Lecture 5.

1) Noetherian rings & modules
2) Hilbert's Basis theorem.

References: [AM], Chapter 6, intro to Chapter 7; [E], Section 1.4.

1) A is commive unital ring

Motivating questions:

Q1: A-modules generalize vector spaces

$$\cup$$

? generalize fin. dim. vector spaces

Q2: Which rings are "reasonable"?

We'll give two related answers to these questions.

Preliminary answer to Q1: finitely generated modules.

Issue: Often need to consider submodules. A submodule in a finitely generated module may fail to be fin. gen'd

Example: I infinite set, $A := \mathbb{F}[x_i]_{i\in I}$

Module $M := A$ (gen'd by 1), $N := (x_i)_{i\in I}$

Claim: N is not finitely generated.

Recall every $f\in A$ is polynomial in fin. many variables. If $f_1,\dots,f_k \in N \rightsquigarrow \sum_{i=1}^k g_i f_i$ $(g_i\in A)$: deg 1 terms in $\sum_{i=1}^k g_i f_i \in \mathrm{Span}_{\mathbb{F}}($deg 1 terms of $f_1,\dots,f_k)$ - only involves fin. many x_i's. $\square$

1.1) Main definitions.

Let M be A-module

i) Say that M is Noetherian if $\forall$ submodule of M (including M) is finitely generated.

ii) A is a <u>Notherian ring</u> if it's Noetherian as a module over itself, i.e. every ideal is finitely generated.

Examples: 0) Every field $\mathbb{F}$ is Noetherian ring (submodules in $\mathbb{F}$ are $\{0\}$, $\mathbb{F}=(1)$).

1) $A=\mathbb{Z}$ is Noetherian: b/c $\forall$ ideal is generated by 1 element.

Non-example: $\mathbb{F}[x_i]_{i\in I}$ for infinite I.

1.2) Basic properties of Noetherian modules.

Want: equivalent conditions of being Noetherian.

Definition: M is A-module. By an <u>ascending chain</u> (AC) of submodules we mean: collection $(N_i)_{i\in\mathbb{Z}_{>0}}$ of submodules of M s.t. $N_i\subseteq N_{i+1}$ $\forall i>0$: $N_1\subseteq N_2\subseteq N_3\subseteq\dots$

We say that the AC $(N_i)_{i\in\mathbb{Z}_{>0}}$ <u>terminates</u>: $\exists\, k>0$ s.t. $N_j=N_k$ $\forall j>k$.

Proposition 1: For an A-module M TFAE:

1) M is Noetherian.

2) $\forall$ AC of submodules in M terminates

3) $\forall$ nonempty set of submodules of M has a maximal element w.r.t. inclusion.

Proof: (2) $\Leftrightarrow$ (3) is easy & general statement about posets. (exercise).

Proof of (1) $\Leftrightarrow$ (2)

(1) $\Rightarrow$ (2): AC $(N_i)_{i>0}$: $N_1\subseteq N_2\subseteq\dots$ $\rightsquigarrow$ $N:=\bigcup\limits_{i>0}N_i$ is a

submodule (compare to proof of $\exists$ of max. ideal, Lec 2).

This N is fin. gen'd: $N = \mathrm{Span}_A(m_1, \dots m_i) \Rightarrow \boxed{\exists\ k \text{ s.t. } m_1, \dots m_i \in N_k}$

$\Rightarrow N = \mathrm{Span}_A(m_1, \dots m_i) = N_k$ so our AC terminates at N_k.

b/c. N_j's form an AC.

$(2) \Rightarrow (1)$. Know $\forall$ AC of submodules terminates. Let N be a submodule that is NOT fin generated: construct N_i's by induction: pick $m_1 \in N \rightsquigarrow N_1 = \mathrm{Span}_A(m_1) = Am_1$. Now suppose I've constructed $m_1, \dots m_i \in N$ & $N_i = \mathrm{Span}_A(m_1, \dots m_i)$

N is not fin.gen. $\Rightarrow N \neq N_i \Rightarrow \exists\ m_{i+1} \in N \setminus N_i$, set $N_{i+1} = \mathrm{Span}_A(m_1, \dots m_{i+1}) \supsetneq N_i$. So $(N_i)_{i>0}$ is AC, doesn't terminate. Contradiction. $\square$

Want: compare properties of being Noetherian for M & its sub- & quotient modules.

Proposition 2: Let M be A-module, $N \subset M$ be a submodule

TFAE (1) M is Noetherian

(2) Both N, M/N are Noetherian.

Proof: $(1) \Rightarrow (2)$: M is Noetherian $\Rightarrow N$ is Noeth'n (tautology)

Check M/N is Noetherian by verifying that $\forall$ AC of submod's of M/N terminates. Let $\pi: M \twoheadrightarrow M/N$, $m \mapsto m+N$.

Let $(\underline{N}_i)_{i>0}$ be an AC of submodules in M/N, $N_i = \pi^{-1}(\underline{N}_i)$

$(N_i)_{i>0}$ form an AC of submodules of M, it must terminate: $\exists\ k>0 \mid N_j = N_k\ \forall j>k$. But $\underline{N}_i = \pi(N_i)$ so

$\underline{N}_j = \pi(N_j) = \pi(N_k) = \underline{N}_k$. So $(\underline{N}_i)_{i>0}$ terminates.

$N_i = \{ n \mid \pi(n) \in \underline{N}_i \} \subset N_{i+1} = \{ n \mid \pi(n) \in \underline{N}_{i+1} \}$ b/c $\underline{N}_i \subset \underline{N}_{i+1}$.

