Lecture 2.

1) Operations with ideals.

2) Maximal ideals.

[AM], Chapter 1, Sections 3 and 6.

1) A is comm've unital ring, $I, J \subset A \rightsquigarrow I\cap J \subset A$.

Definition: The sum $I+J := \{a+b \mid a\in I, b\in J\} \subset A$

The product $IJ := \{\sum_{i=1}^{k} a_i b_i \mid k\in \mathbb{Z}_{>0}, a_i\in I, b_i\in J\}$

The ratio $I:J := \{a\in A \mid aJ\subset I\}$

The radical $\sqrt{I} := \{a\in A \mid \exists\, n\in\mathbb{Z}_{>0} \text{ w. } a^n\in I\}$

Proposition: $I\cap J, I+J, IJ, I:J, \sqrt{I}$ are ideals.

Proof for $\sqrt{I}$: Need to check (an abelian subgroup + (1))

(0) $\sqrt{I}\neq\varnothing$

(1) $a\in A, b\in\sqrt{I} \Rightarrow ab\in\sqrt{I}$

(2) $a,b\in\sqrt{I}\Rightarrow a+b\in\sqrt{I}$

(0) $\Leftarrow \sqrt{I}\supset I$.

b/c A is commutative

(1): $b\in\sqrt{I}\Rightarrow \exists\, n$ w. $b^n\in I\Rightarrow (ab)^n = a^n b^n\in I\Rightarrow ab\in\sqrt{I}$.

(2) $a,b\in\sqrt{I}\Rightarrow\exists\, n$ w. $a^n, b^n\in I$

$$(a+b)^{2n} = \sum_{i=0}^{2n}\binom{2n}{i} a^i b^{2n-i}\in I\Rightarrow a+b\in\sqrt{I}$$

again, use that A is comm've

$\in I$ if $i\geqslant n$ $\in I$ if $i\leqslant n$

$\square$

Examples: 1) $I=(f_1,\dots f_n)$, $J=(g_1,\dots g_m)$. Then:

- $I+J = (f_1,\dots f_n, g_1,\dots g_m)$: $0\in I,J\Rightarrow f_i, g_j\in I+J\Rightarrow$

$(f_1, \dots f_n, g_1, \dots g_m) \subset I+J$; $I+J \subset (f_1, \dots f_n, g_1, \dots g_m)$ manifest.

- $IJ = (f_i g_j \mid i=1,\dots n, j=1,\dots m)$ - exercise.
- $I \cap J$, $I:J$, $\sqrt{I}$ - generators may be tricky...

2) $A = \mathbb{Z}$, $I = (a)$. Want $\sqrt{I}$: $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, p_i primes, $\alpha_i \in \mathbb{Z}_{>0}$
$b \in \sqrt{I} \Leftrightarrow b^n \vdots a$ for some $n \Leftrightarrow b \vdots p_1 \dots p_k \Leftrightarrow \sqrt{(a)} = (p_1 \dots p_k)$.
divisible by

Exercise: general A, I: show $\sqrt{\sqrt{I}} = \sqrt{I}$.

2) Maximal ideals:

2.1) Definition & examples.

Definition: An ideal $\mathfrak{m} \subset A$ is <u>maximal</u> if:

- $\mathfrak{m} \neq A$.
- $\mathfrak{m}'$ another ideal s.t. $\mathfrak{m} \subseteq \mathfrak{m}' \subsetneq A \Rightarrow \mathfrak{m}' = \mathfrak{m}$.

i.e. maximal = maximal w.r.t. inclusion among ideals $\neq A$.

Lemma (equivalent characterization) TFAE:

(1) $\mathfrak{m}$ is maximal

(2) $A/\mathfrak{m}$ is a field

Proof: both are equivalent to:

(3) The only two ideals in $A/\mathfrak{m}$ are $\{0\}$ & $A/\mathfrak{m}$.

(1)$\Leftrightarrow$(3): b/c of bijection $\{$ideals in A containing $\mathfrak{m}\} \xrightarrow{\sim}$ $\{$ideals in $A/\mathfrak{m}\}$, Remarks in Section 3.2 of Lecture 1.

$(3) \Rightarrow (2)$: $a \in A/\mathfrak{m}, a \neq 0 \Rightarrow (a) = (A/\mathfrak{m})a \ni a$

$$\begin{array}{l} \Downarrow \\ \neq \{0\} \\ \Downarrow \\ = A/\mathfrak{m} \\ \Downarrow \\ \ni 1 \end{array}$$

$\Rightarrow$ a is invertible $\Rightarrow A/\mathfrak{m}$ is a field.

$(2) \Rightarrow (3)$: reverse the argument (exercise). $\square$

Examples: 1) $A = \mathbb{Z}$, (a) is maximal $\Leftrightarrow$ a is prime.

2) $A = \mathbb{F}[x]$ ($\mathbb{F}$ is field), (f) is maximal $\Leftrightarrow f$ is irred'ble.

3) $A = \mathbb{C}[x_1, \dots x_n]$ (can replace $\mathbb{C}$ w. any alg. closed field).

