Lecture 9.

1) Localization of rings, cont'd
2) Localization of modules.

Refs: [AM], Intro to Chapter 3.

Bonus: localization in noncommutative rings.

1) A is comm've unital ring, localizable subset $S\subset A$

$(0\notin S, 1\in S;\ s,t\in S \Rightarrow st\in S)$.

$A\times S/\sim =: A_S$

(*) $(a,s)\sim(b,t) \overset{def}{\iff} \exists u\in S \mid uta=usb$

The equiv. class of (a,s) is denoted by $\frac{a}{s}$.

A_S is a comm've unital ring w. usual addition & multipl'n of fractions.

1.1) Examples.

0) $S=\{$all invertible elements in $A\}$ is localizable; $A_S\simeq A$ b/c every equiv. class in $A\times S$ has unique repres've of the form $(a,1)$.

1) A is a domain, $S=A\setminus\{0\}$ is localizable. In A_S every nonzero element is invertible: $\left(\frac{a}{s}\right)^{-1}=\frac{s}{a}$. So A_S is a field known as the fraction field of A, denoted $\mathrm{Frac}(A)$.

E.g. $A=\mathbb{Z} \rightsquigarrow \mathrm{Frac}(A)=\mathbb{Q}$.

1') For general A, $S:=\{$all nonzero divisors $\in A\}$ is localizable

Exer: In A_S every non-zero divisor is invertible.

2) Let $f \in A \rightsquigarrow S := \{f^n \mid n \geq 0\}$ is localizable $\Leftrightarrow S \not\ni 0 \Leftrightarrow$ f is not nilpotent. Resulting localization is also denoted by A_f or $A[f^{-1}]$.

Concrete example: $A = \mathbb{F}[x]$, $f = x$. Then $\mathbb{F}[x]_x = \left\{ \frac{f(x)}{x^n} \right\}$ is the ring of Laurent polynomials $\mathbb{F}[x^{\pm 1}] = \left\{ \sum_{i=-m}^{n} a_i x^i \right\}$. (isomorphism of rings).

2') for $f_1, \ldots, f_k \in A \rightsquigarrow S := \{f_1^{n_1} \ldots f_k^{n_k} \mid n_i \geq 0\}$ is localizable $\Leftrightarrow S \not\ni 0$. An exercise below will show $A_S = A[(f_1 \ldots f_k)^{-1}]$.

3) Let $\mathfrak{p} \subset A$ be prime ideal $\Rightarrow S := A \setminus \mathfrak{p}$ is localizable:

$0 \notin S \Leftrightarrow 0 \in \mathfrak{p}$ $\checkmark$; $1 \in S \Leftrightarrow 1 \notin \mathfrak{p}$ $\checkmark$;

$s, t \in S \Leftrightarrow s, t \notin \mathfrak{p} \Rightarrow st \notin \mathfrak{p} \Leftrightarrow st \in S$.

We write $A_{\mathfrak{p}}$ for A_S.

1.2) Universal property of A_S

Recall ring homomorphism $\iota : A \to A_S$, $a \mapsto \frac{a}{1}$.

Proposition: Let B be an A-algebra, i.e. have ring homom'm $\varphi : A \to B$. Suppose $\varphi(s) \in B$ is invertible $\forall s \in S$.

1) $\exists!$ ring homom'm $\varphi' : A_S \to B$ that makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[d, "\iota"'] \arrow[dr, "\varphi"] & \\ A_S \arrow[r, dashed, "\varphi'"] & B \end{tikzcd}$$

2) φ' is given by $\varphi'\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1}$.

Sketch of proof:

Existence: need to show that formula in 2) indeed gives a well-defined ring homomorphism.

Well-defined: need to check $\frac{a}{s}=\frac{b}{t} \Rightarrow \varphi(a)\varphi(s)^{-1}=\varphi(b)\varphi(t)^{-1}$.