(2) $\Rightarrow$ (1): Have $(N_i)_{i \geq 0}$ is an AC of submodules in M. Want to show it terminates. Then $(N_i \cap N)_{i\geq 0}$ is AC in N & $(\pi(N_i))_{i\geq 0}$ is AC in M/N. We know that both terminate: $\exists\, k \geq 0$ s.t. $N_j \cap N = N_k \cap N$ & $\pi(N_j) = \pi(N_k)$ $\forall j \geq k$.

Want to check: $N_j = N_k$ (show (N_i) terminates):

$n \in N_j \rightsquigarrow \pi(n) \in \pi(N_j) = \pi(N_k)$ so $\exists\, n' \in N_k \mid \pi(n') = \pi(n)$ $\Leftrightarrow \pi(n-n') = 0 \Leftrightarrow n - n' \in N$. But $n, n' \in N_j$ (b/c $n' \in N_k \subset N_j$) $\Rightarrow$ $n - n' \in N_j \Rightarrow n - n' \in N \cap N_j = N \cap N_k \Rightarrow n = n' + (n-n') \in N_k \Rightarrow N_j = N_k$ $\square$

Noetherian $\Rightarrow$ fin. gen'd. But, when A is Noetherian, we also have $\Leftarrow$.

Corollary: Let A be Noetherian. Then $\forall$ fin. gen'd A-module is Noetherian.

Proof: $M = \mathrm{Span}_A(m_1, \dots m_k)$: $\underline{m} = (m_1, \dots m_k) \rightsquigarrow \varphi_{\underline{m}} : A^{\oplus k} \twoheadrightarrow M$ $\Rightarrow M = A^{\oplus k} / \ker \varphi_{\underline{m}}$. By (1) $\Rightarrow$ (2) of Prop 2 enough to show $A^{\oplus k}$ is Noetherian. Proof: by induction on k.

$k = 1$ holds b/c A is Noetherian.

$k > 1$: $A^{\oplus k-1} \subset A^{\oplus k}$ as $\{(a_1, \dots a_{k-1}, 0)\}$ submodule.

by induction.

$A^{\oplus k} / A^{\oplus k-1} \simeq A$ via proj'n to kth coordinate

Both $A^{\oplus k-1}$, A are Noetherian. We use (2) $\Rightarrow$ (1) of Prop 2 to conclude that $A^{\oplus k}$ is Noetherian A-module $\square$

2) Hilbert's basis thm

: Most rings we are dealing with are Noetherian. The 1st result here is the following thm.

Theorem 1: If A is Noetherian, then so is $A[x]$.
Proof: see next lecture. Now: some corollaries.
Definition: Let B be an A-algebra. Say B is finitely generated (as an A-algebra) if $\exists\, b_1, \dots b_k$ s.t. $\forall b \in B\ \exists\, F \in A[x_1, \dots x_k]$ s.t. $b = F(b_1, \dots b_k)$

So $\varphi: A[x_1, \dots x_k] \longrightarrow B$, $F \mapsto F(b_1, \dots b_k)$, is surjective.

Corollary: Let A be Noetherian & B be a finitely generated A-algebra. Then B is a Noetherian ring.
Proof: Use Hilbert's Thm k times to see that $A[x_1, \dots x_k]$ is Noetherian. Let $I \subset B$ be ideal, need to show it's fin. gen'd
$J := \varphi^{-1}(I) \subset A[x_1, \dots x_k]$ is ideal so $J = (F_1, \dots F_\ell)$. But then $I = \varphi(J) = (\varphi(F_1), \dots \varphi(F_\ell))$ is finitely generated $\square$

Since fields, $\mathbb{F}$, & $\mathbb{Z}$ are Noetherian rings, any finitely generated $\mathbb{F}$- or $\mathbb{Z}$- algebra is also Noetherian.

BONUS: Non-Noetherian rings in real (rather, complex) math. life: Most of the rings we deal with in Commutative algebra are Noetherian. Here is, however, a very natural example of a non-Noetherian ring that appears in Complex analysis.

Complex analysis studies holomorphic (a.k.a. complex analytic or complex differentiable functions). Let $Hol(\mathbb{C})$ denote the set of holomorphic functions on $\mathbb{C}$. These can be thought as power series that absolutely converge everywhere.

$Hol(\mathbb{C})$ has a natural ring structure - via addition & multiplication of functions.

Proposition: $Hol(\mathbb{C})$ is not Noetherian

Proof: We'll produce an AC of ideals: $I_j = \{f(z) \in Hol(\mathbb{C}) \mid f(2\pi\sqrt{-1}k) = 0 \ \forall$ integer $k \geq j\}$, $j \in \mathbb{Z}_{\geq 0}$. It's easy to check that all of these are indeed ideals. It is also clear that they form an AC (when we increase j we relax the condition on zeroes). We claim that $I_j \subsetneq I_{j+1}$ hence this AC doesn't terminate & $Hol(\mathbb{C})$ is not Noetherian. Equivalently, we need to show that, for each j, there $f_j(z) \in Hol(\mathbb{C})$ such that $f_j(2\pi\sqrt{-1}k) = 0 \ \forall k > j$ while $f_j(2\pi\sqrt{-1}j) \neq 0$. Consider the function $f(z) = e^z - 1$. This function is periodic with period $2\pi\sqrt{-1}$. Also $f(z) = \sum_{i=1}^{\infty} \frac{1}{i!} z^i$. So $z=0$ is an order 1 zero of $f(z)$. Since $2\pi\sqrt{-1}$ is a period, every $2\pi\sqrt{-1}k$ $(k \in \mathbb{Z})$ is an order 1 zero. We set $f_j(z) = (e^z - 1)/(z - 2\pi\sqrt{-1}j)$. This function is still holomorphic on the entire $\mathbb{C}$, we have $f_j(2\pi\sqrt{-1}j) \neq 0$ & $f_j(2\pi\sqrt{-1}k) = 0$ for $k \neq j$. $\square$