Lecture 23

1) Prime ideals & irreducibility.
2) Geometric interpretation of alg'a homom's.

Bonus: Is this ideal radical?

Refs: [E], Section 1.6, Vinberg, A course in Algebra, §3.6, [AM], Ch. 4 & 7.1.

1) Reminder on prime ideals: A is comm've ring, $I \subset A$ ideal.
Say I is prime (Lec 3, Sect 1) if one of equiv't conditions hold:

1) A/I is domain
2) $a_1, a_2 \notin I \Rightarrow a_1 a_2 \notin I$.
3) if $I_1, I_2 \subset A$ are ideals & $I_1 I_2 \subset I \Rightarrow I_1$ or $I_2 \subset I$.

In particular, prime $\Rightarrow$ radical.
Let $\mathbb{F}$ be alg. closed field. In Section 2 of Lec 2, we've established

$$\begin{array}{ccc} \{\text{radical ideals in } \mathbb{F}[x_1 \dots x_n]\} & \xleftrightarrow{\sim} & \{\text{alg. subsets of } \mathbb{F}^n\} \\ \cup & & \cup \\ \{\text{prime ideals}\} & \xleftrightarrow{\sim} & ? \end{array}$$

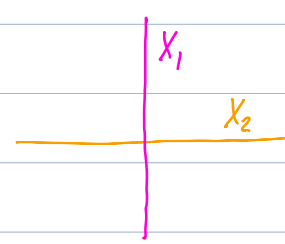
i.e. what's a geometric charact'n of $V(I)$'s for prime I.

1.1) Irreducible algebraic subsets.

Definition: an alg. subset X in $\mathbb{F}^n$ is called

- <u>irreducible</u>: if $X = X_1 \cup X_2$, where $X_i \subset \mathbb{F}^n$ is alg'c, then $X = X_i$ for some i.
- <u>reducible</u>, else.

Example: $n=2$, $X = \{(x_1, x_2) \mid x_1 x_2 = 0\}$ is reducible, $X_i = \{(x_1, x_2) \mid x_i = 0\}$, $i = 1,2$



Prop'n: TFAE: (a) X is irreducible

(b) $I(X)\left(=\{f\in \mathbb{F}[x_1,\dots x_n] \mid f|_X=0\}\right)$ is prime.

(c) $\mathbb{F}[X]\ (=\mathbb{F}[x_1,\dots x_n]/I(X))$ is a domain.

Proof: (b)$\Leftrightarrow$(c) is standard.

(a)$\Rightarrow$(b): assume that $I(X)$ isn't prime, i.e. $\exists f_i\in\mathbb{F}[x_1,\dots x_n]\setminus I(X)$ s.t. $f_1f_2\in I(X)$; $X_i:=\{\alpha\in X \mid f_i(\alpha)=0\}$, $i=1,2$. Then $X_i\subsetneq X$ (properly b/c $f_i\notin I(X)$, i.e. $f_i|_X\neq 0$), is an alg'c subset & $X_1\cup X_2=\{\alpha\in X \mid (f_1f_2)(\alpha)=0\}=[f_1f_2\in I(X)]=X$. Contradiction w. X being irreducible.

(b)$\Rightarrow$(a): assume X is reducible: $X=X_1\cup X_2$ w. $X_i\subsetneq X$ alg'c subset, define $I_i:=I(X_i)\supsetneq I(X)$ (again by Corollary in Sect. 2.2 of Lecture 22). Claim $I_1I_2\subset I(X)\Leftarrow I_1\cap I_2\subset I(X)$ $\Leftrightarrow$ [by that Corollary] $V(I_1\cap I_2)\supset X$. By Lemma in Sect 2.2 in Lecture 22, $V(I_1\cap I_2)=V(I_1)\cup V(I_2)(=X_1\cup X_2=X)$.
This proves the claim leading to contradiction $\square$

Examples: 1) $\mathbb{F}^n$ is irreducible b/c $\mathbb{F}[\mathbb{F}^n]=\mathbb{F}[x_1,\dots x_n]$ is domain.

