Lecture 13

1) Categories.
2) Functors.

Ref: [R], Sections 1.1, 1.3

BONUS: Homotopy category of topological space.

1) Definitions below will have a familiar structure: have data & axioms. For example, here's a basic algebraic structure.

Definition: a **monoid** is

(Data): a set M equipped w. a multipl'n map $M\times M\to M$

(axioms): that is associative and has unit, 1.

For example, a group is exactly a monoid, where all elements are invertible. Every ring is a monoid w.r.t. multiplication.

1.1) Definition of a category.

Definition: A **category**, $\mathcal{C}$, consists of

(Data):
- a "collection" of **objects**, $Ob(\mathcal{C})$.
- $\forall X, Y\in Ob(\mathcal{C}) \rightsquigarrow$ a <u>set</u> of **morphisms**, $Hom_{\mathcal{C}}(X,Y)$
- $\forall X,Y,Z\in Ob(\mathcal{C})$, a map (of sets) called **composition**

$$Hom_{\mathcal{C}}(X,Y)\times Hom_{\mathcal{C}}(Y,Z)\longrightarrow Hom_{\mathcal{C}}(X,Z),\ (f,g)\mapsto g\circ f$$

($\circ$ is often omitted)

These satisfy:

(Axioms): i) composition is associative:
$(f\circ g)\circ h = f\circ(g\circ h)$ for $f\in Hom_{\mathcal{C}}(W,X)$, $g\in Hom_{\mathcal{C}}(X,Y)$, $h\in Hom_{\mathcal{C}}(Y,Z)$.
ii) Units: $\forall X\in Ob(\mathcal{C})\ \exists\ 1_X\in Hom_{\mathcal{C}}(X,X)$ s.t.
- $f\circ 1_X = f\ \forall f\in Hom_{\mathcal{C}}(X,Y)$,
- $1_X\circ g = g\ \forall g\in Hom_{\mathcal{C}}(Z,X)$.

1.2) Examples

1) Category of sets, Sets: objects = sets, morphisms = maps of sets, composition = composition of maps. Axioms: classical (unit $1_X = id_X$).

2) Sets w. additional str're: objects = sets w. addi'l str're, morphisms = maps compatible w. this str're, composition = = comp'n of maps. This includes

a) Category of groups, Groups: objects are groups, morphisms = homomorphisms of groups.

b) Category of rings, Rings.

c) For a ring A, have categories of A-modules, A-Mod, & A-algebras (A-Alg), in the latter morphisms = A-linear homomorphisms of rings.

Not all categories have the form in 2:

3a) Let Γ be an oriented graph w. vertices V & edges E.
$\rightsquigarrow$ category $\mathcal{C}(\Gamma)$, the pass category of Γ.

- Objects $= V$.
- Morphisms $=$ paths in the graph:

$$X \bullet \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_k} Y$$

this includes empty paths, one for every vertex.

- Composition: concatenation of paths.

Axioms: associativity is manifest, $1_X =$ empty path in X.

3b) Note: $\forall X \in Ob(\mathcal{C}) \Rightarrow Hom_{\mathcal{C}}(X,X)$ is a monoid w.r.t. $\circ$
Conversely, every monoid, M, gives a category w. one object, X, & $(Hom_{\mathcal{C}}(X,X), \circ) := M$.

1.3) Remarks:

1) Sometimes, objects in a category form a set (here we say our category is small). In general, they form a "class."
2) 1_X is uniquely determined. Moreover, if $f \in Hom_{\mathcal{C}}(X, Y)$, has a (2-sided) inverse g (i.e. $g \in Hom_{\mathcal{C}}(Y,X) \mid f \circ g = 1_Y, g \circ f = 1_X$) then g is unique, $f^{-1} := g$. In this case, f is called an isomorphism; we say X & Y are isomorphic (X & Y behave the same from the point of view of $\mathcal{C}$, e.g. $Z \in Ob(\mathcal{C}) \rightsquigarrow$

$$\begin{array}{ccc} Hom_{\mathcal{C}}(Z,X) & \xrightarrow{\sim} & Hom_{\mathcal{C}}(Z,Y) \\ \psi & \longmapsto & f \circ \psi \end{array}$$

(inverse is $\psi' \mapsto f^{-1} \circ \psi'$).

Notation: $X \xrightarrow{f} Y$ means $f \in Hom_{\mathcal{C}}(X,Y)$.

1.4) Subcategories: $\mathcal{C}$ is a category.

