Lecture 25.

1) Modules over local rings & Nakayama lemma
2) Geometric meaning of modules.
Refs: [AM], Sect. 2.5

Bonus: graded Nakayama lemma.

1.1) Nakayama lemma: Let A be local ring (i.e. has unique max. ideal, $\mathfrak{m}$). The next result is very important.

Theorem (Nakayama lemma) If M is fin. gen'd A-module, $\mathfrak{m}M = M$, then $M = \{0\}$.

Proof: let $v_1, \dots v_k \in M$ be A-module generators; $M = \mathfrak{m}M$
$\Leftrightarrow \exists\, a_{ij} \in \mathfrak{m},\ i,j = 1,..k \mid v_i = \sum_{j=1}^{k} a_{ij} v_j \ \forall i = 1,.. k.$

$$\Psi = \underset{\text{identity matrix}}{I} - (a_{ij})_{i,j=1}^{k} \in \mathrm{Mat}_{k\times k}(A),\quad \vec{v} = \underset{\text{column vector}}{(v_1, \dots v_k)^T} \in M^{\oplus k}$$

$\rightsquigarrow \Psi\vec{v} = \vec{0}\ (\in M^{\oplus k})$. For "adj't" matrix Ψ' have $\Psi'\Psi = \det(\Psi) I$
$\Psi\vec{v} = \vec{0} \Rightarrow \Psi'\Psi\vec{v} = \vec{0} \Rightarrow \det(\Psi)\vec{v} = \vec{0}$ i.e. $\det(\Psi)v_i = 0\ \forall i$.
Formula for det $\Rightarrow \det(\Psi) \in 1 + \mathfrak{m}$. Since A is local, $1 + \mathfrak{m}$ are invertible in A. Thx to this, $\det(\Psi)v_i = 0 \Rightarrow v_i = 0$ $\forall i$. Since $M = \mathrm{Span}_A(v_1, .. v_k) \Rightarrow M = \{0\}$ $\square$

Notation: • $M(\mathfrak{m}) := M/\mathfrak{m}M$ is an $A/\mathfrak{m}$-module; $A/\mathfrak{m}$ is a field $\Rightarrow M(\mathfrak{m})$ is a vector space.
• For $v \in M$, write $\bar{v}$ for its image in $M/\mathfrak{m}M$.

Corollary: Let $v_1, \dots v_k \in M$. If $\bar{v}_1, \dots \bar{v}_k \in M(\mathfrak{m})$ span $M(\mathfrak{m})$, then $v_1, \dots v_k$ span A-module M.

Proof: Consider A-module $N = M / \mathrm{Span}_A(v_1, \dots v_k)$. Want $N = \{0\}$.
$M(\mathfrak{m}) = \mathrm{Span}_{A/\mathfrak{m}}(\bar{v}_1, \dots \bar{v}_k) \Leftrightarrow \mathrm{Span}_A(v_1, \dots v_k) + \mathfrak{m}M = M$
$\Leftrightarrow \mathfrak{m}N = N$. By Nakayama lemma, $N = \{0\}$ $\square$

1.2) Projective modules over local rings.

Theorem: Let A be local ring, and P a fin. gen'd projective A-module. Then P is free. Moreover, from a finite generating set of P we can select a basis.

Proof: Let $v_1, \dots v_k \in P \mid P = \mathrm{Span}_A(v_1, \dots v_k) \Rightarrow$
$P(\mathfrak{m}) = \mathrm{Span}_{A/\mathfrak{m}}(\bar{v}_1, \dots \bar{v}_k)$. Can pick a basis in $P(\mathfrak{m})$ among $\bar{v}_1, \dots \bar{v}_k$, say $\bar{v}_1, \dots \bar{v}_n$ ($n = \dim_{A/\mathfrak{m}} P(\mathfrak{m})$). By Corollary in Sect. 1.1, $P = \mathrm{Span}_A(v_1, \dots v_n)$. Want to show: $v_1, \dots v_n$ form basis in P.

$$A^{\oplus n} \xrightarrow{\pi} P$$

Since P is projective,
π "splits" $\Rightarrow A^{\oplus n} \simeq P \oplus P'$ (for some A-module P')

$$\Downarrow$$

$$\underbrace{(A/\mathfrak{m})^{\oplus n}}_{\dim = n} \simeq \underbrace{P(\mathfrak{m})}_{\dim = n} \oplus P'(\mathfrak{m}) \text{ - isom. of vect. space}/A/\mathfrak{m}.$$

$\Rightarrow \dim P'(\mathfrak{m}) = 0$. By Nakayama lemma, $P' = \{0\}$ so π is an isomorphism. $\square$

2) Geometric meaning of modules.

