Lecture 4

1) Submodules & quotient modules.
2) Finitely generated, free and projective modules.

References: [AM], Chapter 2, Sections 2,3,5.

1.1) Submodules:
A comm've unital ring.

Definition: let M be an A-module. A <u>submodule</u> in M is an abelian subgroup $N \subset M$ s.t. $a \in A, n \in N \Rightarrow an \in N$.

Rem: N has a natural A-module str're.

Examples:

0) A is a field (so module = vector space): Submodule = subspace.
1) $A = \mathbb{Z}$ (so module = abelian group): Submodule = subgroup.
2) $A = \mathbb{F}[x]$ ($\mathbb{F}$ is a field). A module M = $\mathbb{F}$-vector space w. operator $X: M \to M$. A submodule $N \subset M$ – subspace s.t. $X(N) \subseteq N$. Conversely, every X-stable subspace is submodule b/c $f(x)m = f(X)m$ & $X(N) \subseteq N \Rightarrow f(X)(N) \subset N$.
3) A is any ring, $M = A$: submodule = ideal.

1.2) Constructions w. submodules.

1) $\varphi: M \to N$ A-module homom'm: $\ker\varphi \subset M$ & $\operatorname{im}\varphi \subset N$ are submodules.
2) $m_1, \dots, m_k \in M \rightsquigarrow \operatorname{Span}_A(m_1, \dots, m_k) := \{\sum_{i=1}^{k} a_i m_i \mid a_i \in A\}$ – this is special case of image: $\underline{m} = (m_1, \dots, m_k) \rightsquigarrow \varphi_{\underline{m}}: A^{\oplus k} \to M$
$\varphi_{\underline{m}}(a_1, \dots, a_k) = \sum_{i=1}^{k} a_i m_i$ so $\operatorname{Span}_A(m_1, \dots, m_k) = \operatorname{im}\varphi_{\underline{m}}$

More generally, $(m_i)_{i\in I} \rightsquigarrow \mathrm{Span}_A(m_i \mid i\in I) := \{$finite A-lin. combinations of m_i's$\}$.

3) Sums & intersections: $M_1, M_2 \subset M$ submodules
$M_1 \cap M_2$, $M_1 + M_2 = \{m_1 + m_2 \mid m_i \in M_i\}$ - submodules.

4) Product w. ideal: $N \subset M$ submodule, $I \subset A$ ideal
$IN := \{\sum_{i=1}^{\ell} a_i n_i \mid a_i \in I, n_i \in N\}$ - submodule
(compare to product of ideals).

1.3) Quotient modules: M is A-module, $N \subset M$ submodule
$\rightsquigarrow$ abelian group $M/N = \{m+N \mid m\in M\}$ & abelian group homom'm $\pi: M \twoheadrightarrow M/N$, $\pi(m) := m+N$. Then M/N has a natural A-module str're.

Proposition: 1) The map $A \times (M/N) \to M/N$, $(a, m+N) \mapsto am+N$ is well-defined ($am+N$ only depends on $m+N$ & not on m itself) an equips M/N w. A-module structure.

2) This module str're is unique s.t. $\pi: M \to M/N$ is a module homomorphism.

3) (Universal property of M/N & $\pi: M \twoheadrightarrow M/N$.) Let $\varphi: M \to M'$ be A-module homom'm s.t $N \subset \ker\varphi$. Then $\exists!$ module homom'm $\underline{\varphi}: M/N \to M'$ s.t. diagram commutes:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ M/N \arrow[r, dashed, "\underline{\varphi}"'] & M' \end{tikzcd}$$

$\underline{\varphi}$ is given by:
$\underline{\varphi}(m+N) := \varphi(m)$

Proof: exercise.

Part 3 can be used to understand $\mathrm{Hom}_A(M/N, M')$

Exercise: Consider
$$\pi^*: \mathrm{Hom}_A(M/N, M') \longrightarrow \mathrm{Hom}_A(M, M')$$
$$\psi \longmapsto \psi \circ \pi$$

Then π^* is injective, is a module homomorphism & $\mathrm{im}\,\pi^* = \{\psi \in \mathrm{Hom}_A(M, M') \mid N \subset \ker \psi\}$.