Def'n: (i) By a subcategory, $\mathcal{C}'$, in $\mathcal{C}$ we mean:

(Data) • A subcollection, $Ob(\mathcal{C}')$, in $Ob(\mathcal{C})$.

• $\forall\, X,Y \in Ob(\mathcal{C}')$, a subset $Hom_{\mathcal{C}'}(X,Y) \subset Hom_{\mathcal{C}}(X,Y)$ s.t.

(Axioms) • If $f \in Hom_{\mathcal{C}'}(X,Y)$, $g \in Hom_{\mathcal{C}'}(Y,Z) \Rightarrow g\circ f \in Hom_{\mathcal{C}'}(X,Z)$

• $1_X \in Hom_{\mathcal{C}'}(X,X)$ $\forall\, X \in Ob(\mathcal{C}')$.

(ii) A subcategory $\mathcal{C}'$ in $\mathcal{C}$ is called full if $Hom_{\mathcal{C}'}(X,Y) = Hom_{\mathcal{C}}(X,Y)$, $\forall\, X,Y \in Ob(\mathcal{C}')$.

A subcategory $\mathcal{C}'$ has a natural category str're.

Examples:

1) A monoid M = category w. one object
A nonempty subcategory M' in M = a submonoid.
M' is full $\Leftrightarrow$ $M' = M$.

2) $\mathbb{Z}$-Mod (a.k.a. category of abelian groups) is a full subcategory in Groups

3) The category of commut've rings, CommRings is a full subcategory in Rings.

1.3) Constructions w. categories.

Definition: For a category, $\mathcal{C}$, its opposite category, $\mathcal{C}^{opp}$ consists of

- the same objects as $\mathcal{C}$,
- $Hom_{\mathcal{C}^{opp}}(X,Y) := Hom_{\mathcal{C}}(Y,X)$
- $g \circ^{opp} f := f \circ g$ $\quad (f \in Hom_{\mathcal{C}^{opp}}(X,Y) = Hom_{\mathcal{C}}(Y,X),$
  $g \in Hom_{\mathcal{C}^{opp}}(Y,Z) = Hom_{\mathcal{C}}(Z,Y))$.

Definition: For categories $\mathcal{C}_1, \mathcal{C}_2$, their product $\mathcal{C}_1 \times \mathcal{C}_2$ is defined by: • $Ob(\mathcal{C}_1 \times \mathcal{C}_2) = Ob(\mathcal{C}_1) \times Ob(\mathcal{C}_2)$

- $Hom_{\mathcal{C}_1 \times \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) = Hom_{\mathcal{C}_1}(X_1, Y_1) \times Hom_{\mathcal{C}_2}(X_2, Y_2)$
- composition is componentwise.

Rem: for usual categories we care about (Groups, Rings, A-Mod), the opposite cat'y essentially has no independent meaning, except: $\mathcal{C} = CommRings$, where $\mathcal{C}^{opp}$ is the category of affine schemes, which is of crucial importance for Algebraic geometry.

2) Functors: Motto: functor relative to categories is what a group homomorphism is relative to groups.

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition: A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is

(Data) • an assignment $X \mapsto \mathcal{F}(X): Ob(\mathcal{C}) \to Ob(\mathcal{D})$.

- $\forall X, Y \in Ob(\mathcal{C})$, a map $Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$

$$\begin{array}{ccc} f & \longmapsto & \mathcal{F}(f) \end{array}$$

(Axioms) – compatibility between compositions & units

- $\forall f \in Hom_{\mathcal{C}}(X,Y), g \in Hom_{\mathcal{C}}(Y,Z) \Rightarrow \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$

equality in $Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Z))$.

- $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ $\forall$ $X \in Ob(\mathcal{C})$

Example: Let $\mathcal{C}, \mathcal{D}$ be categories w. single object corresponding to monoids M, N. Then a functor $\mathcal{C} \to \mathcal{D}$ is the same thing as a monoid homomorphism.

Remarks: · Have the identity functor $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

· For functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{E}$ can take the composition $\mathcal{G} \circ \mathcal{F} : \mathcal{C} \to \mathcal{E}$ ($\mathcal{G} \circ \mathcal{F}(X) = \mathcal{G}(\mathcal{F}(X))$).

· A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is the same thing as a functor $\mathcal{C}^{opp} \to \mathcal{D}^{opp}$.

More examples:

1) Let $\mathcal{C}'$ be a subcategory in $\mathcal{C}$. Then have inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ taking objects/morphisms in $\mathcal{C}'$ & sending them to the same objects/morphisms now in $\mathcal{C}$; axioms are clear.