$\mathbb{F}$ is alg. closed field, X is an alg'c subset in $\mathbb{F}^n$ (i.e. an affine alg'c variety), $A = \mathbb{F}[X]$

Q: How to think about A-modules geometrically?

2.1) Fibers

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \{\text{max. ideals in } A\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \alpha & \longmapsto & \mathfrak{m}_\alpha := \{f \in A \mid f(\alpha) = 0\}; \quad A/\mathfrak{m}_\alpha \xrightarrow{\sim} \mathbb{F}. \end{array}$$

Definition: For an A-module M, its *fiber* at α is

$M(\alpha) := M/\mathfrak{m}_\alpha M$, an $\mathbb{F}$-vector space

Rem: if M is fin. gen'd $\Rightarrow \dim M(\alpha) < \infty \ \forall \alpha$

So: from M we get a collection of vector spaces indexed by pts. of X.

Examples: 1) $M = A^{\oplus n} \Rightarrow M(\alpha) = \mathbb{F}^n$.

2) $M = A/I$, where $I \subset A$ is an ideal.

$M(\alpha) = (A/I)/\mathfrak{m}_\alpha (A/I) = A/(I + \mathfrak{m}_\alpha)$.

If $\mathfrak{m}_\alpha \supset I \Rightarrow I + \mathfrak{m}_\alpha = \mathfrak{m}_\alpha \Rightarrow M(\alpha) = \mathbb{F}$.

$\Leftrightarrow \alpha \in V(I)$ - common zeros of I in X

If $\mathfrak{m}_\alpha \not\supset I \Rightarrow I + \mathfrak{m}_\alpha = A \Rightarrow M(\alpha) = \{0\}$.

Warning: In general, we cannot recover M just by knowing the dimensions of the fibers. For example, in 2) above we can only recover $V(I) \Leftrightarrow \sqrt{I}$ from knowing dimensions of fibers, not I itself.

Premium example: $\mathbb{F} = \mathbb{C}$, $P(x) \in \mathbb{C}[x]$ is a cubic polynomial w/o repeated roots $\rightsquigarrow y^2 - P(x) \in \mathbb{C}[x,y]$ is irreducible (exercise).

$X := \{(x,y)\in \mathbb{C}^2 \mid y^2 - P(x) = 0\}$ "elliptic curve."

Pick $\alpha_0 \in X \rightsquigarrow$ max. ideal $\mathfrak{m}_{\alpha_0} \subset A := \mathbb{C}[X] = \mathbb{C}[x,y]/(y^2 - P(x))$.

Exercise: show $\mathfrak{m}_{\alpha_0}(\alpha) \simeq \mathbb{F}$ for all $\alpha \in X$ (for $\alpha \neq \alpha_0$, this is easy; but for $\alpha = \alpha_0$ need to use that P has no repeated roots).

Fact (from Alg. geometry) there are uncountably many pairwise non-isomorphic A-modules among $\mathfrak{m}_{\alpha_0}$'s (Alg. geom. can tell you when 2 such modules are isomorphic).

2.2) Localization of modules vs fibers.

$X \subset \mathbb{F}^n$ alg. subset, $A = \mathbb{F}[X]$, $f \in A \setminus \{0\}$

localization $A_f = A[f^{-1}]$ is the algebra $\mathbb{F}[X_f]$, where $X_f := \{\alpha \in X \mid f(\alpha) \neq 0\}$ (Section 2.1 of Lecture 24)

Now let M be A-module $\rightsquigarrow$ localization M_f, module$/A_f$.

$\alpha \in X_f \rightsquigarrow M(\alpha), M_f(\alpha)$, vector spaces over $\mathbb{F}$

Proposition: $\forall \alpha \in X_f$ have natural isomorphism $M(\alpha) \xrightarrow{\sim} M_f(\alpha)$.

Proof: View $\mathbb{F}$ as an A_f-algebra via $g \in A_f \mapsto g(\alpha) \in \mathbb{F}$.

Recall (Section 2 in Lecture 19)

$A_f \otimes_A M \xrightarrow{\sim} M_f$;

$M_f(\alpha) = M_f / \mathfrak{m}_\alpha M_f$ [Prob 6, HW4] $\xrightarrow{\sim} (A_f/\mathfrak{m}_\alpha)\otimes_{A_f} M_f$

So

$M_f(\alpha) = \mathbb{F} \otimes_{A_f} (A_f \otimes_A M)$.