2) Consider $X=\{(x_1,x_2)\in\mathbb{F}^2 \mid x_1x_2=1\}$. Note that $\mathbb{F}[x_1,x_2]/(x_1x_2-1)$ $\simeq\mathbb{F}[x^{\pm 1}]$ is domain. So the ideal (x_1x_2-1) is prime $\Rightarrow X$ is irreducible

3) $Y:=\{(y_1,y_2)\in\mathbb{F}^2 \mid y_1^2=y_2^3\}$. Can show $y_1^2-y_2^3$ is an irreducible polynomial so $(y_1^2-y_2^3)$ is prime. So it's radical $\Rightarrow I(Y)$ is prime. Hence Y is irreducible.

1.2) Irreducible components.

Theorem: Let X be alg'c subset. Then

a) $\exists$ irreducible alg'c subsets $X_1, \dots X_k$ s.t. $X = \bigcup_{i=1}^{k} X_i$.

b) For $X_1, \dots X_k$ we can take maximal (w.r.t. inclusion) irreducible alg. subsets contained in X.

Note, that (b) recovers $X_1, \dots X_k$ uniquely.

Def'n: These $X_1, \dots X_k$ (from b)) are called <u>irreducible components</u> of X.

Example: $X = \{(x_1, x_2) \mid x_1 x_2 = 0\}$. Irreducible comp's are $\{x_1 = 0\}$, $\{x_2 = 0\}$.

Proof: a) Assume the contrary: $\exists$ $X \neq$ finite union of irreducibles $\Leftrightarrow$ the set $\mathcal{A}$ of all such X's is $\neq \emptyset$. $\rightsquigarrow$ nonempty set $\{I(X) \mid X \in \mathcal{A}\}$. Since $\mathbb{F}[x_1, \dots x_n]$ is Noetherian, every nonempty set of ideals has max'l (w.r.t $\subset$) element. Pick $X' \in \mathcal{A}$ s.t. $I(X')$ is maximal in $\{I(X) \mid X \in \mathcal{A}\}$ $\Longleftrightarrow$ X' is minimal in $\mathcal{A}$ w.r.t. $\subset$. But X' is reducible b/c $X' \in \mathcal{A}$ $\Longleftrightarrow$ $X' = X^1 \cup X^2$ w. $X^i \subsetneq X'$ $\Longrightarrow$ [X' is min'l in $\mathcal{A}$] $X^i \notin \mathcal{A}$ $\rightsquigarrow$ $X^i = \bigcup_j X^i_j$ (finite unions of irreducibles) $\rightsquigarrow$ $X' = \bigcup_j X^1_j \cup \bigcup_j X^2_j$ – contradicts $X' \in \mathcal{A}$.

b) $X = \bigcup_{i=1}^{k} X_i$, where assume that none of X_i's is contained in another. Need to show: if $Y \subset X$ max'l irreducible $\Rightarrow$ $Y = X_i$ (for autom. unique i). To prove this: we observe

$$Y = \bigcup_{i=1}^{k} (Y \cap X_i); \text{ since } Y \text{ is irreducible} \Rightarrow Y = Y \cap X_i \text{ for}$$

some i $\Rightarrow$ $Y \subset X_i$, by since Y is maximal, $Y = X_i$. □

Corollary (alg'c formulation of Thm) Let $I \subset \mathbb{F}[x_1, \dots x_n]$ be radical ideal. Then $I = \bigcap_{i=1}^{k} I_i$, where I_i is prime; and we can recover I_i's uniquely if we assume they are minimal (w.r.t $\subseteq$) w. $I \subset I_i$.

Remark: the same statement is true if $\mathbb{F}[x_1 \dots x_n]$ w. arbitrary Noeth'n ring (premium exercise). Can generalize corollary to arbitrary ideals (primary decomp'n), see e.g. [AM], Chapters 4 & 7.1.