2) Forgetful functors: forget part of a structure

2a) For: Groups $\longrightarrow$ Sets;

On objects: For $(G) = G$ viewed as a set.

On morphisms: For $(f) = f$, viewed as a map of sets.

Axioms: clear.

BONUS: homotopy category of topological spaces.

B1) Equivalence on morphisms.

Let $\mathcal{C}$ be a category. Suppose that $\forall X, Y \in Ob(\mathcal{C})$, the set $Hom_{\mathcal{C}}(X,Y)$ is endowed with an equivalence relation $\sim$ s.t.

(1) If $g, g' \in Hom_{\mathcal{C}}(Y,Z)$ are equivalent & $f \in Hom_{\mathcal{C}}(X,Y)$, then $g \circ f \sim g' \circ f$.

(2) If $f, f' \in Hom_{\mathcal{C}}(X,Y)$ are equivalent and $g \in Hom_{\mathcal{C}}(Y,Z)$, then $g \circ f \sim g \circ f'$.

We write $[f]$ for the equivalence class of f.

Given such an equivalence relation, we can form a new category to be denoted by $\mathcal{C}/\sim$ as follows:

- $Ob(\mathcal{C}/\sim) := Ob(\mathcal{C})$
- $Hom_{\mathcal{C}/\sim}(X,Y) := Hom_{\mathcal{C}}(X,Y)/\sim$ - the set of equivalence classes
- $[g] \circ [f] = [g \circ f]$ - well-defined precisely b/c of (1) & (2)

We note that there is a natural functor $\pi : \mathcal{C} \to \mathcal{C}/\sim$ given by $X \mapsto X$, $f \mapsto [f]$.

Example: Let M be a monoid. Note that the equivalence class of $1 \in M$ is a submonoid, say M_0, moreover, (1) & (2) imply that $mM_0 = M_0 m$ $\forall m \in M$. Such submonoids are called normal (for groups we recover the usual condition). And if $M_0 = [1]$ is normal, then (1) and (2) hold - an exercise. For a normal submonoid M_0 we can M/M_0 with a natural monoid structure - just as we do for groups. The category $\mathcal{C}/\sim$ corresponds to the quotient monoid M/M_0 and the functor π is just the natural epimorphism $M \twoheadrightarrow M/M_0$.

Rem*: $\mathcal{C}/\sim$ looks like a quotient category. But in situations where the term "quotient" is used and that are closer to quotients of abelian groups (Serre quotients of abelian categories) the construction is different - and more difficult.

B2) Homotopy category of topological spaces.

Let's recall the usual category of topological spaces. Let X be a set. One can define the notion of topology on X: we declare some subsets of X to be "open", these are supposed to satisfy certain axioms. A set w. topology is called a topological space. A map $f: X \to Y$ of topological spaces is called continuous if $U \subset Y$ is open $\Rightarrow f^{-1}(U) \subset X$ is open.

We define the category Top of topological spaces w.

$Ob(Top)$ = topological spaces.

$Hom_{Top}(X,Y)$:= continuous maps $X \to Y$

Composition = composition of maps.

One issue: this category is hard to understand - hard to study topological spaces up to homeomorphisms.

Now we introduce our equivalence relation of $Hom_{Top}(X,Y)$

Definition: Continuous maps $f_0, f_1: X \to Y$ are called homotopic if $\exists$ a continuous map $F: X \times [0,1] \to Y$ s.t. $f_0(x) = F(x,0)$ & $f_1(x) = F(x,1)$.

Informally, f_0, f_1 are homotopic if one can continuously deform f_0 to f_1. It turns out that being homotopic is an equivalence relation satisfying (1) & (2) from B1. The corresponding category $Top/\sim$ is known as the <u>homotopy category of topol'l spaces.</u> Note that in this category morphisms are <u>not</u> maps!

Here is why we care about the homotopy category. Isomorphic here means homotopic (X is homotopic to Y if $\exists\ X \xrightarrow{f} Y$, $Y \xrightarrow{g} X$ s.t. fg is homotopic to 1_Y & gf is homotopic to 1_X) and this is easier to understand then being homeomorphic. Second, the classical invariants such as homology and homotopy groups only depend on homotopy type. A more educated way to state this: these invariants are functors from the homotopy category of topological spaces to Groups (true as stated for homology, for homotopy it's more subtle, this requires fixing a point in X and hence need to work w. an auxiliary category of "pointed" topological spaces - up to homotopy).