Lecture 24

1) Algebra homomorphisms vs polynomial maps
2) Geometric significance of localization

Refs: [E], Section 1.6, Intro to Section 2.

BONUS: 1) What is an algebraic variety?
2) Projective varieties.

1) Algebra homomorphisms vs polynomial maps.

Let $F$ be algebraically closed field. Let $X$ be an algebraic subset of $F^n$ (the set of solutions to a system of polynomial equations). Recall that to $X$ we assign the ideal $I(X) = \{ f \in \mathbb{F}[x_1, ..., x_n] | f|_X = 0 \}$ & the algebra $\mathbb{F}[X] = \mathbb{F}[x_1, ..., x_n]/I(X)$ whose elements can be interpreted as polynomial functions on $X$, and the natural projection $\mathbb{F}[x_1, ..., x_n] \rightarrow \mathbb{F}[X]$ is the restriction to $X$. Set $\bar{x}_i = x_i|_X$, note that $\bar{x}_1, ..., \bar{x}_n$ generate the algebra $\mathbb{F}[X]$.

1.1) Polynomial maps.

Definition: Let $X \subseteq \mathbb{F}^n$, $Y \subseteq \mathbb{F}^m$ be algebraic subsets. A map $\varphi: X \rightarrow Y$ is called polynomial if $\exists f_1, ..., f_m \in \mathbb{F}[X]$ s.t.

$\varphi(x) = (f_1(x), ..., f_m(x)) \quad \forall x \in X$ (in particular, $(f_1(x), ..., f_m(x)) \in Y$).

Rem: polynomial map $X \rightarrow \mathbb{F} = $ polynomial function on $X$. 

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Exercise: the composition of polynomial maps is polynomial.

In particular, let \( q: X \to Y \) be a polynomial map & \( g \in \mathbb{F}[y] \), i.e. \( g: Y \to \mathbb{F} \). Consider the composed polynomial map \( g \circ q: X \to \mathbb{F} \). When viewed as an element of \( \mathbb{F}[X] \), \( g \circ q \) will be denoted by \( q^*(g) \) and called the pullback from \( g \) (under \( q \)).

**Lemma:** 1) \( q^*: \mathbb{F}[y] \to \mathbb{F}[x] \) is algebra homomorphism. 2) On the generators \( \bar{y}_j := y_j/y \), \( q^*(\bar{y}_j) = f_j \)

**Proof:**

1) is exercise, compare to Prob 1 in HW4.

2) \( \bar{y}_j := y_j \) \( \Rightarrow \) \( q^*(\bar{y}_j) = \bar{f}_j \)

**Examples:** 1) Inclusion map \( i: X \hookrightarrow \mathbb{F}^n \) is polynomial, \( i^*: \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}[x] \) is the restriction map. More generally, if \( X \subseteq Y \subseteq \mathbb{F}^n \) are algebraic subsets, then the inclusion map \( i: X \hookrightarrow Y \) is polynomial & \( i^*: \mathbb{F}[Y] \to \mathbb{F}[X] \) is \( g \mapsto g|_X \).

2) \( X = \mathbb{F}, Y = \{y_1^2, y_2^3\} \) \( \subseteq \mathbb{F}^2 \), \( q: X \to Y, x \mapsto (x^3, x^2) \) is a polynomial map. Since \( y_1^2 - y_2^3 \) is irreducible, the ideal \( (y_1^2, y_2^3) \) is prime, hence radical, so \( \mathbb{F}[Y] = \mathbb{F}[y_1, y_2]/(y_1^2, y_2^3) \). By 2) of Lemma, \( q^*(\bar{y}_1) = x^3, q^*(\bar{y}_2) = x^2 \) this determines \( q^* \) uniquely because \( y_1, y_2 \) generate the algebra \( \mathbb{F}[Y] \).
1.2) **Main result.**

**Theorem:** \( \varphi \mapsto \varphi^* \) defines a bijection between:

1. Polynomial maps \( \varphi : X \to Y \)
2. Algebra homomorphisms \( \mathbb{F}[Y] \to \mathbb{F}[X] \)

**Proof:** Given \( \varphi = (f_1, \ldots, f_m) \), \( \varphi^* \) is the unique algebra homomorphism \( \mathbb{F}[Y] \to \mathbb{F}[X] \) s.t. \( \varphi^*(y_i) = f_i \). We'll use this observation to construct the inverse map \((\text{II}) \to (\text{I})\).

