1) Integral extensions of rings.
2) Integral closure.
Ref: [AM], Section 5.1.

Intro/recap: We’ve seen a bunch of constructs of rings:
- direct sums
- rings of polynomials
- quotient rings
- completions (HW4)
- localizations
- tensor products
- symmetric algebras (HW4)

Today: another construction: taking integral extensions/closures, motivated by algebraic number theory, generalizes algebraic extensions/closures for fields (see MATH 370).

1) Integral extensions of rings.
Reminder: if $K \leq L$ are two fields, then one can speak about:
- $L$ being finitely generated (as a field) over $K$.
- $L$ being algebraic over $K$.
- $L$ being finite over $K$.

Now suppose that $A$ is a commutative unital ring, let $B$ be a commutative unital $A$-algebra. We’ve already defined what it means for $B$ to be finitely generated (as algebra) over $A$: 

[1]
\[ \exists b_1, \ldots, b_k \in B \text{ s.t. } \forall b \in B \exists F \in A[x_1, \ldots, x_k] \mid b = F(b_1, \ldots, b_k). \]

1) Definition & examples.

Definition: Say \( B \) is finite over \( A \) if \( B \) is a finitely gen'd A-module.

- Say \( b \in B \) is integral over \( A \) if \( \exists \) monic (i.e. leading coeff = 1) \( f \in A[x] \mid f(b) = 0 \).
- \( B \) is integral over \( A \) if \( \forall b \in B \) is integral (over \( A \)).

Examples: 1) \( A = \mathbb{K} < B = \mathbb{L} \) -extension of fields. Then the notions of being finite are equivalent. And integral \( \iff \) algebraic. But \( \mathbb{L} \) is fin. gen'd as an algebra over \( \mathbb{K} \implies \)
fin gen'd as a field over \( \mathbb{K} \) but not vice versa.

2) Let \( d \in \mathbb{Z} \), not a complete square, \( A = \mathbb{K}, B = \mathbb{K}[\sqrt{d}] \)
\( B \) is finite over \( A \) (rk 2 free \( A \)-module w. basis \( 1, \sqrt{d} \)).

Claim: \( B \) is integral over \( A \):
\[ \beta \in B \text{ equals } a + b\sqrt{d} \ (a, b \in \mathbb{K}) \text{ } \iff \text{ conjugate } \overline{\beta} : = a - b\sqrt{d} \]
\[ \beta + \overline{\beta} = 2a, \quad \beta \overline{\beta} = a^2 - b^2d \implies f(x) = (x - \beta)(x - \overline{\beta}) = x^2 - 2ax + (a^2 - b^2d) \in A[x] \text{ & } f(\beta) = 0. \text{ So } \beta \text{ is integral over } A \implies \]
\( B \) is integral over \( A \).

1.2) Properties.

Reminder: for field extensions: finite \( \iff \) algebraic \& fin gen'd.
Thm: Let $B$ be an $A$-algebra. Consider the following conditions:

(a) $B$ is fin. gen'd & integral over $A$.
(b) $B$ is finite over $A$.

Then (a) $\Rightarrow$ (b) and, if $A$ is Noetherian, then (b) $\Rightarrow$ (a).

 Added on 11/6: Can remove Noetherian assumpn: see Remark on page 6.

Proof: (b) $\Rightarrow$ (a) when $A$ is Noetherian

finite $\Rightarrow$ fin. gen'd (by if $b_1, ..., b_k$ generate $B$ as $A$-module $\Rightarrow$ they generate $B$ as $A$-algebra).

finite $\Rightarrow$ integral: $\beta \in B$, want $\exists$ monic $f(x) \in A[x]$ s.t. $f(\beta) = 0$.

For $k > 0 \Rightarrow M_k = \text{Span}_{A}(1, ..., \beta^{k-1}) \subset B$, is an $A$-submodule.

$M_i$'s form an ascending chain of submodules, which has to terminate by $A$ is Noetherian & $B$ is fin. gen'd ($\Rightarrow$ Noeth' $A$-module).

So $\exists k > 0$ s.t. $M_{k+1} = M_k \Rightarrow \beta^k \in M_{k+1} = M_k$, i.e.

$\beta^k = a_{k+1}\beta^{k-1} + a_k\beta^{k-2} + ... + a_1 \Rightarrow \text{set } f(x) = x^k - a_{k+1}x^{k-1} - ... - a_1$. $\checkmark$

(a) $\Rightarrow$ (b): Let $\beta_1, ..., \beta_k$ be generators of $A$-algebra $B$, know all of them are int'l over $A$. Want to show $B$ is fin. gen'd $A$-module; for $i = 0, ..., k \Rightarrow B_i = A[\beta_1, ..., \beta_i]$ (subalgebra gen'd by these elements), $B_0 = A$, $B_k = B$.

We'll show by induc't that $B_i$ is a fin. gen'd $A$-module.

Induction step: $B_{i+1} = B_i[\beta_{i+1}]$, $\beta_{i+1}$ integral over $A$ hence over $B_i$, $f(x) \in B_i[x]$ s.t. $f(\beta_{i+1}) = 0$, $f(x) = x^m + c_{m-1}x^{m-1} + ... + c_0$.

$g \in B_i \Rightarrow B_i[x] \twoheadrightarrow B_i[\beta_i] \xrightarrow{\beta_{i+1}} B_{i+1}/(f(x))$.

Since $f(x)$ is monic, $B_i[x]/(f(x))$ is generated by $1, ..., x^{m-1}$ as a $B_i$-module, in part' its fin. gen'd $\Rightarrow B_{i+1}$ is a
fin. gen'd $B_i$-module.

