) & $N_3 \simeq N_2 / \operatorname{im} \psi_1$ (via ψ_2).

Right square is comm've: φ_3 is identified w. the isom'm $M_2 / \operatorname{im} \tau_1 \xrightarrow{\sim} N_2 / \operatorname{im} \psi_1$ induced by φ_2. $\square$

$\square$ of Thm.

Remarks:

1) Similarly to Lemma if we have comm've diagram w. exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_3 \arrow[r] \arrow[d, "\varphi_3"] & M_4 \arrow[r] \arrow[d, "\varphi_4"] & M_5 \arrow[d, "\varphi_5"] \\ 0 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5 \end{tikzcd}$$

If φ_4 is an isom'm, φ_5 is inj've $\Rightarrow$ φ_3 is an isom'm (exercise).

2) Lemma & Rem 1 are special cases of the following result known as the 5-lemma: for comm've diagram w. exact rows:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\varphi_1"] & M_2 \arrow[r] \arrow[d, "\varphi_2", "\wr"'] & M_3 \arrow[r] \arrow[d, "\varphi_3"] & M_4 \arrow[r] \arrow[d, "\varphi_4", "\wr"'] & M_5 \arrow[d, hook, "\varphi_5"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$

Then φ_3 is an isom'm (premium exercise based on diagram chase).

3) Alternative proof of Thm: both $A_S \otimes_A \bullet$, $\bullet_S$: A-Mod $\longrightarrow$ A_S-Mod are left adj't to For: A_S-Mod $\longrightarrow$ A-Mod (for $A_S \otimes_A \bullet$ this follows from the 1st Corollary in Section 1.3 of Lecture 17, for $\bullet_S$ it's part 3 of Prob 5 in HW3).

3.1) Projective modules

Let P be an A-module. We know $\mathrm{Hom}_A(P,\bullet)$ is left exact, see Example 3 in Section 2.3 of Lecture 18.

Q: For which P is this functor exact $\Leftarrow$[(c) of Lemma in Sect. 1]

$\Rightarrow$ $\mathrm{Hom}_A(P,\bullet)$ sends surjections to surjections.

Example: $P = A^{\oplus I}$. Claim $\mathrm{Hom}_A(A^{\oplus I},\bullet)$ is exact.

$$\mathrm{Hom}_A(A^{\oplus I},\bullet) \xRightarrow{\sim} \bullet^{\times I}$$

in particular, for $\varphi: M \twoheadrightarrow N$, we have commu've diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\varphi\circ ?"] \arrow[d, "\wr"] & \mathrm{Hom}_A(A^{\oplus I}, N) \arrow[d, "\wr"] \\
M^{\times I} \arrow[r, "\varphi^{\times I}"] & N^{\times I}
\end{tikzcd}$$

Since φ is surj've $\Rightarrow \varphi^{\times I}$ is surj've.

Reminder: An A-module P is projective if $\exists$ A-module P' s.t. $P \oplus P'$ is a free A-module ($\simeq A^{\oplus I}$ for some set I).

Thm: TFAE

(1) $\mathrm{Hom}_A(P, \bullet)$ is exact.

(2) $\forall$ A-linear surjection $\pi: M \twoheadrightarrow P$ $\exists$ A-linear $\iota: P \to M$ s.t. $\pi \circ \iota = id_P$

(3) P is projective.

Proof:

(1) $\Rightarrow$ (2): $\mathrm{Hom}_A(P, M) \xrightarrow{\pi\circ ?} \mathrm{Hom}_A(P,P)$ is surjective $\Rightarrow$ $\exists$ $\iota \in \mathrm{Hom}_A(P,M)$ s.t. $\pi \circ \iota = id_P$, which is (2).

(2) $\Rightarrow$ (3): Pick $\iota: P \to M$ w. $\pi \circ \iota = id_P \Rightarrow \iota$ is inj've.

$\Downarrow$ sol'n to Prob. 8(c) in HW1

$$M = \ker \pi \oplus \underbrace{\mathrm{im}\, \iota}_{(\simeq P)}$$

We apply this to $\pi: M := A^{\oplus I} \twoheadrightarrow P$ & we get (3) w. $P' = \ker \pi$.

(3) $\Rightarrow$ (1):

Lemma: Let M, M' be A-modules. TFAE

(a) $\mathrm{Hom}_A(M, \bullet)$, $\mathrm{Hom}_A(M', \bullet)$ are exact

(b) $Hom_A(M\oplus M', \bullet)$ is exact.

