Lecture 24.

1) What is an affine (alg'c) variety?
2) Geometric significance of localization.

Bonus: • What's alg'c variety?
• Projective var's & graded alg's

Refs: [E], Section 1.6, intro to Section 2.

1.0) Recap.: $\mathbb{F}$ is alg. closed field, all algebras are comm've & unital

Def: $\mathbb{F}$-algebra A is <u>reduced</u> if it has no (nonzero) nilp't elements

So for ideal $I \subset \mathbb{F}[x_1,\dots x_n]$: $\mathbb{F}[x_1,\dots x_n]/I$ is reduced $\Leftrightarrow$ I is radical.

Recall nat'l bijections:

$$X \in \{\text{alg'c subsets in } \mathbb{F}^n\} \ni V(J)$$

$$X \mapsto I(X), \qquad V(J) \leftarrow J$$

$$I(X) \in \{\text{radical ideals in } \mathbb{F}[x_1,\dots x_n]\} \ni J$$

$$\ker(\mathbb{F}[x_1,\dots x_n] \twoheadrightarrow A) \leftarrow A, \qquad J \mapsto \mathbb{F}[x_1,\dots,x_n]/J$$

this comes from gener's

$$\begin{pmatrix} \mathbb{F}[X] \text{ w. generators} \\ \bar{x}_i := x_i|_X \end{pmatrix} \in \{\text{reduced } \mathbb{F}\text{-algebras w. fixed collect'n of } n \text{ generators}\} \ni A,\ \mathbb{F}[x_1,\dots,x_n]/J$$

So we get this composite bij'n $X \mapsto \begin{pmatrix} \mathbb{F}[X] \text{ w. generators} \\ \bar{x}_i := x_i|_X \end{pmatrix}$

Point: Usually, when we consider algebras we don't specify generators.

Q: Can we talk about algebraic subsets "on their own" i.e. w/o specifying an embedding into some $\mathbb{F}^n$?

In Lec 23, Sect. 2.1 for alg'c subsets $X \subset \mathbb{F}^n$, $Y \subset \mathbb{F}^m$ we defined the notion of polynomial map: $\varphi: X \to Y$ s.t. $\exists f_1, \dots, f_m \in \mathbb{F}[X]$ s.t. $\varphi = (f_1, \dots, f_m)$. Then define $\varphi^*: \mathbb{F}[Y] \to \mathbb{F}[X]$, $g \mapsto g \circ \varphi$, this is algebra homomorphism uniquely charact'd by $\varphi^*(\bar{y}_j) = f_j$.

We've seen in Section 2.2 of Lecture 23 that

$$\varphi \mapsto \varphi^*: \{\text{polyn'l maps } X \to Y\} \xrightarrow{\sim} \mathrm{Hom}_{Alg}(\mathbb{F}[Y], \mathbb{F}[X]).$$

1.1) Affine var's I.

Observation • For polynomial maps $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \Rightarrow$ $\psi \circ \varphi: X \longrightarrow Z$ is polynomial, & $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*: \mathbb{F}[Z] \to \mathbb{F}[X]$ $\left((\psi \circ \varphi)^*(h) = h \circ \psi \circ \varphi = \varphi^*(\psi^*(h))\right)$

• $(id_X)^* = id_{\mathbb{F}[X]}$.

Definition: Polynomial map $\varphi: X \to Y$ is an <u>isomorphism</u> if it has (autom. unique) inverse polyn'l map $\varphi^{-1}: Y \to X$.

Corollary (of Observation) For $\varphi: X \to Y$ TFAE:

(1) φ is isomorphism

(2) $\varphi^*: \mathbb{F}[Y] \to \mathbb{F}[X]$ is (algebra) isom'm.

Examples (of (non) isom'ms)

1) $X = \mathbb{F}$, $Y = \{(y_1, y_2) \in \mathbb{F}^2 \mid y_2 = y_1^2\}$, $\varphi: X \to Y$, $x \mapsto (x, x^2)$

It's isom'm w. inverse $Y \longrightarrow X$, $(y_1, y_2) \mapsto y_1$.

