Lecture 5.

1) Noetherian rings & modules
2) Hilbert's Basis theorem

References: [AM], Chapter 6, intro to Chapter 7; [E], Section 1.4.

1) A is commutative unital ring

Motivating questions:

Q1: A-module generalize vector spaces

? generalize fin dim vector spaces

Q2: Which rings are "reasonable"?

We'll give two related answers to these questions.

Preliminary answer to Q1: finitely generated modules

Issue: Often need to consider submodules. A submodule in a finitely generated module may fail to be fin gen'd

Example: If infinite set, \( A = \prod_{i \in I} F \)

Module \( M = A \) (gen'd by 1), \( N = (x_i)_{i \in I} \)

Claim: \( N \) is not finitely generated.

Recall every \( f \in A \) is polynomial in fin many variables. If \( f_1, \ldots, f_k \in N \), \( \sum_{i=1}^{\infty} g_i f_i \) (\( g_i \in A \)): deg 1 terms in \( \sum_{i=1}^{\infty} g_i f_i \in \text{Span}_{F} (\text{deg 1 terms of } f_1, \ldots, f_k) \) only involves fin many \( x_i \)'s. □

1.1) Main definitions. Let \( M \) be \( A \)-module

i) Say that \( M \) is Noetherian if it submodule of \( M \) (including \( M \)) is finitely generated.
ii) \( A \) is a Noetherian ring if it's Noetherian as a module over itself, i.e. every ideal is finitely generated.

Examples: 1) Every field \( \mathbb{F} \) is Noetherian ring (submodules in \( \mathbb{F} \) are \( \{0\}, \mathbb{F} = (1) \)).

1) \( A = \mathbb{Z} \) is Noetherian: \( 6/ \mathfrak{a} \) ideal is generated by 1 element.

Non-example: \( \mathbb{F}[x_i]_{i \in I} \) for infinite \( I \).

1.2) Basic properties of Noetherian modules.

\textbf{Want:} equivalent conditions of being Noetherian.

\textbf{Definition:} \( M \) is \( A \)-module. By an ascending chain (AC) of submodules we mean: collection \( (N_i)_{i \in \mathbb{Z}_{\geq 0}} \) of submodules of \( M \) s.t. \( N_i \subseteq N_{i+1} \) \( \forall i \geq 0 \); \( N_i \subseteq N_j \subseteq N_k \).

We say that the AC \( (N_i)_{i \in \mathbb{Z}_{\geq 0}} \) terminates: \( \exists k \geq 0 \) s.t. \( N_j = N_k \) \( \forall j > k \).

\textbf{Proposition 1:} For an \( A \)-module \( M \) TFAE:

1) \( M \) is Noetherian.

2) \( \forall \) AC of submodule in \( M \) terminates

3) \( \forall \) nonempty set of submodules of \( M \) has a maximal element w.r.t. inclusion.

\textbf{Proof:} (2) \( \iff \) (3) is easy & general statement about posets (exercise).

\textbf{Proof of (1) \( \iff \) (2)}

\( 1 \Rightarrow 2 \): AC \( (N_i)_{i \geq 0} : N_i \subseteq N_{i+1} \implies N_i = \bigcup_{i \geq 0} N_i \) is a
submodule (compare to proof of \( I \) of max. ideal, Lec 2). This \( N \) is fin gen'd: \( N = \text{Span}_A(m_1, \ldots, m_i) \Rightarrow \exists k \text{ s.t. } m_1, \ldots, m_i \in N_k \) 
\( \Rightarrow N = \text{Span}_A(m_1, \ldots, m_i) = N_k \) so our AC terminates at \( N_k \).

