Lecture 9 (minor update 10/2)

1) Integral extensions of rings.

2) Integral closure.

Ref: [AM], Section 5.1.

1) Integral extensions of rings.

As usual, $A$ is a commutative ring. The notions of finitely generated and finite $A$-algebras were recalled in Sec 2 of Lec 8.

1.1) Definition & examples.

Definition: Let $B$ be a commutative $A$-algebra.

- $b \in B$ is integral over $A$ if $\exists$ monic (i.e. leading coeff = 1) $f \in A[x]$ s.t. $f(b) = 0$.
- $B$ is integral over $A$ if $\forall b \in B$ is integral (over $A$).

Exercise: If $B$ is integral over $A$ & $C$ is a quotient of $B$, then $C$ is integral over $A$.

Example: 1) Let $f(x) \in A[x]$ be a monic polynomial. Then $\overline{1} := x + (f) \in B := A[x]/(f)$ is integral. Also note that $B$ is finite over $A$ (generated by $1, \overline{x}, \overline{x}^d \ldots$ for $d := \deg f$). Below we'll see that $B$ is integral over $A$.

2) Let $A \subset K \subset B \subset \mathbb{C}$ be a field extension. Then $B$ is integral over $A$ $\iff$ $B$ is algebraic over $A$ (we can divide by the leading coefficient).
Rem: In this example the two meanings of being finite are equivalent. But being finitely generated as an algebra is much stronger than being finitely generated as a field (for the latter we allow taking inverses among operations we use, for the former we only use multiplication & addition/subtraction. We'll later see (under some restrictions although this result holds in general) that if \( L \) is a finitely generated \( K \)-algebra, then \( \dim_k L < \infty \).

1.2) Finite vs integral.

Reminder: for field extensions: finite \( \iff \) [algebraic & finitely generated (as a field extension)].

Thm: Let \( B \) be an \( A \)-algebra. TFAE

(a) \( B \) is integral and finitely generated over \( A \).
(b) \( B \) is finite over \( A \).

The proof of \( (a) \implies (b) \) is based on the following lemma. Note that if \( A_2 \) is an \( A \)-algebra & \( A_2 \) is an \( A_1 \)-algebra, then \( A_2 \) is also an \( A \)-algebra: the homomorphism \( A \to A_2 \) is the composition \( A \to A_1 \to A_2 \).

Lemma 1: Suppose \( A_1 \) is finite over \( A \) & \( A_2 \) is finite over \( A_1 \). Then \( A_2 \) is finite over \( A \).

Proof: Have \( a_1, \ldots, a_k \in A_1 \) & \( b_1, \ldots, b_l \in A_2 \) s.t. \( A_1 = \operatorname{Span}_A (a_1, \ldots, a_k) \), \( A_2 = \operatorname{Span}_{A_1} (b_1, \ldots, b_l) \).

Exercise: \( A_2 = \operatorname{Span}_A (b_i a_j \mid i = 1, \ldots, l, j = 1, \ldots, k) \)
Proof of (a) ⇒ (b) of Thm: B is generated by finitely many elements \( b_1, \ldots, b_k \in B \). Each of them is integral over \( A \). Let \( B_i, i = 1 \ldots k \), be the \( A \)-subalgebra of \( B \) generated by \( b_1, \ldots, b_i \), i.e. \( B_i = \text{Span}_A(b_1, \ldots, b_i, d_i \in \mathbb{N}_{\geq 0}) \). By the construction, we have \( B_k = B \) and \( B_i \subset B_{i+1} \). Set \( B_0 : = A \). Since all \( b_i \) are integral over \( A \), we also have that \( b_i \) is integral over \( B_{i-1} \). By the construction, \( b_i \) generates \( B_i \) as an algebra over \( B_{i-1} \). We claim that \( B_i \) is finite over \( B_{i-1} \). This claim together with Lemma 1 yields (a) ⇒ (b). So let’s prove the claim.

Let \( f_i \in B_{i-1}[x] \) be a monic polynomial s.t. \( f_i(b_i) = 0 \). Then the unique \( B_{i-1} \)-algebra homomorphism \( B_{i-1}[x] \to B_i \) \( x \mapsto b_i \) factors as \( B_{i-1}[x] \to B_{i-1}[x]/(f_i) \to B_i \). But \( b_i \) generates \( B_i \) over \( B_{i-1} \). So \( B_{i-1}[x]/(f_i) \to B_i \). The source is a finite over \( B_{i-1} \), by Example 1. So is the target.

To prove (b) ⇒ (a) we will need the following lemma: a module version of the Cayley-Hamilton theorem from linear algebra. We will prove a more general form that will be used later.