2) $X=\mathbb{F}$, $Y=\{(y_1,y_2)\in\mathbb{F}^2 \mid y_2^2=y_1^3\}$, $\mathbb{F}[Y]=\mathbb{F}[y_1,y_2]/(y_2^2-y_1^3)$

$\varphi: x\mapsto (x^2,x^3)$ – is a bijection, but NOT an isom'm.

$\varphi^*: \mathbb{F}[Y]\to\mathbb{F}[X]$, $g\mapsto g(x^2,x^3)\neq x$ so φ^* isn't surj've.

With notion of isomorphism we can identify different polynomial subsets in different $\mathbb{F}^n$'s so can talk about polynomial subsets irrespective of embedding, those are <u>affine varieties</u>.

Sometimes-an-issue: if there's an isom'm $X\xrightarrow{\sim}Y$, it doesn't need to be unique, and there may be no preferred choice.

Q: Can we define affine varieties intrinsically, starting from reduced fin. gen'd algebra A?

1.2) Affine varieties, II.

Recall: for algebraic subset X have a bijection

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \operatorname{Hom}_{Alg}(\mathbb{F}[X],\mathbb{F}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \alpha & \longmapsto & [f\mapsto f(\alpha)] \end{array}$$

Now given a reduced fin. gen'd $\mathbb{F}$-algebra $A \rightsquigarrow$ set $\operatorname{Hom}_{Alg}(A,\mathbb{F})$

We can view A as sitting inside $\{\operatorname{Hom}_{Alg}(A,\mathbb{F})\to\mathbb{F}\}$:

for $f\in A$, $\alpha\in\operatorname{Hom}_{Alg}(A,\mathbb{F})$, $f(\alpha):=\alpha(f)$.

For $\tau: B\to A \rightsquigarrow \varphi_\tau: \operatorname{Hom}_{Alg}(A,\mathbb{F})\to\operatorname{Hom}_{Alg}(B,\mathbb{F})$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \alpha & \longmapsto & \alpha\circ\tau \end{array}$$

Exercise: For $\tau: \mathbb{F}[Y] \to \mathbb{F}[X]$, $\varphi_\tau: X \to Y$ from Sect 2.2 in Lec 23, coincides w. φ_τ just above.

"Definition": Affine variety assoc'd to A as above is the set $\mathrm{Hom}_{Alg}(A, \mathbb{F})$ w. additional structures "coming" from A.

1.3) Zariski closed/open subsets.

Definition: Let $X \subset \mathbb{F}^n$ be algebraic subset. A subset $Y \subset X$ is called:

- Zariski closed, if it's an alg'c subset in $\mathbb{F}^n$.
- Zariski open, if $X \setminus Y$ is Zariski closed.

Remarks: • Yes, these are open & closed subsets in Zariski topology.

• Every Zariski closed subset is naturally an affine variety. This fails, in general, for Zariski open subsets.

Lemma: Have bijection:

$$\begin{array}{ccc} \{\text{Zariski closed subsets of } X\} & \ni & Y \\ \updownarrow\wr & & \downarrow \\ \{\text{radical ideals in } \mathbb{F}[X]\} & \ni & \{f \in \mathbb{F}[X] \mid f|_Y = 0\} \end{array}$$

Proof: exercise (note that we've seen this for $X = \mathbb{F}^n$; general case follows from there).

Remark: Note that the collection of Zariski closed subsets is recovered from $\mathbb{F}[X]$, and not from inclusion of X into $\mathbb{F}^n$.

In particular, affine varieties in the setting of Sect 1.2 come with topology.

2) Geometric significance of localization.

X is an alg'c subset of $\mathbb{F}^n$, $A := \mathbb{F}[X]$. Goal: understand geom. meaning of localizations of A.

2.1) A_f.

Let $f \in A$ be nonzero $\rightsquigarrow A_f = A[f^{-1}]$ - again fin. gen'd alg'a. Since A has no nilp. el'ts $\Rightarrow$ neither does A_f (Prob 9 in HW5)

$A_f \rightsquigarrow$ affine var'y X_f ($\mathbb{F}[X_f] = A_f$)

Q: What's connection between X & X_f?

