Lecture 20.

1) Integral extensions of rings.
2) Integral closure.

Ref: [AM], Section 5.1.

0) Intro/recap We've seen a bunch of constr'ns of rings:
- direct sums
- rings of polynomials
- quotient rings
- completions (HW1)
- localizations
- tensor products
- symmetric algebras (HW4)

Today: another construction: taking integral extensions/closures, motivated by alg'c number theory, generalizes algebraic extensions/closures for fields (see MATH 370).

1) Integral extensions of rings.

Reminder: if $K \subset L$ are two fields, then one can speak about:
- L being finitely generated (as a field) over K.
- L being algebraic over K.
- L being finite over K.

Now suppose that A is a comm've unital ring, let B be a comm've unital A-algebra. We've already defined what it means for B to be fin. generated (as algebra) over A:

$\exists\ b_1, \dots b_k \in B$ s.t. $\forall\ b \in B\ \exists\ F \in A[x_1, \dots x_k] \mid b = F(b_1, \dots b_k)$.

1.1) Definition & examples.

Definition: • Say B is <u>finite</u> over A if B is a finitely gen'd A-module.

• Say $b \in B$ is <u>integral</u> over A if $\exists$ <u>monic</u> (i.e. leading coeff $= 1$) $f \in A[x] \mid f(b) = 0$.

• B is <u>integral</u> over A if $\forall\ b \in B$ is integral (over A).

Examples: 1) $A = K \subset B = L$ - extension of fields. Then the notions of being finite are equivalent. And integral $\iff$ algebraic. But L is fin. gen'd as an algebra over $K \Rightarrow$ fin. gen'd as a field over K but not vice versa.

2) Let $d \in \mathbb{Z}$, not a complete square, $A = \mathbb{Z}$, $B = \mathbb{Z}[\sqrt{d}]$
B is finite over A (rk 2 free A-module w. basis $1, \sqrt{d}$).

Claim: B is integral over A:

$\beta \in B$ equals $a + b\sqrt{d}$ $(a, b \in \mathbb{Z}) \rightsquigarrow$ conjugate $\bar{\beta} := a - b\sqrt{d}$
$\beta + \bar{\beta} = 2a$, $\quad \beta\bar{\beta} = a^2 - b^2 d \rightsquigarrow f(x) = (x - \beta)(x - \bar{\beta}) = x^2 - 2ax + (a^2 - b^2 d) \in A[x]$ & $f(\beta) = 0$. So β is integral over $A \Rightarrow$ B is integral over A.

1.2) Properties.

Reminder: for field extensions: finite $\iff$ algebraic & fin. gener'd.

Thm: Let B be an A-algebra. Consider the following conditions:

(a) B is fin. gen'd & integral over A.

(b) B is finite over A.

Then (a) $\Rightarrow$ (b) and, if A is Noetherian, then (b) $\Rightarrow$ (a).

Added on 11/6: Can remove Noeth'n assump'n: see Remark on page 6.

Proof: (b) $\Rightarrow$ (a) when A is Noetherian

finite $\Rightarrow$ fin. gen'd (b/c if $b_1, \dots b_k$ generate B as A-module $\Rightarrow$ they generate B as A-algebra).

finite $\Rightarrow$ integral: $\beta \in B$, want $\exists$ monic $f(x) \in A[x] \mid f(\beta)=0$.

For $k \geq 0 \rightsquigarrow M_k = \mathrm{Span}_A(1, \dots \beta^{k-1}) \subset B$, is an A-submodule.

M_i's form an ascending chain of submodules, which has to terminate b/c A is Noetherian & B is fin. gen'd ($\Rightarrow$ Noeth'n) A-module. So $\exists$ $k > 0$ s.t. $M_{k+1} = M_k \Rightarrow \beta^k \in M_{k+1} = M_k$ i.e. $\beta^k = a_{k-1}\beta^{k-1} + a_{k-2}\beta^{k-2} + \dots + a_0$, set $f(x) = x^k - a_{k-1}x^{k-1} - \dots - a_0$. ✓

(a) $\Rightarrow$ (b): Let $\beta_1, \dots \beta_k$ be generators of A-algebra B, know all of them are int'l over A. Want to show B is fin. gen'd A-module; for $i = 0, \dots k \rightsquigarrow B_i := A[\beta_1, \dots \beta_i]$ (subalgebra gen'd by these elements), $B_0 = A$, $B_k = B$.

We'll show by ind'n that B_i is a fin. gen'd A-module.

Induction step: $B_{i+1} = B_i[\beta_{i+1}]$, β_{i+1} is integral over A hence over B_i, $f(x) \in B_i[x]$ s.t. $f(\beta_{i+1}) = 0$, $f(x) = x^m + c_{m-1}x^{m-1} + \dots + c_0$, $c_j \in B_i$, $\rightsquigarrow B_i[x] \twoheadrightarrow B_{i+1}$, $x \mapsto \beta_{i+1}$, factors through $B_i[x]/(f(x))$.

Since $f(x)$ is <u>monic</u>, $B_i[x]/(f(x))$ is generated by $1, \dots x^{m-1}$ as a B_i-module, in part'r it's fin. gen'd $\Rightarrow B_{i+1}$ is a fin. gen'd B_i-module.