\( \Box \)

(2) \( \Rightarrow \) (1). Know AC of submodule terminates. Let \( N \) be a submodule that is NOT fin generated: construct 
\( N_i \)'s by induction: pick \( m_i \in N \Rightarrow N_i = \text{Span}_A(m_i) = A m_i \). Now suppose I've constructed \( m_1, \ldots, m_i \in N \) & \( N_i = \text{Span}_A(m_1, \ldots, m_i) \) 
\( N \) is not fin gen. \( \Rightarrow N + N_i \Rightarrow \exists m_{i+1} \in N \setminus N_i \), set 
\( N_{i+1} = \text{Span}_A(m_1, \ldots, m_{i+1}) \neq N_i \). So \( (N_i)_{i \geq 0} \) is AC, doesn't terminate. Contradiction

Want: compare properties of being Noetherian for \( M \) & its sub- & quotient modules.

Proposition 2: let \( M \) be \( A \)-module, \( N \subset M \) be a submodule
TFAE
(1) \( M \) is Noetherian
(2) Both \( N, M/N \) are Noetherian.

Proof: (1) \( \Rightarrow \) (2): \( M \) is Noetherian \( \Rightarrow N \) is Noeth.'n (tautology)
Check \( M/N \) is Noetherian by verifying that \( \forall \) AC of submods of \( M/N \) terminates. Let \( \varphi: M \to M/N, \ m \mapsto m + N \).

Let \( (N_i)_{i \geq 0} \) be an AC of submodules in \( M/N \), \( N_i = \varphi^i(N_i) \) 
\( (N_i)_{i \geq 0} \) form an AC of submodules of \( M \), it must terminate: 
\( \exists k \geq 0 \ | \ N_j = N_k \ \forall j \geq k \). But \( N_i = \varphi^i(N_i) \) so
\( N_j = \varphi^j(N_j) = \varphi^i(N_k) = N_k \). So \( (N_i)_{i \geq 0} \) terminates.
\[ N_i = \{ n \mid \varpi_i(n) \in N_i \} \subseteq N_i = \{ n \mid \varpi_i(n) \in N_i \} \quad 6/ \quad N_i \subseteq N_i \]

(2) \Rightarrow (1): Have \((N_i)_{i \geq 0}\) is an AC of submodules in \(\mathcal{M}\). Want to show it terminates. Then \((N_i \cap N)_{i \geq 0}\) is AC in \(N\) & \((\varpi_i(N_i))_{i \geq 0}\) is AC in \(N/N\). We know that both terminate:

\[ \exists K \geq 0 \text{ s.t. } N_i \cap N = N_k \cap N \& \varpi_i(N_i) = \varpi_i(N_k) \quad \forall i \geq k. \]

Want to check: \(N_j = N_k \) (show \((N_i)\) terminates):

\[ n \in N_j \Rightarrow \varpi_i(n) \in \varpi_i(N_j) = \varpi_i(N_k) \quad \exists n' \in N_k \mid \varpi_i(n') = \varpi_i(n) \]

\[ \Leftrightarrow \varpi_i(n-n') = 0 \Leftrightarrow n-n' \in N. \quad \text{But } n, n' \in \bigcap_{j \geq K} N_j: (K \ subseteq n' \in N_j) \Rightarrow \]

\[ n-n' \in N_j \Rightarrow n-n' \in \bigcap_{i \geq K} N_j = N \cap N_k \Rightarrow n = n' + (n-n') \in N_k \Rightarrow N_j = N_k \square \]

Noetherian \(\Rightarrow\) fin. gen'd. But, when \(A\) is Noetherian, we also have \(\subset\).

Corollary: Let \(A\) be Noetherian. Then \(\mathcal{M}\) fin. gen'd \(A\)-module is Noetherian.

Proof: \(M = \text{Span}_A (m_1, \ldots, m_k) \quad m = (m_1, \ldots, m_k) \rightarrow \psi_m : A^\oplus_k \rightarrow M \)

\[ \Leftrightarrow M = A^\oplus_k / \ker \psi_m. \quad B(y) (1) \Rightarrow (2) \text{ of Prop 2 enough to show } \]

\(A^\oplus_k\) is Noetherian. Proof: by induction on \(k\).

\(k = 1\) holds b/c \(A\) is Noetherian.