Given an algebra homomorphism \( \tau : \mathbb{F}[Y] \to \mathbb{F}[X] \) define \( \psi : X \to \mathbb{F}^m \) by \( \psi = (\tau(y_1), \ldots, \tau(y_m)) \). We claim \( \text{im} \psi \subset Y \) (so that \( \psi \) can be viewed as a polynomial map \( X \to Y \)). We have \( \text{im} \psi \subset Y \iff G(\text{im} \psi) = 0 \forall C \in I(Y) \iff \)

\[ G(\tau(y_1), \ldots, \tau(y_m)) = 0 \]  

(\*)

Note that \( G(y_1, \ldots, y_m) = 0 \) b/c \( G \in I(Y) \) & \( \tau \) preserves polynomial relations b/c it's an algebra homomorphism. (\*) follows.

So we have maps \( \varphi \mapsto \varphi^* : (\text{I}) \leftrightarrow (\text{II}) : \tau \mapsto \psi \). We have \( \psi^*(y_i) = [\text{ith coordinate in } \tau] = \tau(y_i) \implies \psi^* = \tau \) b/c \( y_i \)'s generate. On the other hand, \( \psi^*(\varphi^*(y_1), \ldots, \varphi^*(y_m)) = (\varphi(\tau(y_1)), \ldots, \varphi(\tau(y_m)) = \varphi \). So these maps are mutually inverse, finishing the proof. \( \square \)

1.3) **Affine varieties:**

Similarly to Prob 1 in HW4, we see that:

- For algebraic subsets \( X \subset \mathbb{F}^n, Y \subset \mathbb{F}^m \subset \mathbb{F}^k \) & polynomial maps \( \varphi : X \to Y, \psi : Y \to Z \), we have \( (\psi \circ \varphi)^* = \varphi^* \circ \psi^* \);

- \( (\text{id}_X)^* = \text{id}_{\mathbb{F}[X]} \).
Consider categories:

1) \( E : \text{Ob}(E) = \{ \text{finitely generated } \mathbb{F}\text{-algebras w/o nonzero nilpotents} \}
\quad \text{Morphisms: homomorphisms of algebras.}

2) \( \widetilde{E} : \text{Ob}(\widetilde{E}) = \{ A \in \text{Ob}(E) \text{ w finite collection of algebra generators} \}
\quad \text{Morphisms: same as in } E.

3) \( D : \text{Ob}(D) = \{ \text{algebraic subsets in some } \mathbb{F}^n \}
\quad \text{Morphisms: polynomial maps.}

We have functors:

\[ \begin{align*}
\cdot & : D \to \widetilde{E}^{\text{opp}} : X \mapsto \mathbb{F}[X] \text{ w generators given by coordinate functions, } \bar{x}_i, x_i \mapsto y_i^* \\
\cdot & : \widetilde{E}^{\text{opp}} \to D : A = \mathbb{F}[x_1, \ldots, x_n]/I \text{ (identification coming from generators)} \mapsto V(I), \bar{x} \mapsto y_c.
\end{align*} \]

Crucially, important exercise: these two functors are inverse to each other.

Definition: The category of affine varieties (over \( \mathbb{F} \)) is \( D = E^{\text{opp}} \).

The objects in \( D \) can be thought of as algebraic subsets "independent" of embedding into \( \mathbb{F}^n \). Morphisms are still polynomial maps.

Example: Let \( X_1 = \mathbb{F} \) \& \( X_2 = V(x_2-x^2) \subset \mathbb{F}^2 \). They have isomorphic algebras of functions: polynomials in one variable but different embeddings into \( \mathbb{F}^n \)'s. From the point of view of algebraic geometry they behave in the same way so can be viewed as the same variety.
While this definition of an affine variety looks like cheating, we can talk about:

- Algebra of polynomial functions \( \mathbb{F}[X] \) of an affine variety \( X \) (\( X \) viewed as an object of \( \mathcal{E} \))
- Points of \( X \): algebra homomorphisms \( \mathbb{F}[X] \to \mathbb{F} \)

And so on.

2) Geometric significance of localization

2.1) Localizing one element.