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \rightsquigarrow \mathfrak{m}_\alpha := \{f \in \mathbb{C}[x_1, \dots x_n] \mid f(\alpha) = 0\}$ is an ideal, it's max'l b/c $\mathbb{C}[x_1, \dots x_n]/\mathfrak{m}_\alpha = [\mathfrak{m}_\alpha = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ so every $f + \mathfrak{m}_\alpha \ni$ scalar$] = \mathbb{C}$ (a field).

In fact, this way we get <u>all</u> max. ideals in $\mathbb{C}[x_1, \dots x_n]$. - to be proved later.

2.2) Existence.

Proposition: Every (comm've unital) ring has at least one maximal ideal.

Proof: based on Zorn's lemma from Set theory ($\Leftrightarrow$ axiom of choice)

Definition: Let X be a set.

- A <u>partial order</u> $\leq$ on X is a binary relation s.t.

  $- x \leq x$

$- \ x \leq y \ \& \ y \leq x \Rightarrow x = y \qquad \forall x, y, z \in X$

$- \ x \leq y \ \& \ y \leq z \Rightarrow x \leq z$

- $Y \subseteq X$ is <u>linearly ordered</u> (under $\leq$) if $\forall x, y \in Y$ have $x \leq y$ or $y \leq x$.
- poset = a set equipped with partial order

  Example: $X = \{\text{ideals } I \subset A \mid I \neq A\}$, $\leq := \subseteq$

Zorn lemma: Let X be a poset. Suppose that:

$(*)$ $\forall$ linearly ordered subset $Y \subseteq X$ $\exists$ an upper bound in X, i.e $x \in X$ s.t. $y \leq x$ $\forall y \in Y$.

Then $\exists$ a maximal element $z \in X$

$z \leq x \Rightarrow x = z$.

Proof of Proposition: $X, \leq$ are as in Example. Want to show $(*)$

Y linearly ordered subset of X : $I, J \in Y \Rightarrow I \subseteq J$ or $J \subseteq I$

$\tilde{I} := \bigcup_{I \in Y} I$ (remark: the union of arbitrary ideals may fail to be an ideal)

Need to show:

(I) $\tilde{I}$ is an ideal $\Leftarrow a + b \in \tilde{I}$ as long as $a, b \in \tilde{I}$.

Check: $a, b \in \tilde{I} = \bigcup_{I \in Y} I \Rightarrow \exists I, J \in Y$ s.t. $a \in I, b \in J$.

Can assume $I \subseteq J \Rightarrow a, b \in J \Rightarrow a + b \in J \subseteq \tilde{I}$

(II) $\tilde{I} \neq A \underset{\tilde{I} \text{ is an ideal}}{\Longleftrightarrow} 1 \notin \tilde{I}$

But $1 \notin I$ for every $I \in Y \Rightarrow \tilde{I} = \bigcup_{I \in Y} I \not\ni 1$

Apply Zorn's lemma to finish the proof of Proposition $\square$

Remark: Why do we care about ideals:

Reason 0: Can be used to produce new examples of rings (quotient construction from Section 3.2 of Lecture 1.

Reason 1: If A is a "ring of alg. integers" then we have unique factorization for non-zero ideals. - will not discuss. Should be discussed in

Reason 2: Connection to geometry - in Topic #5.

$Z \subseteq \mathbb{C}^n \rightsquigarrow I_Z := \{f \in \mathbb{C}[x_1, \dots x_n] \mid f|_Z = 0\}$ - an ideal

BONUS: Non-commutative counterparts, part 2.

B1) Proper generalizations of what we discussed in this lecture will be for two-sided ideals. For two such ideals I, J it still makes sense to talk about $I \cap J$, $I+J$, IJ, $I:J$ - those are still 2-sided ideals. For $\sqrt{I}$ the situation is more interesting: the definition we gave doesn't produce an ideal (look at $I = \{0\}$ in $Mat_2(\mathbb{C})$). Under some addit'l assumptions, still can define a 2-sided ideal. We'll explain this for $I = \{0\}$, for the general case just take the preimage of $\sqrt{\{0\}} \subset A/I$ under $A \twoheadrightarrow A/I$.

Definition: A two-sided ideal $I \subset A$ is called <u>nilpotent</u> if $\exists\, n \in \mathbb{Z}_{>0} \mid I^n = \{0\}$.

Exercise: The sum of two nilpotent ideals is a nilpotent ideal.

Under additional assumption: A is "Noetherian" for 2-sided ideals - this condition for commutative rings will be studied later in the class) there's an automatically unique maximal nilpotent ideal. We take this ideal for $\sqrt{\{0\}}$!

B2) Now we discuss maximal ideals.

Definition: A ring A is called <u>simple</u> if it has only 2 two-sided ideals, $\{0\}$ & A.

Exercise: $Mat_n(\mathbb{F})$ is simple for any field $\mathbb{F}$.

Premium exercise: $Weyl_1 = \mathbb{F}\langle x, y\rangle / (xy - yx - 1)$ is simple if $\operatorname{char} \mathbb{F} = 0$ & not simple if $\operatorname{char} \mathbb{F} > 0$.

A two-sided ideal $\mathfrak{m} \subset A$ is maximal if $A/\mathfrak{m}$ is simple