Lecture 26.

Projective modules vs locally free modules. | BONUS: What's next?

1) Main result.
A is a Noeth'n ring.

Theorem: For a fin. gen'd A-module M TFAE:

(a) M is projective.

(b) $\forall$ max. ideal $\mathfrak{m} \subset A$, the local'n $M_{\mathfrak{m}}$ is free $A_{\mathfrak{m}}$-module

(c) $\exists\, f_1, \ldots, f_k \in A$ s.t. $(f_1, \ldots, f_k) = A$ & M_{f_i} is free A_{f_i}-module $\forall\, i = 1, \ldots k$.

Def'n: Modules satisfying (b) $\Leftrightarrow$ (c) are called <u>locally free</u>.

Example: $A = \mathbb{Z}[\sqrt{-5}]$, $I = (2, 1+\sqrt{-5})$. In Prob 2 of HW3, have seen that the A_2-module I_2, A_3-module I_3 are free, $(2,3) = A$ (example of (a) $\Rightarrow$ (c)).

Rem: $X \subset \mathbb{F}^n$ (alg'c subset, where $\mathbb{F}$ is alg. closed field), $A = \mathbb{F}[X]$; $(f_1, \ldots f_k) = A \Longleftrightarrow \underline{V(f_1, \ldots f_k) \subset X \text{ is empty}}$,

similar to Exer in Lec 22

$$\Updownarrow$$

$$X = \bigcup_{i=1}^{k} X_{f_i}; \quad A_{f_i} = \mathbb{F}[X_{f_i}].$$

In Lec 25, we've argued that M_{f_i} should be thought of as the "restriction" of M to X_{f_i}. So (c) says that M is loc'y free in Zariski topology, i.e. M is "algebraic vector bundle." Vector bundles are objects of primary interest in Geometry & Topology.

Proof of (a) $\Rightarrow$ (b): M is fin. gen'd & projective $\Leftrightarrow \exists\, n>0$ & A-module M' s.t. $M\oplus M' \simeq A^{\oplus n}$. Localize at $\mathfrak{m}$:
$M_{\mathfrak{m}} \oplus M'_{\mathfrak{m}} \simeq A_{\mathfrak{m}}^{\oplus n} \Rightarrow M_{\mathfrak{m}}$ is fin. gen'd proj've $A_{\mathfrak{m}}$-module. The ring $A_{\mathfrak{m}}$ is local. By Thm in Sect. 1.2 of Lec 25, $M_{\mathfrak{m}}$ is free $\square$

2) Technical lemma: (to be used for both (b) $\Rightarrow$ (c) & (c) $\Rightarrow$ (a)).

Lemma: Let $\mathfrak{m}\subset A$ be max. ideal, M is a fin. gen'd A-module. TFAE:

(1) $M_{\mathfrak{m}} = \{0\}$.

(2) $\exists\, f\in A\setminus\mathfrak{m}$ s.t. $M_f=\{0\}$, equivalently, $\exists\, n>0$ s.t. $f^n M=\{0\}$.

Proof: Let $S\subset A$ be some localizable subset.
$M_S=\{0\} \Leftrightarrow \frac{m}{t}=\frac{0}{1}\ \forall\, m\in M, t\in S \Leftrightarrow \forall\, m\in M\ \exists\, s=s(m)\in S$ s.t. $sm=0$.

Proof of (1) $\Rightarrow$ (2): let $m_1,\dots m_k\in M$ be generators. Then $\exists\, s_1,\dots s_k\in A\setminus\mathfrak{m}$ s.t. $s_i m_i=0$. Take $f=s_1\dots s_k$. Since $\mathfrak{m}$ is max'l $\Rightarrow$ prime, $f\notin\mathfrak{m}$; $fm_i=0$. Since $m_1\dots m_k$ generate A-module $M \Rightarrow fM=\{0\}$. Discussion above in the proof $\Rightarrow M_f=\{0\}$.

Proofs of (2) $\Rightarrow$ (1) & equiv. in (2) are exercise $\square$

3) Proof of (b) $\Rightarrow$ (c): If $M_{\mathfrak{m}}$ is free $\forall\,\mathfrak{m} \Rightarrow \exists\, f_1,\dots f_k\in A$ s.t. $(f_1,\dots f_k)=A$ & M_{f_i} is free A_{f_i}-module $\forall\, i$.