\(k > 1: \quad A^\oplus_{k-1} \subseteq A^\oplus_k\) as \(\{a_1, \ldots, a_{k-1}, 0\}\) submodule.

\(A^\oplus_k / A^\oplus_{k-1} \cong A\) via proj to \(k\)th coordinate.

Both \(A^\oplus_{k-1}\), \(A\) are Noetherian. We use \((2) \Rightarrow (2)\) of Prop 2 to conclude that \(A^\oplus_k\) is Noetherian \(A\)-module \(\square\)

2) Hilbert's basis thm : Most rings we are dealing with are Noetherian. The 1st result here is the following thm.
Theorem 1: If $A$ is Noetherian, then so is $A[x]$.  
Proof: see next lecture. Now: some corollaries.

Definition: Let $B$ be an $A$-algebra. Say $B$ is finitely generated (as an $A$-algebra) if $\exists b_1, \ldots, b_k \text{ s.t. } \forall b \in B \exists f \in A[x_1, \ldots, x_k] \text{ s.t. } b = f(b_1, \ldots, b_k)$

So $\Phi: A[x_1, \ldots, x_k] \to B$, $f \mapsto f(b_1, \ldots, b_k)$, is surjective.

Corollary: Let $A$ be Noetherian & $B$ be a finitely generated $A$-algebra. Then $B$ is a Noetherian ring.
Proof: Use Hilbert's Thm 9 times to see that $A[x_1, \ldots, x_k]$ is Noetherian. Let $I \subseteq B$ be ideal, need to show it's fin gen'd

$J := \Phi^{-1}(I) \subseteq A[x_1, \ldots, x_k]$ is ideal, so $J = (f_1, \ldots, f_k)$. But then

$I = \Phi(J) = (\Phi(f_1), \ldots, \Phi(f_k))$ is finitely generated \[ \square \]

Since fields, $F$ & $\mathbb{Z}$ are Noetherian rings, any finitely generated $F$- or $\mathbb{Z}$- algebra is also Noetherian.

BONUS: Non-Noetherian rings in real (rather, complex) math.

Life: Most of the rings we deal with in Commutative Algebra are Noetherian. Here is, however, a very natural example of a non-Noetherian ring that appears in Complex Analysis.

Complex analysis studies holomorphic (a.k.a. complex analytic or complex differentiable functions), let $\text{Hol}(\mathbb{C})$ denote the set of holomorphic functions on $\mathbb{C}$. These can be thought as power series that absolutely converge everywhere.
$\text{Hol}(C)$ has a natural ring structure via addition and multiplication of functions.

**Proposition:** $\text{Hol}(C)$ is not Noetherian.

**Proof:** We'll produce an AC of ideals: $I_j = \{ f(z) \in \text{Hol}(C) \mid f(2\pi j \cdot k) = 0 \}$ for integer $k$, $j \in \mathbb{Z}_o$. It's easy to check that all of these are indeed ideals. It is also clear that they form an AC (when we increase $j$, we relax the condition on zeroes). We claim that $I_j \nsupseteq I_{j+1}$, hence this AC doesn't terminate & $\text{Hol}(C)$ is not Noetherian. Equivalently, we need to show that, for each $j$, there $f(z) \in \text{Hol}(C)$ such that $f(2\pi j \cdot k) = 0 \neq k > j$ while $f_j(2\pi j \cdot i) \neq 0$.

Consider the function $f(z) = e^z - 1$. This function is periodic with period $2\pi j^{-1}$. Also $f(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!}$. So $z=0$ is an order 1 zero of $f(z)$. Since $2\pi j^{-1}$ is a period, every $2\pi j^{-1} \cdot k$ ($k \in \mathbb{Z}$) is an order 1 zero. We set $f_j(z) = (e^z - 1)/(z - 2\pi j \cdot i)$. This function is still holomorphic on the entire $C$, we have $f_j(2\pi j \cdot i) \neq 0$ and $f_j(2\pi j \cdot k) = 0$ for $k \neq j$. \[\square\]