Remarks: 1) $I \subset A$ ideal $\rightsquigarrow$ submodule $IM \subset M \rightsquigarrow$ quotient M/IM, an A-module; $a \in I \Rightarrow a(m + IM) = am + IM = 0$. So M/IM is an A/I-module. For example, if $\mathfrak{m} = I$ is maximal $\Rightarrow A/\mathfrak{m}$ is a field so $M/\mathfrak{m}M$ is vector space over $A/\mathfrak{m}$

2) We have standard "isomorphism theorems":

- for $\psi: M \to N$, A-module homom'm, then $M/\ker\psi \xrightarrow{\sim} \mathrm{im}\,\psi$ (A-module isomorphism).
- for submodules $K \subset N \subset M$, have $(M/K)/(N/K) \xrightarrow{\sim} M/N$
- for submodules $N_1, N_2 \subset M$, have $N_1/N_1 \cap N_2 \xrightarrow{\sim} (N_1 + N_2)/N_2$

Have natural abelian group isomorphisms, they are module homomorphisms.

- There are bijections between:

$$\begin{array}{c} \{\text{submodules } L \subset M \mid N \subset L\} \\ \downarrow\ L \mapsto \pi(L) = L/N \qquad \uparrow\ \underline{L} \mapsto \pi^{-1}(\underline{L}). \\ \{\text{submodules } \underline{L} \subset M/N\} \end{array}$$

2.1) Finitely generated modules

Definition: • Elements $m_i \in M$ $(i \in I)$ are generators if $M = \mathrm{Span}_A(m_i \mid i \in I)$, i.e. $\forall m \in M$ is A-linear combination of finite number of m_i's.

• M is finitely generated if it has a finite collection of generators.

Remarks: • $A^{\oplus I}$ is finitely generated $\Leftrightarrow$ I is finite.

• If M is fin. generated, then so is M/N $\forall N \subset M$:

$M = \mathrm{Span}_A(m_1, \ldots m_k) \Rightarrow M/N = \mathrm{Span}_A(\pi(m_1), \ldots, \pi(m_k))$.

• $\underline{m} = (m_1, \ldots m_k) \rightsquigarrow \varphi_{\underline{m}}: A^{\oplus k} \longrightarrow M$, $\varphi_{\underline{m}}(a_1, \ldots a_k) = \sum_{i=1}^{k} a_i m_i$.

$M = \mathrm{Span}_A(m_1, \ldots m_k) \Leftrightarrow \varphi_{\underline{m}}$ is surj've $\Rightarrow M \xrightarrow{\sim} A^{\oplus k}/\ker \varphi_{\underline{m}}$

So: fin. gen'd modules = quotients of $A^{\oplus k}$ for some $k \in \mathbb{Z}_{\geq 0}$.

2.2) Free modules.

Definition: • Elements m_i, $i \in I$, form a basis in M if $\forall m \in M$ is uniquely written as A-linear combination of m_i, $i \in I$.

• M is free if it has a basis.

Examples: • For set I, $A^{\oplus I}$ is free, for a basis can take coordinate vectors e_i, $i \in I$: $e_i = (0, \ldots, 0, \underset{i\text{th position}}{1}, 0 \ldots)$

• If A is field, then every module (a.k.a. vector space) is free. If A is not a field, there are non-free modules: take $J \subset A$ be ideal, $J \neq \{0\}, A \Rightarrow A/J$ is not free (over A)

Remark: Every free module is isomorphic to $A^{\oplus I}$ for some set I: choose basis $m_i \in M$ $(i \in I)$: $\varphi_{\underline{m}}: A^{\oplus I} \xrightarrow{\sim} M$.

every two bases have the same # of el'ts.

Lemma: If $M \simeq A^{\oplus k}$, then k is uniquely recovered from M.

Proof: $\exists$ max. ideal $\mathfrak{m} \subset A$; $M/\mathfrak{m}M$ is a vector space over the field $A/\mathfrak{m}$: $M \simeq A^{\oplus k} \Rightarrow M/\mathfrak{m}M \simeq (A/\mathfrak{m})^{\oplus k}$ as A-modules & so as $A/\mathfrak{m}$-modules. So $k = \dim_{A/\mathfrak{m}}(M/\mathfrak{m}M)$. r.h.s. is indep't of basis. $\square$

2.3) Projective modules.

Definition: A module M is <u>projective</u> if $\exists$ A-module M' s.t. $M \oplus M'$ is free.

Example: free $\Rightarrow$ projective (take $M' = \{0\}$).

Questions: i) Are there (finitely generated) projective modules that are not free.

ii) if so, why should we care about them.

Partial A to i) Yes (see Problem 8 in HW1). For many rings, the answer is no, for example, for principal ideal domains. For $A = \mathbb{F}[x_1, .., x_n]$ ($\mathbb{F}$ is field) every fin. gen'd projective module is free (conj'd by Serre in 55, proved indep. by Quillen & Suslin in 76)

A to ii): they are nice & useful (will elaborate later in the course).

Why we care about modules: interesting special cases (e.g. for $A = \mathbb{F}[x]$) & ideals in A are modules & so are A-algebras. It's often useful to study them as modules.