Lemma: Let M be fin. gen'd A-module, $\mathfrak{m}\subset A$ be max'l ideal. If $M_{\mathfrak{m}}$ is free $A_{\mathfrak{m}}$-module, then $\exists\, f\in A\setminus\mathfrak{m}$ s.t.

M_f is free A_f-module.

Proof of (b)$\Rightarrow$(c) modulo the lemma: By Lemma: $\forall\, \mathfrak{m} \subset A$ max. ideal $\exists\, f(\mathfrak{m}) \in A \setminus \mathfrak{m}$ s.t. $M_{f(\mathfrak{m})}$ is free $A_{f(\mathfrak{m})}$-module. Need to prove: can pick fin. many of $f(\mathfrak{m})$'s that generate A as an ideal, these will be our $f_1, \dots f_k$.

Let $I \subset A$ be the ideal generated by all $f(\mathfrak{m})$'s. I isn't contained in any max. ideal $\Rightarrow I = A$. So $1 \in I$ is a finite A-linear combination of $f(\mathfrak{m})$'s. We're done. $\square$

Proof of Lemma: Let $m_1, \dots m_k$ be generators of A-module $M \Rightarrow$ $\frac{m_1}{1}, \dots \frac{m_k}{1} \in M_{\mathfrak{m}}$ are generators of $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$. By Thm in Sect. 1.2 of Lec 25, can pick basis of $M_{\mathfrak{m}}$ among $\frac{m_1}{1}, \dots \frac{m_k}{1}$, say it's $\frac{m_1}{1}, \dots \frac{m_n}{1}$. Consider $\varphi: A^{\oplus n} \longrightarrow M$, $(a_1, \dots a_n) \mapsto a_1 m_1 + \dots + a_n m_n$. Know $\varphi_{\mathfrak{m}}: A_{\mathfrak{m}}^{\oplus n} \xrightarrow{\sim} M_{\mathfrak{m}}$. We'll show $\exists\, f \in A \setminus \mathfrak{m}$ s.t. φ_f is isomorphism $A_f^{\oplus n} \longrightarrow M_f$.

$K := \ker \varphi \subset A^{\oplus n}$, $C := \operatorname{coker} \varphi \ (= M/\operatorname{im}\varphi)$ - fin. gen'd modules.

Claim: $(K \oplus C)_{\mathfrak{m}} = \{0\}$: use that $\bullet_{\mathfrak{m}}$ is an exact functor so $(\ker\varphi)_{\mathfrak{m}} = \ker(\varphi_{\mathfrak{m}}) \underset{\substack{\uparrow \\ \text{b/c } \varphi_{\mathfrak{m}} \text{ is isom'm}}}{=} \{0\}$, $(\operatorname{coker}\varphi)_{\mathfrak{m}} = \operatorname{coker}(\varphi_{\mathfrak{m}}) = \{0\}$.

Applying (1)$\Rightarrow$(2) of Techn. Lemma (Sect 2) see $\exists\, f \in A \setminus \mathfrak{m}$ s.t. $(K \oplus C)_f = \{0\}$.

Reversing the argument proving Claim, see $\varphi_f: A_f^{\oplus n} \xrightarrow{\sim} M_f$. $\square$

4) Proof (c) $\Rightarrow$ (a): $\exists\, f_1, \dots f_k \in A$ s.t. $(f_1 \dots f_k) = A$ & M_{f_i} is free A_{f_i}-module $\forall i \Rightarrow M$ is projective.

Recall (Section 3.1 of Lec 19) TFAE:

- M is projective
- $\forall$ surj've A-linear map $N^1 \twoheadrightarrow N^2 \Rightarrow$
  $$\mathrm{Hom}_A(M, N^1) \twoheadrightarrow \mathrm{Hom}_A(M, N^2)$$

Recall (Problem 5 in HW3): $\forall$ localizable $S \subset A$ have natural isomorphism $\mathrm{Hom}_A(M,N)_S \xrightarrow{\sim} \mathrm{Hom}_{A_S}(M_S, N_S)$ (for M fin.gen'd & A Noeth'n). Apply to $S = \{f_i^n\}$

Get comm've diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(M, N^1)_{f_i} \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(M, N^2)_{f_i} \arrow[d, "\wr"] \\
\mathrm{Hom}_{A_{f_i}}(M_{f_i}, N^1_{f_i}) \arrow[r] & \mathrm{Hom}_{A_{f_i}}(M_{f_i}, N^2_{f_i})
\end{tikzcd}$$

Since M_{f_i} is free, bottom arrow is surjective, so the top arrow is surjective. Let C be coker of $\mathrm{Hom}_A(M, N^1) \to \mathrm{Hom}_A(M, N^2)$. Need to show $C = \{0\}$. By above, $C_{f_i} = \{0\}$. By Techn. lemma from Sect 2, $\exists\, n_i > 0$ $f_i^{n_i} C = \{0\}$.