Now we view $\mathbb{F}$ as an A-algebra via composition

$A \to A_f \to \mathbb{F}$ that equals $g' \in A \mapsto g'(\alpha) \in \mathbb{F}$.
Then $M(\alpha) = \mathbb{F} \otimes_A M$.
Now: need a natural isomorphism between $\mathbb{F} \otimes_{A_f} (A_f \otimes_A M)$ & $\mathbb{F} \otimes_A M$. $\square$

Lemma: Let A be a ring, B be an A-algebra, C be a B-algebra (hence C is also an A-algebra). Then the functors $C \otimes_A \bullet$, $C \otimes_B (B \otimes_A \bullet): A\text{-Mod} \to C\text{-Mod}$ are isomorphic.
(apply to $B = A_f$, $C = \mathbb{F}$).

Proof: Recall that $F_1 = B \otimes_A \bullet: A\text{-Mod} \to B\text{-Mod}$ is left adjoint to forgetful functor, $G_1: B\text{-Mod} \to A\text{-Mod}$ (Sect 1.2 in Lect. 17). Similarly, $F_2 = C \otimes_B \bullet: B\text{-Mod} \to C\text{-Mod}$ is left adj't to forgetful $G_2: C\text{-Mod} \to B\text{-Mod}$.
$G_1 \circ G_2: C\text{-Mod} \to A\text{-Mod}$ is forgetful so its left adj't is $C \otimes_A \bullet: A\text{-Mod} \to C\text{-Mod}$. By Prob 3 in HW4,
$F_2 \circ F_1: A\text{-Mod} \to C\text{-Mod}$ is left adj't to $G_1 \circ G_2$.
By uniqueness of adj't functors (Section 1.3 in Lecture 14),
$$F_2 \circ F_1 \xrightarrow{\sim} C \otimes_A \bullet . \qquad \square$$

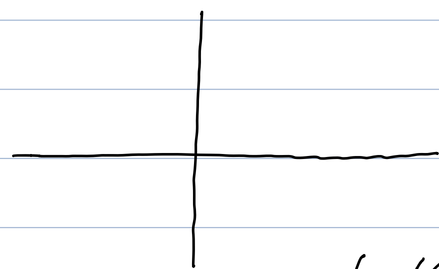
Rem: A way to think about the claim of proposition: on the level of fibers, localization just restricts the module to the Zariski open subset X_f.

2.3*) Fibers of projective modules.

Q: For which fin. gen'd A-modules M, all fibers $M(\alpha)$ have the same dimension?

Theorem: If all fibers have the same dimension, then the module is proj've. The converse is true if X is connected in Zariski topology (i.e. cannot be represented as $X = X_1 \coprod X_2$ for X_1, X_2 Zariski closed).

Rem/example:



$X_1 = \{(x,y) \mid xy = 0\}$

reducible but connected

X_2

$X_2 = \{(x,y) \mid x^2 = x\}$ is disconnected.

Rem*: If M is a fin. gen'd $A = \mathbb{F}[X]$-module, then
$\{\alpha \in X \mid \dim M(\alpha) \geq d\}$ is Zariski closed $\forall d \geq 0$
(fun problem on the stuff in the next lecture – and the Nakayama lemma).

BONUS: Graded Nakayama lemma:

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded ring.

Definition: a grading on an A-module M is a decomposition $M = \bigoplus_{j \in \mathbb{Z}} M_j$ into a direct sum of abelian groups such that
$$A_i M_j \subset M_{i+j} \quad \forall i, j.$$

For example, for a homogeneous ideal $I \subset A$ both I and A/I are graded modules.

Now we are going to discuss an analog of the Nakayama lemma for graded modules. Let M be a finitely generated graded module.

Exer: M is generated by finitely many homogeneous elements.

Proposition (graded Nakayama lemma): Let $v_1, \dots v_k \in M$ be homogeneous elements. If $M = A_{>0} M$, then $M = \{0\}$.

Proof: exercise

Here's a geometric reason to care about graded modules. From a graded $\mathbb{F}$-algebra A we can construct a projective alg'c variety. And from a graded A-module M we can construct a "quasi-coherent" sheaf on that variety. (Quasi) coherent sheaves is a central object of study in Algebraic geometry.