Let \( X \subseteq \mathbb{F}^n \) be an algebraic subset & \( f \in \mathbb{F}[X] \). We want to find a geometric interpretation of the localization \( \mathbb{F}[X][f^{-1}] \).

Let \( f_1, \ldots, f_m \) be generators of \( I(X) \). Then Lemma in Sec 1.1 of Lecture 11 tells us that

\[
\mathbb{F}[X][f^{-1}] = \mathbb{F}[X][t]/(tf-1) \approx \mathbb{F}[x_1, \ldots, x_n, t]/(f_1, \ldots, f_m, tf-1).
\]

The corresponding algebraic subset of \( \mathbb{F}^n \) is

\[
\{(x_1, \ldots, x_n, t) \in \mathbb{F}^n \mid f_i(x_1, \ldots, x_n) = 0 \quad \forall i = 1, \ldots, m; \quad tf_1(x_1, \ldots, x_n) = 1\}.
\]

The projection \( \mathbb{F}^n \to \mathbb{F}^n \) forgetting the \( t \) coordinate identifies this algebraic subset with \( \{x \in X \mid f(x) \neq 0\} \). Denote this subset by \( X_p \). We note that it's not an algebraic subset of \( \mathbb{F}^n \) in our terminology. This subset of \( X \) is called a principal open subset.

Here's an explanation of the terminology.

Definition: a subset \( Y \subseteq X \) is called Zariski closed if it's an algebraic subset of \( \mathbb{F}^n \).
A subset $U \subseteq X$ is Zariski open if $X \setminus U$ is Zariski closed.

**Example:** $X_p \subseteq X$ is Zariski open.

**Exercise:** Any Zariski open subset of $X$ is the union of, in fact, finitely many, principal open subsets.

Zariski open/closed subsets are open/closed subsets in a topology (called Zariski topology). Principal open subsets form a "base of topology."

### 2.2) Localization at the complement of a maximal ideal

Let $X \subseteq \mathbb{F}$ be algebraic subset, $A := \mathbb{F}[X], \mathfrak{m} \subseteq A$ a maximal ideal.

Recall, Sec 2 of Lec 12, that for any prime ideal $\mathfrak{p} \subseteq A$, subset $S := A \setminus \mathfrak{p} \subseteq A$ is multiplicative. We write $A_{\mathfrak{p}} := A[S^{-1}]$. Take $\mathfrak{p} := \mathfrak{m}$.

Note that $A_{\mathfrak{p}}$ is not finitely generated (in general) so is not the algebra of functions of an algebraic subset. It still has a geometric meaning that we are going to discuss now.

For simplicity, assume $X$ is irreducible $\iff A = \mathbb{F}[X]$ is domain.

Fraction field $\text{Frac}(A) = \left\{ \frac{f}{g} \mid g \not\equiv 0 \right\}$, every localization of $A$ is contained in $\text{Frac}(A)$ as a subring, Corollary in Sec 12 of Lec 11.

By Corollary in Sec 14 of Lec 22, the maximal ideals of $A$ are in bijection with $X$: $\mathfrak{m} \leftrightarrow \mathfrak{p}$ with $\mathfrak{p} = \{ f \in A \mid f(x) = 0 \}$. Then

$$A_{\mathfrak{m}} = \left\{ \frac{f}{g} \mid g(x) \neq 0 \right\} = U A_{\mathfrak{p}} = U \mathbb{F}[X_g]$$

(recall that $X_g$
is an algebraic subset (in \(\mathbb{F}^n\)) so it makes sense to speak about its algebra of functions.

**Conclusion:**

Every element of \(A_m\) is a function on a Zariski open subset containing \(d\), but which subset we choose depends on this element.

**Remark:** When \(X\) is reducible, conclusion still holds but

\[ A_m = \bigcup_g \mathbb{F}[X_g] \]

makes no sense (i.e. \(\mathbb{F}[X_g] = A_g\) are, a priori, not subrings in a fixed ring (in general). To fix this one replaces the union with the "direct limit."

**Remark (on terminology):** Recall (Sec 2 of Lec 12) that a commutative ring \(B\) is called \textbf{local} if it has unique maximal ideal.

For example, \(A_y\) is local (w. maximal ideal \(p_x\)), in particular \(A_m\) is local. The discussion above gives a geometric justification to this terminology; this algebra controls what happens locally (in Zariski topology) near \(d \in X\).