2) Geometric meaning of algebra homomorphisms.

X alg'c subset in $\mathbb{F}^n \rightsquigarrow I(X) \rightsquigarrow \mathbb{F}[X] = \mathbb{F}[x_1 \dots x_n]/I(X)$, elements of $\mathbb{F}[X]$ are (polynomial) functions $X \to \mathbb{F}$.

$\mathbb{F}[x_1, \dots x_n] \twoheadrightarrow \mathbb{F}[X]$, $F \mapsto F|_X$, $\bar{x}_i := x_i|_X$ -generate $\mathbb{F}[X]$.

2.1) Polynomial maps.

Definition: $X \subset \mathbb{F}^n$, $Y \subset \mathbb{F}^m$ alg'c subsets. A map $\varphi: X \to Y$ (of sets) is called <u>polynomial</u> if $\exists\, f_1, \dots f_m \in \mathbb{F}[X]$ s.t. $\varphi = (f_1, \dots, f_m)$.

Notice: polynomial map $X \to \mathbb{F}$ = polynomial function on X.

Point: polynomial map $X \to Y$ gives algebra homom'm $\mathbb{F}[Y] \to \mathbb{F}[X]$.

Constr'n: $g \in \mathbb{F}[Y]$ is function $Y \to \mathbb{F} \rightsquigarrow X \xrightarrow{\varphi} Y \xrightarrow{g} \mathbb{F}$

$$\varphi^*(g) := g \circ \varphi.$$

Lemma: 1) $\varphi^*(g) \in \mathbb{F}[X]$.

2) $\varphi^*: \mathbb{F}[Y] \to \mathbb{F}[X]$ is alg. homom'm.

3) On the generators $\bar{y}_j (:= y_j|_Y)$, $\varphi^*(\bar{y}_j) = f_j$

Proof:

1: $g \in \mathbb{F}[Y]$ means $g = G|_Y$ w. $G \in \mathbb{F}[y_1 \dots y_m]$ so $g \circ \varphi = G(f_1 \dots f_n) \in \mathbb{F}[X]$.

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2: Check $\varphi^*(g_1 g_2) = \varphi^*(g_1)\varphi^*(g_2)$. For all $\alpha \in X$ we have

$$\varphi^*(g_1 g_2)(\alpha) = (g_1 g_2)(\varphi(\alpha)) = g_1(\varphi(\alpha)) g_2(\varphi(\alpha)) = \varphi^*(g_1)(\alpha) \cdot \varphi^*(g_2)(\alpha).$$

This establishes the required equality.

3: $\varphi^*(\bar{y}_j)(\alpha) = \bar{y}_j(\varphi(\alpha)) = f_j(\alpha) \Rightarrow \varphi^*(\bar{y}_j) = f_j$ $\square$

Examples: 1) Inclusion map $\iota: X \hookrightarrow \mathbb{F}^n$ is polynomial, $\iota^*: \mathbb{F}[x_1, \dots x_n] \twoheadrightarrow \mathbb{F}[X]$, $F \mapsto F|_X$.

More generally, if $X \subset Y \subset \mathbb{F}^n$ alg'c subsets, then inclusion $\iota: X \hookrightarrow Y$ is polynomial & $\iota^*: \mathbb{F}[Y] \twoheadrightarrow \mathbb{F}[X]$, $g \mapsto g|_X$.

2) $X = \mathbb{F}$, $Y = \{(y_1, y_2) \in \mathbb{F}^2 \mid y_1^2 = y_2^3\}$, $\varphi: X \to Y$, $x \mapsto (x^3, x^2)$ is a polynomial m. Let's compute φ^*

By Example 3 in Sect. 1.1, $\mathbb{F}[Y] = \mathbb{F}[y_1, y_2]/(y_1^2 - y_2^3)$.

By 3) of Lemma, $\varphi^*(\bar{y}_1) = x^3$, $\varphi^*(\bar{y}_2) = x^2$.

2.2) Main result.