Proof of Lemma: $M\oplus M'$ is the coproduct of M, M' is A-Mod so $Hom_A(M\oplus M', \bullet) \xrightarrow{\sim} Hom_A(M,\bullet)\times Hom_A(M',\bullet)$. We apply this to A-module surj'n $\varphi: N_2 \twoheadrightarrow N_3$. Get a comm've diagram

$$\begin{tikzcd} Hom_A(M\oplus M', N_2) \arrow[r, "\sim"] \arrow[d, "\psi\mapsto\varphi\circ\psi"] & Hom_A(M,N_2)\times Hom_A(M',N_2) \arrow[d, "(\psi_2,\psi_3)\mapsto(\varphi\circ\psi_2,\varphi\circ\psi_3)"] \\ Hom_A(M\oplus M', N_3) \arrow[r, "\sim"] & Hom_A(M,N_3)\times Hom_A(M',N_3) \end{tikzcd}$$

(b) $\Leftrightarrow$ $\psi\mapsto\varphi\circ\psi$ is surj've $\Leftrightarrow$ right vertical map is surj've $\Leftrightarrow$ $\psi_2\mapsto\varphi\circ\psi_2$ & $\psi_3\mapsto\varphi\circ\psi_3$ are surj've $\Leftrightarrow$ (a). $\square$

Know $P\oplus P'\simeq A^{\oplus I} \Rightarrow Hom_A(P\oplus P', \bullet)$ is exact (by Example above) By (b)$\Rightarrow$(a) of Lemma, $Hom_A(P,\bullet)$ is exact.

$\square$ of Thm.

3.2) Flat modules

Definition: An A-module F is <u>flat</u> if $F\otimes_A \bullet: A\text{-Mod}\to A\text{-Mod}$ is exact ($\Leftrightarrow$ sends injections to injections)

Examples:

(I) $A^{\oplus I}$ is flat (a complete analog of Example in Sect. 3.1 b/c $A^{\oplus I}\otimes_A \bullet \xrightarrow{\sim} \bullet^{\oplus I}$).

(II) Projective $\Rightarrow$ flat (by complete analog of Lemma in 3.1+

example (I)).

(III) $\forall$ localizable subset S, A_S is a flat A-module b/c $A_S \otimes_A \bullet \overset{\sim}{\Rightarrow} \bullet_S$ (Section 2) & $\bullet_S$ is exact (example 1 in Section 2.3 of Lecture 18).

BONUS: injective modules.

Let A be a (comm've unital) ring.

Definition: An A-module I is <u>injective</u> if $\mathrm{Hom}_A(\bullet, I)$: $A\text{-Mod}^{opp} \longrightarrow A\text{-Mod}$ is exact (equivalently, for an inclusion $N \hookrightarrow M$ the induced homomorphism

$$\mathrm{Hom}_A(I, M) \longrightarrow \mathrm{Hom}_A(I, N) \text{ is surjective}).$$

The definition looks very similar to that of projective modules, however the properties of injective & projective modules are very different! Projective modules - especially finitely generated ones - are nice, but injective modules are quite ugly, they are almost never finitely generated.

The simplest ring is $\mathbb{Z}$. Let's see what being injective means for $\mathbb{Z}$.

Definition: An abelian group M is <u>divisible</u> if $\forall m \in M, a \in \mathbb{Z}$ $\exists m' \in M$ s.t $am' = m$.

Example: The abelian group $\mathbb{Q}$ is divisible. So is $\mathbb{Q}/\mathbb{Z}$.

Proposition 1: For an abelian group M TFAE:

(a) M is injective

(b) M is divisible

Sketch of proof: (a) $\Rightarrow$ (b): apply
$$N \hookrightarrow M \Rightarrow \mathrm{Hom}_A(I,M) \twoheadrightarrow \mathrm{Hom}_A(I,N) \qquad (*)$$
to $M=\mathbb{Z}$, $N=a\mathbb{Z}$.

(b) $\Rightarrow$ (a) is more subtle. The first step is to show that if $(*)$ holds for $N\subset M$, then it holds for $N+\mathbb{Z}m\subset M$ $\forall\, m\in M$. So $(*)$ holds for all fin. gen'd submodules $N\subset M$. Then a clever use of transfinite induction yields $(*)$ for all submodules of M. $\square$

We can get examples of injective modules for more general rings as follows. Note that for an abelian group M, the group $\mathrm{Hom}_{\mathbb{Z}}(A,M)$ is an A-module

Proposition 2: If M is injective as an abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A,M)$ is an injective A-module.

This is approached similarly to Prob 3 in HW5.

Finally, using this proposition one can show that every A-module embeds into an injective one (the corresponding statement for projectives - that every module admits a surjection from a projective module - is easy b/c every free module is proj've).