Lecture 11: Categories, functors & functor morphisms, I

1) Categories.
2) Functors.

Ref: [R], Sections 1.1, 1.3

BONUS: Homotopy category of topological space.

1) Our exposition of Category theory will start w. exploring basic notions: categories, functors & functor morphisms.

Definitions below will have a familiar structure: have data & axioms. E.g. here's a basic algebraic structure.

Definition: a **monoid** is
(Data): a set M equipped w. a multipl'n map $M\times M \to M$
(axioms): that is associative and has unit, 1.

For example, a group is exactly a monoid, where all elements are invertible. Every ring is a monoid w.r.t. multiplication.

1.1) Definition of a category.

Definition: A **category**, $\mathcal{C}$, consists of
(Data): • a "collection" of **objects**, $Ob(\mathcal{C})$.
• $\forall X,Y\in Ob(\mathcal{C}) \rightsquigarrow$ a set of **morphisms**, $Hom_{\mathcal{C}}(X,Y)$
• $\forall X,Y,Z\in Ob(\mathcal{C})$, a map (of sets) called **composition**

$$Hom_{\mathcal{C}}(X,Y)\times Hom_{\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{C}}(X,Z),\ (f,g)\mapsto g\circ f$$

($\circ$ is often omitted)

These satisfy:

(Axioms): i) composition is associative:

$(f\circ g)\circ h = f\circ(g\circ h)$ for $f\in Hom_{\mathcal{C}}(W,X)$, $g\in Hom_{\mathcal{C}}(X,Y)$, $h\in Hom_{\mathcal{C}}(Y,Z)$.

ii) Units: $\forall X\in Ob(\mathcal{C})\ \exists\ 1_X\in Hom_{\mathcal{C}}(X,X)$ s.t.

- $f\circ 1_X = f\ \forall f\in Hom_{\mathcal{C}}(X,Y)$,
- $1_X\circ g = g\ \forall g\in Hom_{\mathcal{C}}(Z,X)$.

1.2) Examples

1) Category of sets, **Sets**: objects = sets, morphisms = maps of sets, composition = composition of maps. Axioms: classical (unit $1_X = id_X$).

2) Sets w. additional str're: objects = sets w. add'l str're, morphisms = maps compatible w. this str're, composition = = comp'n of maps. This includes

a) Category of groups, **Groups**: objects are groups, morphisms = homomorphisms of groups.

b) Category of rings, **Rings**.

c) For a ring A, have categories of A-modules, **A-Mod**, & A-algebras (**A-Alg**), in the latter morphisms = A-linear homomorphisms of rings.

Not all categories have the form in 2:

3a) Let Γ be an oriented graph w. vertices V & edges E.
$\rightsquigarrow$ category $\mathcal{C}(\Gamma)$, the pass category of Γ.

- Objects $= V$.
- Morphisms = paths in the graph:

$$X \bullet \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_k} Y$$

this includes empty paths, one for every vertex.

- Composition: concatenation of paths.

Axioms: associativity is manifest, 1_X = empty path in X.

3b) Note: $\forall X \in Ob(\mathcal{C}) \Rightarrow Hom_{\mathcal{C}}(X,X)$ is a monoid w.r.t. $\circ$
Conversely, every monoid, M, gives a category w. one object, X, & $(Hom_{\mathcal{C}}(X,X), \circ) := M$.

1.3) Remarks:

1) Sometimes, objects in a category form a set (here we say our category is small). In general, they form a "class", a notion defined in Set theory. We'll ignore this issue.

2) 1_X is uniquely determined. Moreover, if $f \in Hom_{\mathcal{C}}(X\, Y)$, has a (2-sided) inverse g (i.e. $g \in Hom_{\mathcal{C}}(Y,X) \mid f \circ g = 1_Y, g \circ f = 1_X$) then g is unique, $f^{-1} := g$. In this case, f is called an isomorphism; we say X & Y are isomorphic (X & Y behave the same from the point of view of $\mathcal{C}$, e.g. $Z \in Ob(\mathcal{C}) \rightsquigarrow$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Z,X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(Z,Y) \\ \psi & \longmapsto & f\circ\psi \end{array}$$

(inverse is $\psi' \mapsto f^{-1}\circ\psi'$).

Notation: $X \xrightarrow{f} Y$ means $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

1.4) Subcategories: $\mathcal{C}$ is a category.

