Lecture 12: Categories, functors & functor morphisms, II.

1) Functors, cont'd.

2) Functor morphisms.

Ref: [R], Sec. 1.3, 1.4.

1) Functors

Let $\mathcal{C}, \mathcal{D}$ be categories. Recall that a functor $F: \mathcal{C} \to \mathcal{D}$ is a pair of an assignment $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$ & for $X, Y \in Ob(\mathcal{C})$ a map $F: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X), F(Y))$ preserving compositions ($F(f \circ g) = F(f) \circ F(g)$) & unit morphisms ($F(1_X) = 1_{F(X)}$).

1.1) More examples of functors.

1) More forgetful functors.

1a) Let A be a commutative ring. Then have the forgetful functor For: A-$Alg \longrightarrow A$-Mod, forgetting the ring multiplication.

1b) Functor Rings $\longrightarrow$ Monoids forgetting the addition.

1c) Let A, B be commutative rings & $\varphi: A \to B$ be a ring homom'm. Then can consider the pullback functor $\varphi^*: B$-$Mod \longrightarrow A$-Mod. It sends $M \in Ob(B\text{-}Mod)$ to M viewed as an A-module & $\psi \in Hom_B(M,N)$ to $\psi \in Hom_A(M,N)$. Forgets part of the action.

2) Let $\mathcal{C}$ be a category. For $X \in Ob(\mathcal{C})$ define the Hom functor
$\mathcal{F}_X\ (:= Hom_{\mathcal{C}}(X, \cdot)): \mathcal{C} \longrightarrow Sets$.

On objects: $\mathcal{F}_X(Y) := Hom_{\mathcal{C}}(X,Y)$, a set.

On morphisms: (we'll use the notation $X \xrightarrow{\psi} Y$ to denote $\psi \in Hom_{\mathcal{C}}(X,Y)$)

$$Y_1 \xrightarrow{f} Y_2 \rightsquigarrow \text{map } \mathcal{F}_X(f): \begin{array}{ccc} Hom_{\mathcal{C}}(X,Y_1) & \rightarrow & Hom_{\mathcal{C}}(X,Y_2) \\ \psi & \longmapsto & f\circ\psi \end{array}$$

Check axioms: composition: $\mathcal{F}_X(g\circ f) = \mathcal{F}_X(g)\circ\mathcal{F}_X(f)$ for
$Y_1 \xrightarrow{f} Y_2 \xrightarrow{g} Y_3$. For $\psi \in Hom_{\mathcal{C}}(X, Y_1)$ have
$[\mathcal{F}_X(g\circ f)](\psi) = (g\circ f)\circ\psi \in Hom_{\mathcal{C}}(X, Y_3)$.
$[\mathcal{F}_X(g)\circ\mathcal{F}_X(f)](\psi) = [\mathcal{F}_X(g)](f\circ\psi) = g\circ(f\circ\psi)$.
By associativity axiom for morphisms, the two coincide.

The unit axiom is left as exercise.

2^{opp}) We can apply this construction to $\mathcal{C}^{opp} \rightsquigarrow$
$\mathcal{F}_X^{opp}: Y \longmapsto Hom_{\mathcal{C}^{opp}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$

$f \in Hom_{\mathcal{C}^{opp}}(Y_1, Y_2) = Hom_{\mathcal{C}}(Y_2, Y_1) \rightsquigarrow$

$$\mathcal{F}_X^{opp}(f): \begin{array}{ccc} Hom_{\mathcal{C}}(Y_1, X) & \longrightarrow & Hom_{\mathcal{C}}(Y_2, X) \\ \psi & \longmapsto & \psi\circ f \end{array} \text{ - map of sets}$$

We can view $\mathcal{F}_X^{opp}$ as a functor $\mathcal{C} \longrightarrow Sets^{opp}$
(a traditional name: contravariant functor $\mathcal{C} \rightarrow Sets$)

3) Algebra constructions as functors:

3a) The "free" functor: $Sets \rightarrow A\text{-}Mod$.

