Lecture 10: Localization, II.

1) Localization of modules, cont'd.

2) Local rings.

Ref: [AM], Sections 3, 3.1

1.0) Reminder

Let $S\subset A$ be a multiplicative subset in a commutative ring, so that we can form the localization $A[S^{-1}]$. Let M be an A-module. We form the $A[S^{-1}]$-module

$$M[S^{-1}]=\{\tfrac{m}{s} \mid m\in M, s\in S\},$$

where $\frac{m}{s}$ is the equivalence class for the relation $\sim$ on $M\times S$: $(m,s)\sim(n,t) \Leftrightarrow \exists\, u\in S \mid utm=usn$.

It comes w. an A-linear map $\iota_M: M\to M[S^{-1}]$, $m\mapsto \frac{m}{1}$.

The pair $(M[S^{-1}],\iota_M)$ has the following universal property: for an $A[S^{-1}]$-module N & A-linear map $\xi: M\to N$ $\exists!$ $A[S^{-1}]$-linear $\tilde{\xi}: M[S^{-1}]\to N$ w. $\xi=\tilde{\xi}\circ\iota_M$, it's given by $\tilde{\xi}(\frac{m}{s})=\frac{1}{s}\xi(m)$.

In particular, to $\varphi\in \mathrm{Hom}_A(M_1,M_2)$ we can assign $\varphi[S^{-1}]\in \mathrm{Hom}_{A[S^{-1}]}(M_1[S^{-1}],M_2[S^{-1}])$ w. $\varphi[S^{-1}](\frac{m}{s})=\frac{\varphi(m)}{s}$.

See Sec 2 in Lec 9 for details.

1.1) Localization of submodules.

In the next 3 sections we study the interaction of localization and some operations w. (sub)modules.

Let M be an A-module, $M'\subset M$ A-submodule. Note that for

$m,n\in M'$, $s,t\in S$ we have $(m,s)\sim(n,t)$ in $M'\times S \Leftrightarrow (m,s)\sim(n,t)$ in $M\times S$. So $M'[S^{-1}]$ can be viewed as a subset in $M[S^{-1}]$, in fact, it's an $A[S^{-1}]$-submodule (exercise).

Note that the localization of the regular A-module A is the regular $A[S^{-1}]$-module $A[S^{-1}]$. So, for an ideal $I\subset A$, get an ideal $I[S^{-1}]\subset A[S^{-1}]$.

Exercise: Show that for submodules $M_1, M_2\subset M$ we have $(M_1+M_2)[S^{-1}] = M_1[S^{-1}]+M_2[S^{-1}]$ (hint: common denom'r), and similarly for intersections. Also $M_1\subset M_2 \Rightarrow M_1[S^{-1}]\subset M_2[S^{-1}]$.

1.2) Localizations vs direct sum.

Let I be a set and M_i, $i\in I$, be A-modules so that we can form the direct sum $\bigoplus_{i\in I} M_i$.

Lemma: There's a natural isomorphism $\bigoplus_{i\in I}(M_i[S^{-1}]) \xrightarrow{\sim} \left(\bigoplus_{i\in I} M_i\right)[S^{-1}]$.

Proof:

Set $M=\bigoplus_{i\in I} M_i$. Consider the map $\xi: M\to \bigoplus_{i\in I}(M_i[S^{-1}])$, $(m_i)\mapsto \left(\frac{m_i}{1}\right)$, it's A-linear. By the universal property, it lifts to the $A[S^{-1}]$-linear map $\tilde{\xi}: M[S^{-1}]\to \bigoplus_{i\in I}(M_i[S^{-1}])$, $\frac{(m_i)}{s}\mapsto\left(\frac{m_i}{s}\right)$.

• $\tilde{\xi}$ is injective: $\tilde{\xi}\left(\frac{(m_i)}{s}\right)=0 \Leftrightarrow \frac{m_i}{s}=0\ \forall i$. Let $I_0=\{i \mid m_i\neq 0\}$. This is a finite subset of I. For $i\in I_0$, $\frac{m_i}{s}=0 \Leftrightarrow \exists\, u_i\in S \mid u_i m_i=0$. Take $u=\prod_{i\in I_0} u_i$ so that $um_i=0\ \forall i\in I \Rightarrow \frac{(m_i)}{s}=0$

• $\tilde{\xi}$ is surjective: take $\left(\frac{m_i}{s_i}\right) \in \bigoplus_{i\in I}(M_i[S^{-1}])$, need: $\left(\frac{m_i}{s_i}\right) \in \operatorname{im}\tilde{\xi}$. Let $I_1 := \{i \in I \mid \frac{m_i}{s_i} \neq 0\}$ – finite set. Set $s := \prod_{i\in I_0} s_i$, $\tilde{m}_i = \left(\prod_{j\in I_0\setminus\{i\}} s_j\right) m_i$ so that $\frac{m_i}{s_i} = \frac{\tilde{m}_i}{s}$ $\forall\, i \in I_1$. Set $\tilde{m}_i := 0$ for $i \notin \tilde{I}_1$. Then $\frac{(\tilde{m}_i)}{s} \mapsto \left(\frac{m_i}{s_i}\right)$, showing the surjectivity. $\square$

Example: $M = A^{\oplus I}$. So $M[S^{-1}] \simeq A[S^{-1}]^{\oplus I}$ – the localization of a free module is free.