Have homom'm $A \xrightarrow{\iota} A_f : a \mapsto \frac{a}{1}$.

Lemma: The polyn'l map $X_f \xrightarrow{\varphi} X$ induced by ι is injective & $\operatorname{im}(\iota) = \{\alpha \in X \mid f(\alpha) \neq 0\}$.

Proof:

$$X \xrightarrow{\sim} \operatorname{Hom}_{Alg}(A, \mathbb{F}), \quad X_f \xrightarrow{\sim} \operatorname{Hom}_{Alg}(A_f, \mathbb{F})$$

$$\beta \circ \iota \xleftarrow[\varphi]{} \beta$$

Univ. property of A_f (Sect 1.2 in Lec 9) implies: a homom'm $\alpha: A \to \mathbb{F}$ factors as $\beta \circ \iota \Leftrightarrow \alpha(f) \neq 0$ & β is uniquely determined by this. This is exactly the claim of the lemma $\square$

Remarks: (i) $\{\alpha \in X \mid f(\alpha) \neq 0\}$ is Zariski open

(ii) Every Zariski open subset of X is the union (of fin. many) subsets as in (i).

2.2) $A_{\mathfrak{m}}$ Let $\mathfrak{m} \subset A$ be max. ideal

Recall, for prime ideal $\mathfrak{p} \subset A$, $S := A \setminus \mathfrak{p}$ - localizable

$A_{\mathfrak{p}} := A_S$. Note $A_{\mathfrak{m}}$ is not finitely generated (in general) so doesn't correspond to any affine variety. It still has a geom'c meaning that we are going to discuss.

For simplicity, assume X is irreducible $\Leftrightarrow A = \mathbb{F}[X]$ is domain $\rightsquigarrow$ fraction field $\mathrm{Frac}(A) = \{\frac{f}{g} \mid g \neq 0\}$, every localization of A is contained in $\mathrm{Frac}(A)$ as subring.

Choice of $\mathfrak{m} \longleftrightarrow \alpha \in X$ so that $\mathfrak{m} = \{f \in A \mid f(\alpha) \neq 0\}$

$$A_{\mathfrak{m}} = \{\tfrac{f}{g} \mid g(\alpha) \neq 0\} = \bigcup_{g \mid g(\alpha) \neq 0} A_g = \bigcup_g \mathbb{F}[X_g]$$

Conclusion:

Every element of $A_{\mathfrak{m}}$ is a function on a Zariski open subset containing α, but which subset we choose depends on this element.

Remark: When X is reducible, conclusion still holds but $A_{\mathfrak{m}} = \bigcup_g \mathbb{F}[X_g]$ makes no sense b/c $\mathbb{F}[X_g] = A_g$ are not subrings in any given ring (in general). To fix this, replace $\bigcup$ w. "direct limit".

Exercise*: For X irreducible, give a similar descr'n for $A_{\mathfrak{p}}$ with arbitrary prime ideal $\mathfrak{p}$.

Remark (on terminology): Recall (Sect. 2.2 in Lec 10): a comm've unital ring B is called <u>local</u> if it has unique maximal ideal.

For example, $A_{\mathfrak{p}}$ is local (w. maximal ideal $\mathfrak{p}_{\mathfrak{p}}$) e.g. $A_{\mathfrak{m}}$ is local. The discussion above gives a geometric justification to this terminology: this algebra controls what happens locally (in Zariski topology) near $\alpha \in X$.

BONUS:

B1) What is an algebraic variety?

We've discussed affine (alg'c) varieties. Now we are going to address the question in the title.