**Lemma 2**: Let \( M \) be an finitely generated \( A \)-module, \( I \subseteq A \) an ideal, \( q: M \to M \) \( A \)-linear map s.t. \( q(M) \subseteq IM \). Then there is a polynomial \( f(x) \in A[x] \) of the form

\[
(*) \quad f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \quad \text{with} \quad a_k \in I^k \quad \forall k
\]

s.t. \( f(q) = 0 \).
Proof: Note that $M$ becomes an $A[x]$-module w. x acting by $q$. Pick generators $m_1,...,m_n \in M$. We have elements $a_{ij} \in I$, $i=1,...,n$ s.t.

\[(1) \quad x m_i = \sum_{j=1}^{n} a_{ij} m_j\]

Form the matrix $X = x I - (a_{ij})$. Then $\det(X) \in A[x]$. Note that $\det(X)$ is a polynomial for as in condition (*) (use the initial definition of $\det$, left as exercise.)

Also note that $\det(X)$ acts by $f(q)$ on $M$. So it’s enough to show that $\det(X)$ acts by 0.

Let $\vec{m} = (m_1,...,m_n)$ viewed as a column vector. Then $X \vec{m} = 0$ by (1). Consider the "adjoint" matrix $X' = (x'_{ij})$ w. $x'_{ij} = (-1)^{i+j} \det (\text{the matrix obtained from } X \text{ by removing row } i \text{ & column } j)$ so that $X'X = \det(X) I$. Then $X \vec{m} = 0 \Rightarrow \det(X) \vec{m} = X'X \vec{m} = \vec{0}$. So

\[(2) \quad \det(X) m_i = 0 \neq i.

Since $m_1,...,m_n$ span the $A$- (and hence $A[x]$-) module $M$, 
\[(2) \Rightarrow f(q)m = \det(X)m = 0 \quad \forall \; m \in M. \text{ This finishes the proof} \]

Proof of (6) $\Rightarrow$ (a): Let $B$ be a finite $A$-algebra. It’s finitely generated as module generators are also algebra generators. We need to show that $\forall \; b \in B$ is integral over $A$.

In Lemme 2 we take $M := B, I = A$ and $q : M \rightarrow M, \; \eta \rightarrow 6m$.

We conclude: $\exists$ monic polynomial $f(x) \in A[x] \; s.t. \; f(q) = 0$.

But applying $f(q)$ to $1 \in B$ we get $f(q)1 = f(6) = 0$. So $6$ is integral over $A.$
Exercise: Under the assumptions of Thm, if $A$ is Noetherian, then $B$ is Noetherian.

1.3) Consequences of Thm.

Corollary 1: i) If $f(x) \in A[x]$ is monic, then $A[x]/(f(x))$ is integral over $A$.
   ii) If $g \in B$ is integral over $A$, then $A[x]$, the $A$-subalgebra of $B$ generated by $g$, is integral over $A$.

Proof: Using Example 1 & (6) $\Rightarrow$ (a) of Thm we get (i). In (ii) if $f(x) \in A[x]$ is a monic polynomial w. f.g. $f(\alpha) = 0$, then $A[x]/(f(x)) \rightarrow A[\alpha]$. Since $A[x]/(f(x))$ is integral (by (i)), $A[\alpha]$ is also integral. □

Corollary 2 (transitivity of integral extensions): If $B$ is an $A$-algebra integral over $A$, and $C$ is a $B$-algebra integral over $B$, then $C$ is integral over $A$ (as an $A$-algebra).

Proof: Take $\gamma \in C$, it's integral over $B$ $\rightarrow$ $\exists b_0,.., b_{k-1} \in B$ s.t. $\gamma^k - b_{k-1} \gamma^{k-1} - .. - b_0 = 0$. We write $A[b_0,.., b_{k-1}]$ for the subalgebra of $B$ generated by $b_0,.., b_{k-1}$. So $\gamma$ is integral over $A[b_0,.., b_{k-1}] \subset B$. But $b_0,.., b_{k-1}$ are integral over $A$. We use (a) $\Rightarrow$ (6) of Thm to show that $A[b_0,.., b_{k-1}]$ is finite over $A$, while $A[b_0,.., b_{k-1}, \gamma]$, the subalgebra of $C$ generated by $b_0,.., \gamma$, is finite over $A[b_0,.., b_{k-1}]$. Using Lemma 1, we see that $A[b_0,.., b_{k-1}, \gamma]$ is finite over $A$. By (6) $\Rightarrow$ (a) of Thm, $\gamma$ is integral over $A$ and we are done. □
2) Integral closure.

**Proposition 1:** Let $B$ be an $A$-algebra. If $d, \beta \in B$ are integral over $A$, then so are $d+\beta, d\beta, ad$ ($a \in A$).


**Corollary/Definition:** The elements in $B$ integral over $A$ form an $A$-subalgebra of $A$ called the integral closure of $A$ in $B$. We'll denote the integral closure $\overline{A}$.

Note that this is a direct generalization of algebraic closures of fields.

**Example:** If $A = K \subset B = L$ are fields, then $\overline{K}$ is the algebraic closure of $K$ in $L$.

**Proposition 2:** The integral closure of $\overline{A}$ in $B$ is $\overline{A}$.

**Proof:** apply Corollary 2, left as exercise.