Exercise: Let's give an example of $\varphi[S^{-1}]$. The linear maps $\varphi: A^{\oplus k} \to A^{\oplus \ell}$ are given by matrices $\Psi \in Mat_{\ell\times k}(A)$: if we view elements of $A^{\oplus k}$, $A^{\oplus \ell}$ as column vectors, then $\varphi(v) = \Psi v$. Show that $\varphi[S^{-1}]: A[S^{-1}]^{\oplus k} \to A[S^{-1}]^{\oplus \ell}$ is given by the matrix $\left(\frac{a_{ij}}{1}\right)$.

1.3) Localization vs kernels and images.

Our next task is to relate $\ker\varphi[S^{-1}]$, $\operatorname{im}\varphi[S^{-1}]$ to $\ker\varphi$, $\operatorname{im}\varphi$.

Proposition: Let M, N be A-modules & $\varphi \in \operatorname{Hom}_A(M,N)$

i) $\ker(\varphi[S^{-1}]) = (\ker\varphi)[S^{-1}]$

ii) $\operatorname{im}(\varphi[S^{-1}]) = (\operatorname{im}\varphi)[S^{-1}]$

Proof: i) First, we check $\ker(\varphi[S^{-1}]) \subset (\ker\varphi)[S^{-1}]$

$\ker(\varphi[S^{-1}]) = \{\frac{m}{s} \in M[S^{-1}] \mid \varphi[S^{-1}](\frac{m}{s}) = 0 \Leftrightarrow$ [def'n of $\varphi[S^{-1}]$] $\frac{\varphi(m)}{s} = 0 \Longleftrightarrow \exists\, u \in S \mid u\varphi(m) = 0 \Longleftrightarrow um \in \ker\varphi\} \subseteq$ $\left[\frac{um}{us} = \frac{m}{s}\right] \subseteq (\ker\varphi)[S^{-1}]$. Now $(\ker\varphi)[S^{-1}] = \{\frac{m}{s} \mid \varphi(m) = 0\}$ $\subset \ker(\varphi[S^{-1}])$, finishing (i).

(ii) $\operatorname{im}(\varphi[S^{-1}]) = \{\varphi[S^{-1}](\frac{m}{s}) = \frac{\varphi(m)}{s}\} = (\operatorname{im}\varphi)[S^{-1}]$. $\square$

Corollary: Let M be A-module, $M' \subset M$ be an A-submodule. Then there's a natural $A[S^{-1}]$-module isomorphism $(M/M')[S^{-1}] \xrightarrow{\sim} M[S^{-1}]/M'[S^{-1}]$.

Proof: Apply Proposition to $\psi: M \twoheadrightarrow M/M'$, $m \mapsto m+M'$.
Then $\operatorname{im}(\psi[S^{-1}]) = (\operatorname{im}\psi)[S^{-1}] = (M/M')[S^{-1}]$
$\ker(\psi[S^{-1}]) = (\ker\psi)[S^{-1}] = M'[S^{-1}]$, $M[S^{-1}]/M'[S^{-1}] \xrightarrow{\sim} (M/M')[S^{-1}]$ $\square$

1.4) Submodules in $M[S^{-1}]$.

Let M be an A-module. A-submodule $N \subset M$ gives $N[S^{-1}] \subset M[S^{-1}]$, an $A[S^{-1}]$-submodule. On the other hand, for an $A[S^{-1}]$-submodule $N' \subset M[S^{-1}]$, consider $\iota_M^{-1}(N') \subset M$, this is an A-submodule b/c ι_M is A-linear & N' is A-submodule of $M[S^{-1}]$ (details are exercise).

Proposition: The maps $N \mapsto N[S^{-1}]$ & $N' \mapsto \iota^{-1}(N')$ are mutually inverse bijections between:

$\{A[S^{-1}]\text{-submodules } N' \subset M[S^{-1}]\}$ &

$\{A\text{-submodules } N \subset M \mid \underbrace{sm \in N \text{ for } s \in S, m \in M \Rightarrow m \in N}_{(\dagger)}\}$

Proof: Step 1: Show that $\iota^{-1}(N')$ satisfies $(\dagger)$:
$sm \in \iota_M^{-1}(N') \Leftrightarrow \iota_M(sm) \in N' \Leftrightarrow \frac{s}{1}\iota_M(m) \in N' \Leftrightarrow [\frac{s}{1}$ is invertible in $A[S^{-1}]] \Leftrightarrow \iota_M(m) \in N' \Leftrightarrow m \in \iota_M^{-1}(N')$.