Indeed: $\frac{a}{s}=\frac{b}{t} \iff \exists\, u\in S$ s.t $uta = usb \xRightarrow{\text{apply } \varphi} \varphi(u)\varphi(t)\varphi(a) = \varphi(u)\varphi(s)\varphi(b)$

But $\varphi(u), \varphi(t), \varphi(s)$ are invertible $\Downarrow$

$$\varphi(a)\varphi(s)^{-1}=\varphi(b)\varphi(t)^{-1} \quad \checkmark$$

Ring homomorphism: is exercise (on addition & multiplication of fractions). Done w. existence.

Uniqueness: φ' makes diagram comm've $\iff \varphi'(\frac{a}{1})=\varphi(a)\ \forall a\in A$
$\Rightarrow \varphi'(\frac{s}{1})=\varphi(s)$ - invertible $\Rightarrow \varphi'(\frac{1}{s})=\varphi(s)^{-1} \Rightarrow$
$\varphi'(\frac{a}{s}) = \varphi'(\frac{a}{1})\varphi'(\frac{1}{s}) = \varphi(a)\varphi(s)^{-1}$. $\square$

Exercise: let A be is a domain, $S\subset A$ be localizable. Then $A_S \simeq \{\frac{a}{s}\in Frac(A) \mid s\in S\}$, a ring isomorphism

Exercise: for $S=\{f_1^{n_1}\cdots f_k^{n_k}\}$ (Example 2') show $A_S\simeq A_{f_1\cdots f_k}$

Corollary: A is arbitrary, $f\in A$ non-nilpotent. Then $A_f \simeq A[t]/(tf-1)$.

Proof: · homomorphism $A_f \longrightarrow A[t]/(tf-1)$

$$\begin{tikzcd} A \arrow[d, "\iota"'] \arrow[dr, "\varphi"] & \\ A_f \arrow[r, dashed, "\varphi'"] & A[t]/(tf-1) \end{tikzcd}$$

$\varphi(a):=a+(tf-1)$, $\varphi(f)=f+(tf-1)$ - invertible w. inverse $t+(tf-1)$
$\Rightarrow \varphi(f^n)$ is invertible $\forall n \rightsquigarrow \varphi'$ by univ. property of A_f

• homomorphism $A[t]/(tf-1) \longrightarrow A_f$

$\tilde{\varphi}'': A[t] \to A_f$, $g \in A[t] \mapsto g(\frac{1}{f}) \in A_f$

$\tilde{\varphi}''(tf-1) = 0$, so $\tilde{\varphi}''$ factors through $A[t]/(tf-1)$

$$\begin{tikzcd} A[t] \arrow[d, "\pi"'] \arrow[dr, "\tilde{\varphi}''"] & \\ A[t]/(tf-1) \arrow[r, dashed, "\varphi''"'] & A_f \end{tikzcd}$$

• check $\varphi'' \circ \varphi' = id$, $\varphi' \circ \varphi'' = id$; enough to do this on generators of our rings. E.g. A_f is generated by $\frac{a}{1}$ $(a \in A)$, $\frac{1}{f}$

$$\varphi'' \circ \varphi'(\tfrac{1}{f}) = \varphi''(t + (tf-1)) = \tfrac{1}{f}.$$

The rest is an exercise. □

Example of computation: $A = \mathbb{C}[x,y]/(xy)$, $f = x$ (zero divisor)

$$[\mathbb{C}[x,y]/(xy)]_x = [\text{corollary}] = (\mathbb{C}[x,y]/(xy))[t]/(tx-1)$$

$$= \mathbb{C}[x,y,t]/(xy, tx-1) = [\,t(xy) = y(tx-1) + y$$

in my ideal ⇒ so does y

$$\text{so } (xy, tx-1) = (y, tx-1)\,] = \mathbb{C}[x,y,t]/(y, tx-1)$$

$$\simeq \mathbb{C}[x,t]/(tx-1) = [\text{corollary}] \simeq \mathbb{C}[x]_x = \mathbb{C}[x^{\pm 1}]$$

– Laurent polynomials.

2) Localization of modules.