Def'n: (i) By a subcategory, $\mathcal{C}'$, in $\mathcal{C}$ we mean:

(Data) • A subcollection, $Ob(\mathcal{C}')$, in $Ob(\mathcal{C})$.

• $\forall X,Y \in Ob(\mathcal{C}')$, a subset $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ s.t.

(Axioms) • If $f \in \mathrm{Hom}_{\mathcal{C}'}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}'}(Y,Z) \Rightarrow g\circ f \in \mathrm{Hom}_{\mathcal{C}'}(X,Z)$

• $1_X \in \mathrm{Hom}_{\mathcal{C}'}(X,X)$ $\forall X \in Ob(\mathcal{C}')$.

(ii) A subcategory $\mathcal{C}'$ in $\mathcal{C}$ is called full if $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $\forall X,Y \in Ob(\mathcal{C}')$.

A subcategory $\mathcal{C}'$ has a natural category str're.

Examples:

1) A monoid M = category w. one object

A nonempty subcategory M' in M = a submonoid.

M' is full $\Leftrightarrow$ $M' = M$.

2) $\mathbb{Z}$-Mod (a.k.a. category of abelian groups) is a full subcategory in Groups

3) The category of commutative rings, CommRings is a full subcategory in Rings.

1.5) Constructions w. categories.

Definition: For a category, $\mathcal{C}$, its opposite category, $\mathcal{C}^{opp}$ consists of

- the same objects as $\mathcal{C}$,
- $Hom_{\mathcal{C}^{opp}}(X,Y) := Hom_{\mathcal{C}}(Y,X)$
- $g \circ^{opp} f := f \circ g$ $\quad (f \in Hom_{\mathcal{C}^{opp}}(X,Y) = Hom_{\mathcal{C}}(Y,X),$
  $g \in Hom_{\mathcal{C}^{opp}}(Y,Z) = Hom_{\mathcal{C}}(Z,Y)).$

Definition: For categories $\mathcal{C}_1, \mathcal{C}_2$, their product $\mathcal{C}_1 \times \mathcal{C}_2$ is defined by:
- $Ob(\mathcal{C}_1 \times \mathcal{C}_2) = Ob(\mathcal{C}_1) \times Ob(\mathcal{C}_2)$
- $Hom_{\mathcal{C}_1 \times \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) = Hom_{\mathcal{C}_1}(X_1, Y_1) \times Hom_{\mathcal{C}_2}(X_2, Y_2)$
- composition is componentwise.

Rem: for usual categories we care about (Groups, Rings, A-Mod), the opposite cat'y essentially has no independent meaning, except: $\mathcal{C} = $ CommRings, where $\mathcal{C}^{opp}$ is the category of affine schemes, which is of crucial importance for Algebraic geometry.

2) Functors: Motto: a relation between a category and a functor is analogous to a relation between a group and a

homomorphism.

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition: A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is

(Data) • an assignment $X \mapsto \mathcal{F}(X): Ob(\mathcal{C}) \to Ob(\mathcal{D})$.

• $\forall X, Y \in Ob(\mathcal{C})$, a map

$$\begin{array}{ccc} Hom_{\mathcal{C}}(X,Y) & \longrightarrow & Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \mathcal{F}(f) \end{array}$$

(Axioms) – compatibility between compositions & units

• $\forall f \in Hom_{\mathcal{C}}(X,Y), g \in Hom_{\mathcal{C}}(Y,Z) \Rightarrow \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$

equality in $Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Z))$.

• $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)} \ \forall \ X \in Ob(\mathcal{C})$

Example: Let $\mathcal{C}, \mathcal{D}$ be categories w. single object corresponding to monoids M, N. Then a functor $\mathcal{C} \to \mathcal{D}$ is the same thing as a monoid homomorphism.

Remarks: • Have the identity functor $Id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$

• For functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{E}$ can take the composition $\mathcal{G} \circ \mathcal{F}: \mathcal{C} \to \mathcal{E}$ $(\mathcal{G} \circ \mathcal{F}(X) = \mathcal{G}(\mathcal{F}(X)))$, it's a functor.

• A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is the same thing as a functor $\mathcal{C}^{opp} \to \mathcal{D}^{opp}$.

More examples:

1) Let $\mathcal{C}'$ be a subcategory in $\mathcal{C}$. Then have inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ sending objects/morphisms in $\mathcal{C}'$ to the same objects/morphisms now in $\mathcal{C}$; axioms are clear.