**BONUS:** 1) What is an algebraic variety?

We've discussed affine (algebraic) varieties. Now we are going to address the question in the title.

A common approach to constructing geometric objects is to...
"glue" them from simpler objects. For example, $C^\infty$ manifolds are glued from balls in Euclidean spaces: $M = \bigcup D_\alpha$, where $D_\alpha = \{ x \in \mathbb{R}^n \mid \| x \| < 1 \}$ for all $\alpha \in \mathbb{N}$. The condition is, roughly, that for all $a, b$ in the index set, the images of $D_\alpha \cap D_b$ under $\varphi_a, \varphi_b$ are open subsets in $\{ x \in \mathbb{R}^n \mid \| x \| < 1 \}$ and the resulting composition $\varphi_b \circ \varphi_a^{-1} : \varphi_a(D_\alpha \cap D_b) \xrightarrow{\cong} D_\alpha \cap D_b \xrightarrow{\varphi_b} \varphi_b(D_\alpha \cap D_b)$ is $C^\infty$ (which makes sense b/c this is a map between open subsets in $\mathbb{R}^n$). Thanks to this definition it makes to speak about various $C^\infty$ objects, e.g. $C^\infty$ maps $M \to N$.

Similarly, it makes sense to speak about complex analytic manifolds: we use balls in $\mathbb{C}^n$ and require that $\varphi_b \circ \varphi_a^{-1}$ is complex analytic (you might have studied that for $n=1$ — in which case the resulting objects appear when you study analytic continuation of holomorphic functions).

Something like that happens for algebraic varieties. The building blocks are affine algebraic varieties and they are glued together using polynomial isomorphisms: if the variety of interest is reasonable ("separated" in a suitable sense) the intersection of two open affine subvarieties is again affine so we can just use what we have in this lecture.

We can define the notion of a polynomial map (a.k.a. morphism): $\varphi : X \to Y$ is a morphism if we can cover $X = \bigcup U_i$, $Y = \bigcup V_j$ w. open affine varieties s.t. $\forall i \exists j \mid \varphi(U_i) \subset V_j$ & $\varphi : U_i \to V_j$ is a polynomial map of affine varieties.
B2) Projective varieties and graded algebras.

Here comes the most important example of the construction sketched above.

We start with $\mathbb{F}^n$ (viewed as a vector space). The projective space $\mathbb{P}^n(=\mathbb{P}(\mathbb{F}^n))$ as a set consists of 1-dimensional subspaces in $\mathbb{F}^n$. In other words, it consists of equivalence classes $[x:...:x] \in \mathbb{F}^n \setminus \{0\}$, where equivalent means proportional. Let us explain how gluing works.

Let $U_i = \{[x:...:x] | x_i \neq 0 \}, i = 0,...,n$. Then the map

$$U_i \ni [x:...:x] \mapsto (\frac{x_0}{x_i}, \frac{x_1}{x_i}, ..., \frac{x_n}{x_i})$$

is a bijection that will be used to identify $U_i$ with $\mathbb{F}^n$. Note that $g_i(U_i, \mathbb{F}^n)$ is given by non-vanishing of a single coordinate, so is an affine variety. And one can show that

$$q_j \circ q_i^{-1}: q_i(U_i, \mathbb{F}^n) \rightarrow q_j(U_j, \mathbb{F}^n)$$

is a polynomial isomorphism.

Example: let $n=2$. Let $y_0 = \frac{x_0}{x_1}$ & $y_1 = \frac{x_0}{x_1}$ be coordinates on $q_0(U_0) \cong \mathbb{F}$ & $q_1(U_1) \cong \mathbb{F}$. Then $q_i(U_i, \mathbb{F}^n)$ is given by $y_i \neq 0$ & $q_0 \circ q_i^{-1}$ sends $y_0$ to $y_1^{-1}$, which is a polynomial isomorphism as we have inverted $y_0$.

So $\mathbb{P}^n$ is an algebraic variety in the sense of the 1st part.