2.1) Definition: A, S as before. Let M be an A-module. Define it's <u>localization</u> M_S as the set of equivalence classes $M \times S/\sim$ w. $\sim$ defined by:

(*) $(m,s)\sim(n,t) \overset{def}{\Longleftrightarrow} \exists\, u\in S \mid utm = usn$

Equiv. class of (m,s) will be denoted by $\frac{m}{s}$.

Proposition: M_S has a natural A_S-module str're (w. addition of fractions) & $A_S \times M_S \to M_S$ given by $\frac{a}{s}\frac{m}{t} := \frac{am}{st}$.

Proof: the same price as ring str're on A_S.

Remark: M_S is A_S-module, ring homom'm $\iota: A\to A_S$ $(a\mapsto \frac{a}{1})$ $\rightsquigarrow$ an A-module str're on M_S: $a\frac{m}{s} = \frac{am}{s}$.

The map $M \xrightarrow{\iota} M_S$, $m\mapsto \frac{m}{1}$, is A-module homom'm. ($\iota: A\to A_S$ is a special case)

Important exercise: For $\iota: M\to M_S$, $\ker \iota = \{m\in M \mid \exists\, s\in S$ s.t. $sm=0\}$. -use (*).

2.2) From homomorphism $\varphi: M\to N$, get $\varphi_S: M_S \to N_S$

Stuff added/modified for Oct 1 is in dark blue.

Observation: M is an A-module, $M'\subset M$ A-submodule. Then M'_S is naturally an A_S-submodule in M_S: $\frac{m'}{s}\mapsto \frac{m'}{s}$: $M'_S \to M_S$

is well-def'd & injective (equiv. rel'n of $M'\times S$ is the restric'n of equiv. rel'n on $M\times S$) & easily seen to be A_S-linear.

Now let $\varphi\in \mathrm{Hom}_A(M,N)$.

Proposition: i) φ_S given by $\varphi_S(\frac{m}{s}):=\frac{\varphi(m)}{s}$ is a well-defined map $M_S\to N_S$ & is A_S-linear.

ii) $\ker(\varphi_S)=(\ker\varphi)_S$ as A_S-submodules

iii) $\mathrm{im}(\varphi_S)=(\mathrm{im}\,\varphi)_S$ — · — · — · — · — · —

Proof: i) - exercise.

ii) First, we check $\ker(\varphi_S)\subset(\ker\varphi)_S$.

$\ker(\varphi_S)=\{\frac{m}{s}\in M_S \mid \varphi_S(\frac{m}{s})=0 \Leftrightarrow$ [def'n of φ_S]

$\frac{\varphi(m)}{s}=0 \Leftrightarrow$ [(*)] $\exists\, u\in S \mid u\varphi(m)=0$

$\Leftrightarrow um\in\ker\varphi\}\subseteq [\frac{um}{us}=\frac{m}{s}]\subseteq(\ker\varphi)_S$.

Now $(\ker\varphi)_S=\{\frac{m}{s}\mid \varphi(m)=0\}\subset\ker(\varphi_S)$, finishing (ii).

(iii) $\mathrm{im}(\varphi_S)=\{\varphi_S(\frac{m}{s})=\frac{\varphi(m)}{s}\}=(\mathrm{im}\,\varphi)_S$. $\square$

Example: • $M=A^{\oplus k}$. Claim: M_S is identified $(A_S)^{\oplus k}$.

$(A^{\oplus k})_S\longrightarrow(A_S)^{\oplus k}$, $\frac{(a_1,\dots,a_k)}{s}\mapsto(\frac{a_1}{s},\dots,\frac{a_k}{s})$

$(A_S)^{\oplus k}\longrightarrow(A^{\oplus k})^S$, $(\frac{a_1}{s_1},\dots,\frac{a_k}{s_k})\mapsto\left(\frac{\prod_{j\neq i}s_j\,a_i}{\prod_j s_j}\right)_{i=1}^k$ ← common denominator

These two maps are mutually inverse.

• $\varphi: A^{\oplus k}\to A^{\oplus \ell}\longleftrightarrow$ matrix $\Psi=(a_{ij})_{i=1,\,j=1}^{\ell\quad k}$ (so that if elements of our modules are columns, φ = multiplication by φ). Then $\varphi_S : A_S^{\oplus k} \to A_S^{\oplus \ell}$ is given by matrix $\left(\frac{a_{ij}}{1}\right)_{i=1, j=1}^{\ell \quad k}$.