We know $B_i = \mathrm{Span}_A(b_1, \dots b_n)$, $B_{i+1} = \mathrm{Span}_{B_i}(h_1, \dots h_\ell) \Rightarrow$ $B_{i+1} = \mathrm{Span}_A(b_i h_j \mid i=1,\dots n, j=1,\dots \ell)$. Finishes induction step — and the proof. $\square$

Corollary 1: Suppose A is Noeth'n ring. If (a) $\Leftrightarrow$ (b) holds, then B is a Noeth'n ring.

— b/c B is a Noeth'n A-module $\Rightarrow$ Noeth'n B-module

Corollary 2: If A is Noeth'n & $f(x) \in A[x]$ is monic, then $A[x]/(f(x))$ is integral over A.

This is b/c $A[x]/(f(x))$ is finite over A (see the proof). Here we can also remove Noeth'n assumption, see Rmk on page 6.

Corollary 3 (transitivity) Let B be an A-algebra, C be a B-algebra. Then:

(a) B fin. gen'd over A & C fin. gen'd over $B \Rightarrow C$ fin. gen'd over A.

(b) — finite — — — finite — — — finite —

(c) — integral — — — integral — — — integral — —

(if A is Noeth'n) — this assumption can be removed.

Proof of (c): $\gamma \in C$ integral over $B \rightsquigarrow \exists\, b_0, \dots b_{k-1} \in B$ s.t. $\gamma^k - b_{k-1}\gamma^{k-1} - \dots - b_0 = 0 \Rightarrow \gamma$ is integral over $A[b_0, \dots b_{k-1}]$. Since $b_0, \dots b_{k-1}$ are integral over $A \Rightarrow A[b_0, \dots b_{k-1}]$ is finite over A; $A[b_0, \dots b_{k-1}, \gamma]$ is finite over $A[b_0, \dots b_{k-1}]$. By (b), $A[b_0, \dots, b_{k-1}, \gamma]$

is finite over A, hence integral $\Rightarrow \gamma$ is integral $\square$

2) Integral closure.

Proposition 1: Let B be an A-algebra. Suppose A is Noeth'n. If $\alpha, \beta \in B$ are integral over A, then so are $\alpha+\beta, \alpha\beta, a\alpha$ ($\forall a \in A$). Again: can remove the Noeth'n assumption: Rmk on page 6.

Proof: Consider subalgebras $A[\alpha] \subset A[\alpha,\beta] \subset B$, $A[\alpha]$ is integral over A, $A[\alpha,\beta]$ is integral over $A[\alpha] \Longrightarrow$ over A as well. Since $\alpha\beta, \alpha+\beta, a\alpha \in A[\alpha,\beta]$, they are integral over A. $\square$

Corollary/definition: The subset $\bar{A}^B$ of all integral over A elements in B form an A-subalgebra. This subalgebra is called the <u>integral closure</u> of A in B.

Note that this is a direct generalization of algebraic closures of fields.

Prop 2: If A is Noeth'n, then the integral closure of $\bar{A}^B$ in B is $\bar{A}^B$.

Proof: Let $\beta \in B$ be integral over $\bar{A}^B$. Need to show β is integral over A ($\Rightarrow \beta \in \bar{A}^B$). Let $f(x) = x^k + b_{k-1}x^{k-1} + \ldots + b_0$ w. $f(\beta)=0$. Then $b_0, \ldots b_{k-1}$ are integral over $A \Rightarrow A[b_0, \ldots b_{k-1}]$ is finite over $A \Rightarrow A[b_0, \ldots b_{k-1}, \beta]$ is finite over A. Hence β is integral over A. $\square$

Once again, can remove the Noeth'n assumption.

Remark (added 11/6): we can remove the assumption that A is Noetherian throughout. This isn't particularly important, as most of rings we encounter are Noetherian.

It's enough to do this in Theorem from Section 1.2, the assumption that A is Noetherian propagates from there.

So let B be finite over A. We need to show $\forall \beta \in B$ is integral over A. This turns out to be a consequence of the Cayley-Hamilton theorem.

We can replace A w. its image in B and assume A is a subring of B. The multiplication by β is an A-linear operator on B, denote this operator by x. Let $b_1 \dots b_k$ be generators of the A-module B. Then $x(b_i) = \sum_{j=1}^{k} a_{ij} b_j$ for some $a_{ij} \in A$. Let $\Psi = (a_{ij}) \in \mathrm{Mat}_{k\times k}(A)$. So x sends the collection $\vec{b} := (b_1 \dots b_k)$ viewed as a column vector to $\Psi \vec{b}$.

Now view B as an $A[x]$-module. The matrix $\tilde{\Psi} := xI - \Psi \in \mathrm{Mat}_{k\times k}(A[x])$ sends $\vec{b}$ to 0. We know that for $\tilde{\Psi}'$ consisting of $(k-1)\times(k-1)$ minors of $\tilde{\Psi}$ (sometimes called the adjoint matrix of $\tilde{\Psi}$ - although this terminology is not the best) we have $\tilde{\Psi}'\tilde{\Psi} = \det(\tilde{\Psi}) I$. It follows that the element $\det(\tilde{\Psi}) = \det(xI - \Psi) \in A[x]$ acts on B by 0. Set $f(x) := \det(xI - \Psi)$, this is a monic polynomial. Recall that x acts on B as multiplication by $\beta \Rightarrow f(\beta) = 0$ in B. So β is integral over A.