One can generalize this construction. Let $F_1,...,F_k \in \mathbb{F}[x_0,...,x_n]$ be homogeneous polynomials of degree $>0$. If $F_i$ vanishes at a nonzero point in $\mathbb{F}^n$, then it also vanishes on the line between this point & 0. So it makes sense to speak about
the zero locus of $F_i$ in $\mathbb{P}^n$ (note that $F_i$ is NOT a function $\mathbb{P}^n \rightarrow \mathbb{C}$). This gives rise to the zero locus $V(F_1, \ldots, F_k)$ and hence to the notion of an algebraic subset of $\mathbb{P}^n$.

Exercise: $V(F_1, \ldots, F_k) \cap U_i$ is an algebraic subset in $U_i \rightarrow \mathbb{C}^n$.

So $V(F_1, \ldots, F_k)$ is an algebraic variety, varieties of that kind are called projective.

Here's a reason why we care about them. Let $\mathbb{F} = \mathbb{C}$. So $\mathbb{C}^n$ has the usual topology. And so does $\mathbb{P}^n$ with $U_i$'s being open subsets.

Important exercise: $\mathbb{P}^n$ is compact in the usual topology.

And so every $V(F_1, \ldots, F_k)$ is compact. In Geometry & Topology, we like compact spaces more than noncompact as they behave better in many ways. And while not all compact (in the usual topology) algebraic varieties are projective, the projective ones are nicest.

Now we discuss a connection between projective varieties & graded algebras. The vanishing locus of $V(F_1, \ldots, F_k)$ depends only on $(F_1, \ldots, F_k)$, a homogeneous ideal.

Exercise: If $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous ideal, then so is its radical.

In fact, $V(F_1, \ldots, F_k)$ only depends on $\sqrt{(F_1, \ldots, F_k)}$, similarly to the affine case. This gives rise to a bijection between

- Algebraic subsets of $\mathbb{P}^n$
- And radical homogeneous ideals in $\mathbb{C}[x_0, \ldots, x_n]$ not containing 1.
Exercise: What ideal corresponds to $P$?

So starting from an algebraic subset in $\mathbb{P}^n$ we get a finitely generated reduced graded algebra, the quotient of $\mathbb{F}[x_0, \ldots , x_n]$ by the corresponding ideal. Note that the elements of this algebra are not functions on the initial algebraic subset of $\mathbb{P}^n$.

Conversely, let $A = \bigoplus A_i$ be a finitely graded reduced $\mathbb{F}$-algebra w. $A_0 = \mathbb{F}$. From this algebra we can construct a projective variety. Namely, if $A$ is generated by $A_+$ ($\iff A$ is a graded quotient of $\mathbb{F}[x_0, \ldots , x_n]$), then we consider the algebraic subset of $\mathbb{P}^n$ defined by the kernel of $\mathbb{F}[x_0, \ldots , x_n] \to A$, which is a homogeneous ideal.

In general—if $A$ isn’t generated by $A_+$—we have the following:

Exercise: $\exists \ d > 0 \ s.t. \ A(d) = \bigoplus_{i=0}^{\infty} A_i$ is generated by $A_d$.

A fun fact: the projective variety we get is independent of the choice of $d$ up to an isomorphism.

Example: Take $A = \mathbb{F}[x_0, x_1]$ (w. usual grading). It gives rise to the projective line $\mathbb{P}^1$. Now consider $A_{(x)}$. It’s generated by $y_i = x^2_0$, $y_i = x_0x_i$, $y_i = x_i^2$. All relations between these elements are generated by $y_0y_i - y_i^2$. The corresponding algebraic subset is $\{[y_0 : y_1 : y_2] | y_0y_2 = y_1^2\}$. Denote it by $X$.

We are going to construct two mutually inverse polynomial maps between $\mathbb{P}^1$ & $X$. Let $\phi : \mathbb{P}^1 \to X$ be given by $[x_0 : x_1] \mapsto [x_0^2 : x_0x_1 : x_1^2]$. Now we define $\psi : X \to \mathbb{P}^1$: ...
\[ \psi([y_0:y_1:y_2]) = \begin{cases} [y_0:y_1], \text{ if } y_2 \neq 0 \\ [y_1:y_2], \text{ if } y_0 \neq 0 \end{cases} \]

**Exercise:** Check if \( y \) are well-defined & mutually inverse maps. Furthermore, check that \( y \) are morphisms (in the sense explained in the end of B1).

A connection with projective varieties is one of the reasons to care about graded algebras.