A common approach to constructing geometric objects is to "glue" them from simpler objects. For example, C^∞ manifolds are glued from balls in Euclidian spaces: $M = \bigcup_\alpha D_\alpha$, where $D_\alpha \xrightarrow[\sim]{\varphi_\alpha} \{v \in \mathbb{R}^n \mid \|v\| < 1\}$. The condition is, roughly, that for all α, β in the index set, the images of $D_\alpha \cap D_\beta$ under $\varphi_\alpha, \varphi_\beta$ are open subsets in $\{v \in \mathbb{R}^n \mid \|v\| < 1\}$ and the resulting composition

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(D_\alpha \cap D_\beta) \xrightarrow[\sim]{\varphi_\alpha^{-1}} D_\alpha \cap D_\beta \xrightarrow[\sim]{\varphi_\beta} \varphi_\beta(D_\alpha \cap D_\beta)$$

is C^∞ (which makes sense b/c this is a map between open subsets in $\mathbb{R}^n$). Thank to this definition it makes to speak about various C^∞-objects, e.g. C^∞-maps $M \to N$.

Similarly, it makes sense to speak about complex analytic manifolds: we use balls in $\mathbb{C}^n$ and require that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is complex analytic (you might have studied that for $n=1$ — in which case the resulting objects appear when you study analytic continuation of holomorphic functions).

Something like that happens for algebraic varieties. The building blocks are affine algebraic varieties and they are glued together using polynomial isomorphisms: if the variety of interest is reasonable ("separated" in a suitable sense) the intersection of two open affine subvarieties is again affine so we can just use what we have in this lecture.

We can define the notion of a polynomial map (a.k.a. morphism): $\varphi: X \to Y$ is a morphism if we can cover $X = \bigcup U_i$, $Y = \bigcup V_j$ w. open affine varieties s.t. $\forall i\ \exists j \mid \varphi(U_i) \subset V_j$ & $\varphi: U_i \to V_j$ is a polynomial map of affine varieties.

B2) Projective varieties and graded algebras.

Here comes the most important example of the construction sketched above.

We start with $\mathbb{F}^{n+1}$ (viewed as a vector space). The <u>projective space</u> $\mathbb{P}^n (= \mathbb{P}(\mathbb{F}^{n+1}))$ as a set consists of 1-dimensional subspaces in $\mathbb{F}^{n+1}$. In other words, it consists of equivalence classes $[x_0 : \ldots : x_n]$ w. $(x_0, \ldots x_n) \in \mathbb{F}^{n+1} \setminus \{0\}$, where equivalent means proportional. Let us explain how gluing works.

Let $U_i = \{[x_0 : \ldots : x_n] \mid x_i \neq 0\}$, $i = 0, \ldots n$. Then the map $U_i \xrightarrow[\sim]{\varphi_i} \mathbb{F}^n$: $[x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i}\right)$ is a bijection that will be used to identify U_i w. $\mathbb{F}^n$. Note that $\varphi_i(U_i \cap U_j)$ is given by non-vanishing of a single coordinate so is an affine variety. And one can show that

$$\varphi_j \circ \varphi_i^{-1}: \quad \varphi_i(U_i \cap U_j) \xrightarrow{\sim} \varphi_j(U_i \cap U_j)$$

is a polynomial isomorphism.

Example: let $n=2$. Let $y_0 = \frac{x_1}{x_0}$ & $y_1 = \frac{x_0}{x_1}$ be coordinates on $\varphi_0(U_0) \simeq \mathbb{F}$ & $\varphi_1(U_1) \simeq \mathbb{F}$. Then $\varphi_i(U_0 \cap U_1)$ is given by $y_i \neq 0$ & $\varphi_1 \circ \varphi_0^{-1}$ sends y_0 to y_0^{-1}, which is a polynomial isomorphism as we have inverted y_0.

So $\mathbb{P}^n$ is an algebraic variety.

One can generalize this construction. Let $F_1, \dots F_k \in \mathbb{F}[x_0, \dots x_n]$ be <u>homogeneous</u> polynomials of degree >0. If F_i vanishes at a nonzero point in $\mathbb{F}^{n+1}$, then it also vanishes on the line between this point & 0. So it makes sense to speak about the zero locus of F_i in $\mathbb{P}^n$ (note that F_i is NOT a function $\mathbb{P}^n \longrightarrow \mathbb{F}$). This gives rise to the zero locus $V(F_1, \dots F_k)$ and hence to the notion of an algebraic subset of $\mathbb{P}^n$.