Let A be a ring. Want to define a functor Free: Sets $\to$ A-Mod
I, set, $\rightsquigarrow$ Free $(I) := A^{\oplus I}$
$f: I \to J \rightsquigarrow$ Free$(f): A^{\oplus I} \to A^{\oplus J}$ – the unique A-linear map sending the basis element e_i $(i \in I)$ to $e_{f(i)} \in A^{\oplus J}$.
Checking axioms of functor: exercise.

3b) Localization of modules is a functor: $S \subset A$ multiplicative $\rightsquigarrow \bullet[S^{-1}]: A\text{-Mod} \to A[S^{-1}]\text{-Mod}$, a functor that sends an A-module M to the $A[S^{-1}]$-module $M[S^{-1}]$ and an A-module homomorphism $\varphi: M \to N$ to $\varphi[S^{-1}]: M[S^{-1}] \to N[S^{-1}]$ (see Sec 2 of Lec 9), $\varphi[S^{-1}]\left(\frac{m}{s}\right) := \frac{\varphi(m)}{s}$. Checking the axioms was a part of the very important exercise in Sec 2 of Lec 9.

2) Functor morphisms.

Motto: A relation between functors & functor morphisms is like a relation between modules & module homomorphisms.

2.1) Definition: Let $\mathcal{C}, \mathcal{D}$ be categories & $F, G: \mathcal{C} \to \mathcal{D}$ be functors.
Def'n: A functor morphism $\eta: F \Rightarrow G$ is
Functors F, G send the objects $X \in Ob(\mathcal{C})$ to $F(X), G(X) \in Ob(\mathcal{D})$. We can relate $F(X), G(X)$ by taking a morphism between them:
(Data) $\forall X \in Ob(\mathcal{C})$, a morphism $\eta_X \in Hom_{\mathcal{D}}(F(X), G(X))$

Picking morphisms which are totally unrelated is pointless. We need to relate η_X, η_Y for $X, Y \in Ob(\mathcal{C})$. The relations we need come from morphisms between X, Y: $f \in Hom_{\mathcal{C}}(X, Y) \rightsquigarrow$ $F(f) \in Hom_{\mathcal{D}}(F(X), F(Y))$, $G(f) \in Hom_{\mathcal{D}}(G(X), G(Y))$

(axiom) s.t. $\forall\ X, Y \in Ob(\mathcal{C})$, $f \in Hom_{\mathcal{C}}(X, Y)$, the following diagram is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

Remarks:

1) In many (but not all) examples, η_X is "natural" meaning it's "uniform" & "independent of additional choices". Hence the name "natural transformation" for a functor morphism that was used in the past.

2) An analogy w. module homomorphisms is as follows. Let A be a ring, M, N be A-modules. For $a \in A$, we write a_M, a_N for the operators of multiplication by a in M, N. Then a group homomorphism $\eta: M \to N$ is an A-module homomorphism iff $\forall\ a \in A$, the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "a_M"] \arrow[d, "\eta"] & M \arrow[d, "\eta"] \\ N \arrow[r, "a_N"] & N \end{tikzcd}$$

Exercise: Let M, N be categories w. one object a.k.a. monoids & $F: M \to N$ be a functor (a.k.a. monoid homomorphism). Then a functor endomorphism $\eta: F \Rightarrow F$ is the same thing as an element $\eta \in N \mid \eta F(m) = F(m)\eta \ \forall\, m \in M$.

2.2) Important example.

Let $X, X' \in Ob(\mathcal{C}) \rightsquigarrow$ functors

$$F_X := Hom_{\mathcal{C}}(X, \cdot),\ F_{X'} := Hom_{\mathcal{C}}(X', \cdot) : \mathcal{C} \to Sets.$$

Goal: from $g \in Hom_{\mathcal{C}}(X', X)$ produce a functor morphism $\eta^g: F_X \Rightarrow F_{X'}$ (note the order!)

i.e. for each $Y \in Ob(\mathcal{C})$ we need to define a map

$$\eta^g_Y: Hom_{\mathcal{C}}(X, Y) \longrightarrow Hom_{\mathcal{C}}(X', Y): \qquad X' \xrightarrow{g} X \xrightarrow{\psi} Y$$

$$\psi \longmapsto \psi \circ g$$

← essentially the only natural way to give such a map.