Theorem: $\varphi \mapsto \varphi^*$ defines a bijection between:

- $\{$polynomial maps $\varphi: X \to Y\}$
- $\{$algebra homomorphisms $\mathbb{F}[Y] \to \mathbb{F}[X]\}$.

Proof: Recall that given $\varphi = (f_1, \dots f_m)$, φ^* is the unique alg. homom'm $\mathbb{F}[Y] \to \mathbb{F}[X]$ s.t. $\varphi^*(\bar{y}_j) = f_j$.

Now given alg. homom. $\tau: \mathbb{F}[Y] \to \mathbb{F}[X]$ define $\varphi_\tau: X \to \mathbb{F}^m$ by $\varphi_\tau = (\tau(\bar{y}_1), \dots, \tau(\bar{y}_m))$.

Need to check $\operatorname{im}\varphi_\tau \subset Y \iff G(\operatorname{im}\varphi_\tau)=0\ \forall\, G\in I(Y)$ $\iff G(\tau(\bar{y}_1),\dots\tau(\bar{y}_m))=0$, which follows from $G(\bar{y}_1,\dots,\bar{y}_m)=0$ & τ is an algebra homomorphism (so preserves polynomial relations).

By constr'n, $\varphi\mapsto\varphi^*$ & $\tau\mapsto\varphi_\tau$ are inverse to each other (details are an exercise) $\square$

BONUS: Is this ideal radical?

We've talked about various properties of ideals (being radical/prime) and rings (being a normal domain). We work w. the ring $\mathbb{F}[x_1,\dots x_n]$, where $\mathbb{F}$ is a field, its ideals & quotients. Usually, the ideals are specified by their generators. So we can ask the following questions:

I) Given $F_1,\dots F_k\in\mathbb{F}[x_1,\dots x_n]$ is the ideal, can we determine whether $(F_1,\dots F_k)$ is radical or prime?

II) Assume $(F_1,\dots F_k)$ is prime. Can we determine whether $\mathbb{F}[x_1,\dots x_n]/(F_1,\dots,F_k)$ is normal?

As usual, the answer is both Yes & No.

Yes: for given n,k (& $F_1,\dots F_k$) there are algorithms (often implemented in Computer Algebra software) that allow to answer these and related questions. The main approach is via Gröbner bases. For more on them, see [E], Chapter 15.

No: if we care about the situation where we have a family of ideals with varying n,k.

Here's a famous example. Consider the space of pairs of square matrices, $\mathrm{Mat}_n(\mathbb{C})^2 \simeq \mathbb{F}^{2n^2}$. We have n^2 quadratic polynomials in these $2n^2$ variables - the entries of the matrix commutator $[A,B]=AB-BA$. For example, for $n=2$ we have

$$\left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\right] = \begin{pmatrix} x_{12}y_{21} - y_{12}x_{21} & x_{11}y_{12} + x_{12}y_{22} - y_{11}x_{12} - y_{12}x_{22} \\ x_{21}y_{11} + x_{22}y_{21} - y_{21}x_{11} - y_{22}x_{21} & y_{12}x_{21} - x_{12}y_{21} \end{pmatrix}$$

In fact, as this example indicates, the n^2 polynomials we get are linearly dependent - $\mathrm{tr}\,[A,B]=0$. In any case, let I be the ideal generated by these polynomials so that
$V(I) = \{(A,B) \in \mathrm{Mat}_n(\mathbb{C})^2 \mid AB=BA\}$, a.k.a the commuting variety.

Open problem 1: is I radical?

One can show $V(I)$ is irreducible, but there's

Open problem 2: Is $\mathbb{C}[V(I)]$ normal?

A fun fact: the normalization of $\mathbb{C}[V(I)]$ is Cohen-Macaulay (see Bonus 2 to Lec 21). This is a result of Victor Ginzburg from some 10 years ago but techniques of proof go way beyond Commutative algebra... Note also that these questions are related to Bonus for Lecture 7.