We know $B_i = \text{Span}_A (b_1, \ldots, b_n)$,
$B_i h = \text{Span}_B (b_1, \ldots, h_e) \Rightarrow$
$B_i h = \text{Span}_A (b_i h_j | i=1, \ldots, l, j=1, \ldots, l)$. Finishes induction step —
and the proof.

\[ \square \]

Corollary 1: Suppose $A$ is Noethan ring. If (a) $\iff$ (6) holds,
then $B$ is a Noethan ring.
- 6/c $B$ is a Noethan $A$-module $\Rightarrow$ Noethan $B$-module

Corollary 2: If $A$ is Noethan & $f(x) \in A[x]$ is mono, then
$A[x]/(f(x))$ is integral over $A$.
This is 6/c $A[x]/(f(x))$ is finite over $A$ (see the proof). Here we
can also remove Noethan assumption, see Rmk on page 6.

Corollary 3 (transitivity) Let $B$ be an $A$-algebra, $C$ be a $B$-
algebra. Then:
(a) $B$ fin. gen'd over $A$ & $C$ fin. gen'd over $B$ $\Rightarrow$ $C$ fin. gen'd over $A$.
(b) $\cdots$ finite $\cdots$ finite $\cdots$ finite $\cdots$
(c) $\cdots$ integral $\cdots$ finite $\cdots$ finite $\cdots$
(if $A$ is Noethan) — this assumption can be removed.

Proof of (c): $Y \in C$ integral over $B$ $\Rightarrow$ $\exists b_0, \ldots, b_{k-1} \in B$ s.t.
$y^{k} b_{k-1} \cdots b_0 = 0 \Rightarrow y$ is integral over $A[b_0, \ldots, b_{k-1}]$. Since
$b_0, \ldots, b_{k-1}$ are integral over $A$ $\Rightarrow A[b_0, \ldots, b_{k-1}]$ is finite over $A$;
$A[b_0, \ldots, b_{k-1}, y]$ is finite over $A[b_0, \ldots, b_{k-1}]$. By (6), $A[b_0, \ldots, b_{k-1}, y]$
\[ 4 \]
is finite over $A$, hence integral $\implies \beta$ is integral

2) Integral closure.

Proposition 1: Let $B$ be an $A$-algebra. Suppose $A$ is Noetherian.
If $d, \beta \in B$ are integral over $A$, then so are $d+\beta$, $d\beta$, $ad$ ($a \in A$). Again: can remove the Noetherian assumption. See on page 6.
Proof: Consider subalgebras $A[x] \subset A[\alpha, \beta] \subset B$, $A[x]$ is integral over $A$, $A[\alpha, \beta]$ is integral over $A[x] \implies$ integral over $A$ as well. Since $d\beta$, $d+\beta$, $ad \in A[\alpha, \beta]$, they are integral over $A$. □

Corollary/definition: The subset $\overline{A}^B$ of all integral over $A$ elements in $B$ form an $A$-subalgebra. This subalgebra is called the integral closure of $A$ in $B$.

Note that this is a direct generalization of algebraic closures of fields.

Prop 2: If $A$ is Noetherian, then the integral closure of $\overline{A}^B$ in $B$ is $\overline{A}^B$.
Proof: Let $\beta \in B$ be integral over $\overline{A}^B$. Need to show $\beta$ is integral over $A$ ($\implies \beta \in \overline{A}^B$). Let $f(x) = x^k + b_kx^{k-1} + \ldots + b_0 \in \mathbb{Z}$, $f(\beta) = 0$.
Then $b_0, \ldots, b_{k-1}$ are integral over $A \implies A[b_0, \ldots, b_{k-1}]$ is finite over $A \implies A[b_0, \ldots, b_{k-1}, \beta]$ is finite over $A$. Hence $\beta$ is integral over $A$. □

Once again, can remove the Noetherian assumption.
Remark (added 11/6): we can remove the assumption that $A$ is Noetherian throughout. This isn’t particularly important, as most of rings we encounter are Noetherian.

It’s enough to do this in Theorem from Section 1.2, the assumption that $A$ is Noetherian propagates from there.

So let $B$ be finite over $A$. We need to show if $\beta \in B$ is integral over $A$. This turns out to be a consequence of the Cayley-Hamilton theorem.

We can replace $A$ with its image in $B$ and assume $A$ is a subring of $B$. The multiplication by $\beta$ is an $A$-linear operator on $B$, denote this operator by $x$. Let $b_1, \ldots, b_k$ be generators of the $A$-module $B$. Then $x(b_i) = \sum_{j=1}^k a_{ij} \beta$, for some $a_{ij} \in A$. Let $\Psi = (a_{ij}) \in \text{Mat}_{k \times k}(A)$. So $x$ sends the collection $b = (b_1, \ldots, b_k)$ viewed as a column vector to $\Psi \vec{b}$.

Now view $B$ as an $A[x]$-module. The matrix $\tilde{\Psi} = xI - \Psi \in \text{Mat}_{k \times k}(A[x])$ sends $\vec{b}$ to $0$. We know that for $\tilde{\Psi}$ consisting of $(k-1) \times (k-1)$ minors of $\Psi$ (sometimes called the adjoint matrix of $\Psi$—although this terminology is not the best) we have $\tilde{\Psi}^* \tilde{\Psi} = \det(\tilde{\Psi})I$. It follows that the element $\det(\tilde{\Psi}) = \det(xI - \Psi) \in A[x]$ acts on $B$ by $0$.

Set $f(x) = \det(xI - \Psi)$, this is a monic polynomial. Recall that $x$ acts on $B$ as multiplication by $\beta \Rightarrow f(\beta) = 0$ in $B$. So $\beta$ is integral over $A$. 

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