So we have two maps between the two sets, need to show that they are mutually inverse.

Step 2: $\iota_M^{-1}(N[S^{-1}])=N$ for $\forall$ A-submodule N satisfying $(\dagger)$:
$\iota_M^{-1}(N[S^{-1}])=\{m\in M \mid \iota_M(m)\in N[S^{-1}] \Leftrightarrow \frac{m}{1}=\frac{n}{s}$ for some $n\in N, s\in S$ $\Leftrightarrow \exists\, u\in S \mid usm=un\in N \Leftrightarrow [(\dagger)]\; m\in N\}=N$.

Step 3: $(\iota_M^{-1}(N'))[S^{-1}]=N'$: $(\iota^{-1}(N'))[S^{-1}]=\{\frac{n}{s} \mid \frac{n}{1}\in N' \Leftrightarrow [\frac{s}{1}$ is invertible$] \Leftrightarrow \frac{n}{s}\in N'\}=N'$. $\square$

Corollary: Suppose M is a Noetherian (resp. Artinian) A-module. Then $M[S^{-1}]$ is a Noetherian (resp. Artinian) $A[S^{-1}]$-module. In particular, if A is a Noetherian (resp. Artinian) ring, then so is $A[S^{-1}]$.

Proof (of M is Noetherian $\Rightarrow$ so is $M[S^{-1}]$; everything else is exercise.) Let $N_1'\subset N_2'\subset\ldots\subset N_i'\subset\ldots$ be an AC of submodules in $M[S^{-1}]$. Set $N_i:=\iota_M^{-1}(N_i')$. Then $N_1\subset N_2\subset\ldots$ is AC of submodules in M so $\exists\, k$ s.t. $N_k=N_i$ $\forall i\geq k$. By Proposition, $N_i'=N_i[S^{-1}]\Rightarrow N_k'=N_i'$. So the AC in $M[S^{-1}]$ terminates, hence $M[S^{-1}]$ is Noetherian. $\square$

2) Local rings.

Here's an important example of a multiplicative subset. Let $\mathfrak{p}\subset A$ be a prime ideal. The equivalent characterization ($ab\in\mathfrak{p}\Rightarrow a\in\mathfrak{p}$ or $b\in\mathfrak{p}$) means that $A\setminus\mathfrak{p}$ is multiplicative.

We write $A_{\mathfrak{p}}$ for $A[(A\setminus\mathfrak{p})^{-1}]$. For an A-module M, we write $M_{\mathfrak{p}}$ for $M[(A\setminus\mathfrak{p})^{-1}]$.

Proposition: $\mathfrak{p}_\mathfrak{p}$ is the unique maximal ideal of $A_\mathfrak{p}$.

Proof: Pick an ideal $I' \subsetneq A_\mathfrak{p}$. First, we need to show $I' \subseteq \mathfrak{p}_\mathfrak{p}$. Set $I := \iota^{-1}(I')$, an ideal in A. By Prop'n in Sec 1.4,

$$sa \in I \text{ for } s \notin \mathfrak{p} \implies a \in I \qquad (\heartsuit)$$

Assume $I \not\subset \mathfrak{p} \Leftrightarrow S \cap I \neq \emptyset$. Pick $s \in S \cap I$, $a := 1$, so $sa \in I$ but $a \notin I$. This contradicts $(\heartsuit)$ showing $I \subset \mathfrak{p}$. By Prop'n in Sec 1.4, $I' = I_\mathfrak{p}$, so $I' = I_\mathfrak{p} \subset \mathfrak{p}_\mathfrak{p}$

Second, we need to show $\mathfrak{p}_\mathfrak{p} \neq A_\mathfrak{p}$. Indeed, $\frac{1}{1} \in \mathfrak{p}_\mathfrak{p} \Leftrightarrow \exists\, a \in \mathfrak{p}$, $s \notin \mathfrak{p}$ s.t. $\frac{1}{1} = \frac{a}{s} \Leftrightarrow \exists\, u \notin \mathfrak{p}$ s.t. $us = ua$. The l.h.s is not in $\mathfrak{p}$, while the r.h.s. is in $\mathfrak{p}$, a contradiction. $\square$

Definition: A commutative ring B is local if it has a unique maximal ideal.

Example: $A_\mathfrak{p}$ is local.

Local rings are important because they have nice properties that general rings do not, while some questions about general rings can be reduced to those of local rings — by passing from A to $A_\mathfrak{p}$. Nice properties of modules over local rings will be studied later in the course.