Note $(f_1^{n_1}, \dots f_k^{n_k}) \supset \underbrace{(f_1, \dots, f_k)^n}_{= A}$ for $n = n_1 + n_2 + \dots + n_k$, so $(f_1^{n_1}, \dots f_k^{n_k}) = A$

Since $f_i^{n_i} C = \{0\} \Rightarrow (f_1^{n_1}, \dots f_k^{n_k}) C \Rightarrow C = \{0\}$.

End of proof of Thm! $\square$

5) Constant dimensions of fibers $\Rightarrow$ projective

X be affine variety, $A = \mathbb{F}[X]$, M a fin. gen'd A-module.
$\alpha \in X \longleftrightarrow \mathfrak{m}_\alpha \subset A$ max. ideal $\rightsquigarrow$ fiber $M(\alpha) := M/\mathfrak{m}_\alpha M$, a vector space over $\mathbb{F}\ (\simeq A/\mathfrak{m}_\alpha)$.

Thm: If all $\dim_{\mathbb{F}} M(\alpha)$ have the same dimension, then M is projective.

Proof (a bit sketchy): We'll check M is locally free. Pick $\alpha_0 \in X$, let $v_1, \dots v_n$ be a basis in $M(\alpha_0)$. Fix lifts $m_1, \dots m_n \in M$ be lifts of $v_1, \dots v_n$. Let $\mathfrak{m}$ be the unique max. ideal in $A_{\mathfrak{m}_{\alpha_0}}$. Take $M' := M_{\mathfrak{m}_{\alpha_0}}$. Consider $M'(\mathfrak{m}) = M'/\mathfrak{m}M'$, $\mathbb{F}$-vector space.
Similarly to Sect 2.2 in Lec 25, $M'(\mathfrak{m}) \xleftrightarrow{\sim} M(\alpha_0)$.
In particular, the images of $\frac{m_1}{1}, \dots, \frac{m_n}{1}$ generate $M'(\mathfrak{m}) \Longrightarrow$ [corollary of Nakayama lemma] $\frac{m_1}{1}, \dots \frac{m_n}{1}$ generate $M' = M_{\mathfrak{m}_{\alpha_0}}$.
Equiv. the corresp. map $A_{\mathfrak{m}_{\alpha_0}}^{\oplus n} \longrightarrow M_{\mathfrak{m}_{\alpha_0}}$ is surj've.

We can argue similarly to proof of (b) $\Rightarrow$ (c) of prev. Thm to see $\exists f \notin \mathfrak{m}_{\alpha_0}$ s.t. $A_f^{\oplus n} \twoheadrightarrow M_f$. Want to show it's isomorphism.

Assume contrary: $\exists\ 0 \neq \underbrace{(g_1, \dots g_n)}_{\text{functions on } X_f} \in \ker$ of this map.

$\exists\ \alpha \in X_f \mid (g_1(\alpha), \dots g_n(\alpha)) \neq 0$; $A_f^{\oplus n} \longrightarrow M_f \rightsquigarrow A_f(\alpha)^{\oplus n} \twoheadrightarrow M_f(\alpha)$ has $(g_1(\alpha), \dots g_n(\alpha))$ in the kernel. So $\dim M_f(\alpha) < n = \dim M(\alpha_0)$

Lec 25, Sect 2.2 $\longrightarrow$ $\| \ \dim M(\alpha)$

Conclude $\dim M(\alpha) < \dim M(\alpha_0)$.

By assumption, $\dim \mathcal{M}(\alpha) = \dim \mathcal{M}(\alpha_0)$. Contradiction. $\square$

BONUS: What's next (in studying Commutative Algebra)?

A short answer: a whole lot, see Eisenbud's book

- we've discussed finite ring extensions a bit. There's more to it, like going up & going down theorems. These are best understood geometrically.
- we've briefly touched upon completions. There's more to it, incl. the Artin-Rees lemma, Hensel lemma etc.
- we haven't discussed the dimension theory at all, but it's very important. Neither we talked about regular rings, algebraic counterparts of smooth (a.k.a. nonsingular) affine varieties.
- various homological algebra considerations starting with Hilbert's Syzygy Theorem.
- and so on.