Corollary of Prop'n: Let M be A-module, $M' \subset M$ be an A-submodule. Then $(M/M')_S \xrightarrow{\sim} M_S/M'_S$ (nat'l).

Proof: Apply (ii) & (iii) of Prop'n to $\varphi : M \twoheadrightarrow M/M'$, $m \mapsto m+M'$. Then $\operatorname{im}(\varphi_S) = (\operatorname{im}\varphi)_S = (M/M')_S$

$\ker(\varphi_S) = (\ker\varphi)_S = M'_S \rightsquigarrow M_S/M'_S \xrightarrow{\sim} (M/M')_S$. $\square$

Very important exercise:

- $id_S : M_S \to M_S$ is the identity map.
- for $\varphi : M \to N$, $\varphi' : N \to P$, then $(\varphi'\varphi)_S = \varphi'_S \varphi_S$.

BONUS: Localization in noncommutative rings.

When we define the ring structure on A_S it's important that the elements of S commute w. all elements of A. Otherwise, assume for simplicity that all elements of S are invertible. We are trying to multiply right fractions as^{-1} and bt^{-1} and get a right fraction. We get $as^{-1}bt^{-1}$ – and we are stuck...

How to do localization in noncommutative rings was discovered by Ore (who was a faculty at Yale 1927-1968)

Let S be a subset of a (noncommutative) ring A such that $0 \notin S$, $1 \in S$; $s,t \in S \Rightarrow st \in S$ as before. There are so called Ore conditions that guarantee that there is a localization A_S consisting of right, equivalently, of left fractions. Namely if S doesn't contain zero divisors we need to require:

(O1) $\forall\, a\in A, s\in S\ \exists\ b\in A, t\in S$ s.t. $ta=bs$ (think, $as^{-1}=t^{-1}b$).

+ its mirror analog (left $\leftrightarrow$ right)

When S contains zero divisors we also should require:

(O2) if $sa=0$ for $a\in A, s\in S$, then $\exists\ t\in S$ w. $at=0$

– and its mirror condition.

In fact, (O2) allows to reduce to the case when there are no zero divisors in S: $J:=\{a\in A \mid \exists\ s\in S \text{ s.t. } sa=0\}$ is a two-sided ideal thx to (O2)+its mirror, so we replace A w. A/J, and S with its image in A/J. So we can just assume there are no zero divisors in S & (O1) and its mirror.

The we can define the set A_S of equivalence classes in A_S: $(a,s)\sim(a',s')$ (think $as^{-1}=a's'^{-1}$): we find b,t w $ta=bs$ (think $as^{-1}=t^{-1}b$) and declare $(a,s)\sim(a',s')$ if $ta'=bs'$.

Here we already see that everything becomes more painful: even to see that this doesn't depend on the choice of b,t requires a check. And there's more of this. Eventually, one gets the localization A_S consisting of right fractions (equivalently left) fractions) w. natural ring structure. It has a universal property similar to what we have in the commutative case.

Checking the Ore conditions is hard. And they are not always satisfied. For example, they aren't satisfied when

$A = \mathbb{F}\langle x, y\rangle$ is a free $\mathbb{F}$-algebra & $S = A \setminus \{0\}$.

Still, they are satisfied in a number of examples. Namely, recall that A is <u>prime</u> if for any two-sided ideals I, J we have $IJ = \{0\} \Rightarrow I \neq \{0\}$ or $J \neq \{0\}$. We say A is <u>Noetherian</u> if all left & right ideals are finitely generated.

Theorem (Goldie) Let A be a prime Noetherian ring. Then the set S of all non-zero divisors in A satisfies the Ore conditions. The localization A_S is of the form $\mathrm{Mat}_n(D)$, where $n>0$ & D is a skew-field (a.k.a. divison ring).

In particular, A has no zero divisors $\Leftrightarrow n=1$.