Now we need to check the axiom (commutative diagram):
$\forall\, f \in Hom_{\mathcal{C}}(Y_1, Y_2)$, $F_X(f) = f \circ ?$, $F_{X'}(f) = f \circ ?$ & we have that

$$\left.\begin{array}{ccc} \psi \in Hom_{\mathcal{C}}(X, Y_1) & \xrightarrow{f \circ ?} & Hom_{\mathcal{C}}(X, Y_2) \\ \Big\downarrow \eta_{Y_1}(?) = ? \circ g & & \Big\downarrow \eta_{Y_2}(?) = ? \circ g \\ Hom_{\mathcal{C}}(X', Y_1) & \xrightarrow{f \circ ?} & Hom_{\mathcal{C}}(X', Y_2) \end{array}\right\} \text{is commutative}$$

$\downarrow \longrightarrow: \psi \mapsto \psi\circ g \mapsto f\circ(\psi\circ g)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| \longleftarrow$ b/c composition in a category is associative.

$\longrightarrow \downarrow: \psi \mapsto f\circ\psi \mapsto (f\circ\psi)\circ g$

We've checked: η^g is a functor morphism.

2.3) Yoneda lemma.

It turns out that we have described all functor morphisms between these functors. We write $Hom_{Fun}(F,G)$ for the collec- of functor morphisms $F \Rightarrow G$.

Thm (Yoneda lemma): $g \mapsto \eta^g$ is a bijection $Hom_{\mathcal{C}}(X',X) \xrightarrow{\sim} Hom_{Fun}(F_X, F_{X'})$.

Proof of Thm:

Step 1: Construct a map $Hom_{Fun}(F_X, F_{X'}) \longrightarrow Hom_{\mathcal{C}}(X',X)$

$$\eta \longmapsto g_\eta$$

$\eta \rightsquigarrow$ map $\eta_X: Hom_{\mathcal{C}}(X,X) \longrightarrow Hom_{\mathcal{C}}(X',X)$

$$1_X \longmapsto g_\eta$$

i.e. $g_\eta = \eta_X(1_X)$.

Step 2: Now we check that $g \mapsto \eta^g$, $\eta \mapsto g_\eta$ are mutually inverse starting w. $g_{\eta^g} = \eta^g_X(1_X) = 1_X\circ g = g$.

Step 3: We show that $\eta^{g_\eta} = \eta \Leftrightarrow \forall\, Y \in Ob(\mathcal{C})$ have $(\eta^{g_\eta})_Y = \eta_Y$ equality of maps $Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{C}}(X',Y)$

Note that $(\eta^{g\eta})_Y$ sends $\psi \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ to $\psi \circ \eta_X(1_X)$.
We'll use comm've diagram from the def'n of functor morphism that gives an equality of compositions that we then apply to 1_X.

$$\begin{tikzcd}
1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[r, "F_X(\psi)=\psi\circ ?"] \arrow[d, "\eta_X"] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[d, "\eta_Y"] \\
\mathrm{Hom}_{\mathcal{C}}(X',X) \arrow[r, "F_{X'}(\psi)=\psi\circ ?"] & \mathrm{Hom}_{\mathcal{C}}(X',Y)
\end{tikzcd}$$

$\downarrow \longrightarrow : \psi \circ \eta_X(1_X) = (\eta^{g\eta})_Y(\psi) \in \mathrm{Hom}_{\mathcal{C}}(X',Y)$.

$\longrightarrow \downarrow : \eta_Y(\psi \circ 1_X) = \eta_Y(\psi) \in \mathrm{Hom}_{\mathcal{C}}(X',Y)$

The equality $(\eta^{g\eta})_Y = \eta_Y$ follows finishing the proof. $\square$