2) Forgetful functors: forget part of a structure

2a) $For: Groups \to Sets$;

On objects: $For(G) = G$ viewed as a set.

On morphisms: $For(f) = f$, viewed as a map of sets.

Axioms: clear.

BONUS: homotopy category of topological spaces.

B1) Equivalence on morphisms.

Let $\mathcal{C}$ be a category. Suppose that $\forall X, Y \in Ob(\mathcal{C})$, the set $Hom_{\mathcal{C}}(X,Y)$ is endowed with an equivalence relation $\sim$ s.t.

(1) If $g, g' \in Hom_{\mathcal{C}}(Y,Z)$ are equivalent & $f \in Hom_{\mathcal{C}}(X,Y)$, then $g \circ f \sim g' \circ f$.

(2) If $f, f' \in Hom_{\mathcal{C}}(X,Y)$ are equivalent and $g \in Hom_{\mathcal{C}}(Y,Z)$, then $g \circ f \sim g \circ f'$.

We write $[f]$ for the equivalence class of f.

Given such an equivalence relation, we can form a new category to be denoted by $\mathcal{C}/\sim$ as follows:

- $Ob(\mathcal{C}/\sim) := Ob(\mathcal{C})$
- $Hom_{\mathcal{C}/\sim}(X,Y) := Hom_{\mathcal{C}}(X,Y)/\sim$ – the set of equivalence classes
- $[g]\circ[f] = [g\circ f]$ – well-defined precisely b/c of (1) & (2)

We note that there is a natural functor $\pi: \mathcal{C} \to \mathcal{C}/\sim$ given by $X \mapsto X$, $f \mapsto [f]$.

Example: Let M be a monoid. Note that the equivalence class of $1 \in M$ is a submonoid, say M_0, moreover, (1) & (2) imply that $mM_0 = M_0 m \ \forall\, m \in M$. Such submonoids are called normal (for groups we recover the usual condition). And if $M_0 = [1]$ is normal, then (1) and (2) hold – an exercise. For a normal submonoid M_0 we can M/M_0 with a natural monoid structure – just as we do for groups. The category $\mathcal{C}/\sim$ corresponds to the quotient monoid M/M_0 and the functor π is just the natural epimorphism $M \twoheadrightarrow M/M_0$.

Rem*: $\mathcal{C}/\sim$ looks like a quotient category. But in situations where the term "quotient" is used and that are closer to quotients of abelian groups (Serre quotients of abelian categories) the construction is different – and more difficult.

B2) Homotopy category of topological spaces.

Let's recall the usual category of topological spaces. Let

X be a set. One can define the notion of topology on X: we declare some subsets of X to be "open", these are supposed to satisfy certain axioms. A set w. topology is called a topological space. A map $f: X \to Y$ of topological spaces is called continuous if $U \subset Y$ is open $\Rightarrow f^{-1}(U) \subset X$ is open.

We define the category Top of topological spaces w.

$Ob(Top)$ = topological spaces.

$\mathrm{Hom}_{Top}(X,Y) :=$ continuous maps $X \to Y$

Composition = composition of maps.

One issue: this category is hard to understand - hard to study topological spaces up to homeomorphisms.

Now we introduce our equivalence relation of $\mathrm{Hom}_{Top}(X,Y)$

Definition: Continuous maps $f_0, f_1: X \to Y$ are called homotopic if $\exists$ a continuous map $F: X \times [0,1] \to Y$ s.t. $f_0(x) = F(x,0)$ & $f_1(x) = F(x,1)$.

Informally, f_0, f_1 are homotopic if one can continuously deform f_0 to f_1. It turns out that being homotopic is an equivalence relation satisfying (1) & (2) from B1. The corresponding category $Top/\sim$ is known as the homotopy category of topol'l spaces. Note that in this category morphisms are not maps!

Here is why we care about the homotopy category. Isomorphic here means homotopic (X is homotopic to Y if $\exists\ X \xrightarrow{f} Y$, $Y \xrightarrow{g} X$ s.t. fg is homotopic to 1_Y & gf is homotopic to 1_X) and this is easier to understand then being homeomorphic. Second, the classical invarients such as homology and homotopy groups only depend on homotopy type. A more educated way to state this: these invarients are functors from the homotopy category of topological spaces to Groups (true as stated for homology, for homotopy it's more subtle, this requives fixing a point in X and hence need to work w. an auxiliary category of "pointed" topological spaces - up to homotopy).