Exercise: $V(F_1, \dots F_k) \cap U_i$ is an algebraic subset in $U_i \xrightarrow{\sim} \mathbb{F}^n$.

So $V(F_1, \dots F_k)$ is an algebraic variety, varieties of that kind are called <u>projective</u>.

Here's a reason why we care about them. Let $\mathbb{F} = \mathbb{C}$. So $\mathbb{C}^n$ has the usual topology. And so does $\mathbb{P}^n$ with U_i's being open subsets.

Important exercise: $\mathbb{P}^n$ is compact — in the usual topology.

And so, every $V(F_1, \dots F_k)$ is compact. In Geometry & Topology we like compact spaces more than noncompact as they behave better in many ways. And while not all compact (in the usual topology) algebraic varieties are projective, the projective ones are nicest.

Now we discuss a connection between projective varieties & graded algebras. The vanishing locus of $V(F_1, \dots F_k)$ depends only on $(F_1, \dots F_k)$, a homogeneous ideal.

Exercise: If $I \subset \mathbb{F}[x_0, \dots x_n]$ is a homogeneous ideal, then so is its radical.

In fact, $V(F_1, \dots F_k)$ only depends on $\sqrt{(F_1, \dots F_k)}$, similarly to the affine case. This gives rise to a bijection between

- Algebraic subsets of $\mathbb{P}^n$
- and radical homogeneous ideals in $\mathbb{F}[x_0, \dots x_n]$ that do not contain 1.

Exercise: What ideal corresponds to $\emptyset$?

So starting from an algebraic subset in $\mathbb{P}^n$ we get a finitely generated reduced <u>graded</u> algebra, the quotient of $\mathbb{F}[x_0, \dots x_n]$ by the corresponding ideal. Note that the elements of this algebra are not functions on the initial algebraic subset of $\mathbb{P}^n$.

Conversely, let $A = \bigoplus_{i=0}^{\infty} A_i$ be a fin. gen'd graded reduced $\mathbb{F}$-algebra w. $A_0 = \mathbb{F}$. From this algebra we can construct a projective variety. Namely, if A is generated by A_1 ($\Leftrightarrow$ A is a graded quotient of $\mathbb{F}[x_0, \dots x_n]$), then we consider the algebraic subset of $\mathbb{P}^n$ defined by the kernel of $\mathbb{F}[x_0, \dots x_n] \twoheadrightarrow A$, which is a homogeneous ideal.

In general – if A isn't generated by A_1 – we have the following:

Exercise: $\exists\, d > 0$ s.t. $A_{(d)} := \bigoplus_{i=0}^{\infty} A_{di}$ is generated by A_d.

A fun fact: the projective variety we get is independent of the choice of d up to an isomorphism.

Example: Take $A = \mathbb{F}[x_0, x_1]$ (w. usual grading). It gives rise to the projective line $\mathbb{P}^1$. Now consider $A_{(2)}$. It's generated by $y_0 := x_0^2$, $y_1 := x_0 x_1$, $y_2 := x_1^2$. All relations between these elements are generated by $y_0 y_2 - y_1^2$. The corresponding algebraic subset is $\{[y_0 : y_1 : y_2] \mid y_0 y_2 = y_1^2\}$. Denote it by X.

We are going to construct two mutually inverse polynomial maps between $\mathbb{P}^1$ & X. Let $\varphi : \mathbb{P}^1 \to X$ be given by $[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_1^2]$. Now we define $\psi : X \to \mathbb{P}^1$:

$$\psi([y_0 : y_1 : y_2]) = \begin{cases} [y_0 : y_1], \text{ if } y_2 \neq 0 \\ [y_1 : y_2], \text{ if } y_0 \neq 0. \end{cases}$$

Exercise: Check φ, ψ are well-defined & mutually inverse maps. Furthermore, check that φ, ψ are morphisms (in the sense explained in the end of B1).

A connection with projective varieties is one of the reasons to care about graded algebras.