Lecture 13: categories, functors & functor morphisms, III.

1) Yoneda lemma vs compositions & inverses.
2) Objects representing functors
3) Products in categories.

Refs: [R], Secs 2.1-2.3; [HS], Sec II.5.

BONUS: category equivalences.

1) Yoneda lemma vs compositions & inverses.

1.0) Reminder.

Let $F, G: \mathcal{C} \to \mathcal{D}$ be functors. By a functor morphism $\eta: F \Rightarrow G$ we mean a collection $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)), X \in Ob(\mathcal{C})$ s.t. $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following is commutative

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

We write $\mathrm{Hom}_{Fun}(F, G)$ for the collection of functor morphisms $F \Rightarrow G$.

In Sec 2.2 of Lec 12 we have constructed the following example. For $X \in Ob(\mathcal{C})$, consider the Hom functor $F_X = \mathrm{Hom}_{\mathcal{C}}(X, \bullet): \mathcal{C} \to$ Sets. Then for $g \in \mathrm{Hom}_{\mathcal{C}}(X', X)$ we have

$$\eta^g: F_X \Rightarrow F_{X'}: \eta^g_Y: \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X', Y), \ \eta^g_Y(\varphi) := \varphi \circ g.$$

Thm (Yoneda lemma): $g \mapsto \eta^g : \mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}}(F_X, F_{X'})$

As we've seen in the proof, the inverse is given by
$$\mathrm{Hom}_{\mathrm{Fun}}(F_X, F_{X'}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X', X) (= F_{X'}(X)),\ \eta \mapsto \eta_X(1_X)$$

Remark: More generally, for arbitrary $F: \mathcal{C} \to \mathrm{Sets}$
$$\mathrm{Hom}_{\mathrm{Fun}}(F_X, F) \xrightarrow{\sim} F(X),\ \eta \mapsto \eta_X(1_X)$$
The inverse map is given by $a \mapsto \eta^a$ w. $\eta^a_Y : \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow F(Y)$, given by $\psi \mapsto [F(\psi)](a)$. Details are a premium exercise.

1.1) Yoneda lemma vs compositions.

$\forall$ functor $F: \mathcal{C} \to \mathcal{D}$, have the identity morphism $id_F: F \Rightarrow F$. If $\mathcal{D} = \mathrm{Sets}$ & $F = F_X$, then $id_F = \eta^{1_X}$.

Also we can take compositions of functor morphisms
$$\tau: G \Rightarrow H,\ \eta: F \Rightarrow G \rightsquigarrow \tau\circ\eta : F \Rightarrow H,\ (\tau\circ\eta)_X := \tau_X \circ \eta_X$$

Exercise: • check that $(\tau\circ\eta)$ is indeed a functor morphism (hint: stack the commutative diagrams).

If $F = F_X,\ G = F_{X'},\ H = F_{X''}$ for $X, X', X'' \in Ob(\mathcal{C})$, & $g \in \mathrm{Hom}_{\mathcal{C}}(X', X)$, $g' \in \mathrm{Hom}_{\mathcal{C}}(X'', X')$, then

(1) $$\eta^{g'} \circ \eta^{g} = \eta^{g\circ g'}$$

equality of functor morphisms $F_X \Rightarrow F_{X''}$. Indeed, for $\psi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$
$\eta^{g'}_Y \circ \eta^{g}_Y(\psi) = \eta^{g'}_Y(\psi\circ g) = (\psi\circ g)\circ g' = \psi\circ(g\circ g') = \eta^{g\circ g'}_Y(\psi)$.

Remark: If $\mathcal{C}$ is small ($Ob(\mathcal{C})$ is a set), then the collection of functor morphisms $F \Rightarrow G$, denoted $Hom_{Fun}(F,G)$, is a set $\rightsquigarrow$ category of functors $Fun(\mathcal{C},\mathcal{D})$: objects are functors, morphisms = morphisms of functors.

We then can consider a functor $\mathcal{C}^{opp} \longrightarrow Fun(\mathcal{C}, Set)$ $X \mapsto F_X$ $\forall X \in Ob(\mathcal{C})$, $g \in Hom_{\mathcal{C}^{opp}}(X,Y) = Hom_{\mathcal{C}}(Y,X) \mapsto \eta^g : F_X \Rightarrow F_Y$. The Yoneda lemma says that

$$Hom_{\mathcal{C}^{opp}}(X,Y) \xrightarrow{\sim} Hom_{Fun(\mathcal{C},Sets)}(F_X, F_Y)$$

Functors that induce isomorphisms between the morphism sets have a name: fully faithful ("faithful" refers to the injectivity, and "full" to the surjectivity).

1.2) Yoneda vs functor isomorphisms.

Let $\mathcal{C},\mathcal{D}$ be categories & $F,G : \mathcal{C} \to \mathcal{D}$ be functors.

By def'n, $\eta : F \Rightarrow G$ is a functor isomorphism if it has a (two-sided) inverse: $\eta^{-1} : G \Rightarrow F$ w. $\eta \circ \eta^{-1} = id_G$, $\eta^{-1} \circ \eta = id_F$, which, as usual is unique if it exists.

Lemma: 1) η is an isomorphism $\Leftrightarrow$ $\eta_Y \in Hom_{\mathcal{D}}(F(Y), G(Y))$ is an isomorphism for all $Y \in Ob(\mathcal{C})$.

2) Let $\mathcal{D} = Sets$, $F = F_X$, $G = F_{X'}$, $g \in Hom_{\mathcal{C}}(X', X)$. Then $\eta^g : F_X \Rightarrow F_{X'}$ is an isomorphism (of functors) $\Leftrightarrow$ g is an isomorphism (of objects).

Proof: 1) $\Rightarrow$: from def'n.

$\Leftarrow$: define $(\eta^{-1})_Y := \eta_Y^{-1}$.

Exercise: check the axiom of a functor morphism.
Then η^{-1} is automatically inverse of η.
2): exercise (from Yoneda lemma & $\eta^{g'\circ g} = \eta^{g}\circ\eta^{g'}$). $\square$

2) Objects representing functors

Definition: Let $F: \mathcal{C} \to Sets$ be a functor. We say $X \in Ob(\mathcal{C})$ **represents** F if F is isomorphic to F_X.

A representing object may fail to exist (this is the case for the forgetful functor from <u>finite</u> groups to sets (cf. HW4). If a representing object exists, we say that F is **representable**.

Lemma: An object, X, representing F is unique up to isom'm if it exists.

Proof:

Let $X, X' \in Ob(\mathcal{C})$ represent F: $F_X \overset{\sim}{\underset{\eta}{\Rightarrow}} F \overset{\sim}{\underset{\eta'}{\Leftarrow}} F_{X'}$ $\rightsquigarrow$ a functor isomorphism $\eta'^{-1}\circ\eta: F_X \overset{\sim}{\Rightarrow} F_{X'}$. By 2) of Lemma in Sec 1.2, X & X' are isomorphic. $\square$

Example: The forgetful functor For: $\mathcal{C} = Groups \to Sets$ is represented by $\mathbb{Z}$. Indeed, for any group G we have a bijection (of sets) $\eta_G: Hom_{Groups}(\mathbb{Z}, G) \xrightarrow{\sim} G$, $\varphi \mapsto \varphi(1)$. (η_G) is a functor morphism - what we need to check is that $\forall$ group homomorphism $f: G \to H$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{Groups}(\mathbb{Z}, G) \arrow[r, "\varphi \mapsto f\circ\varphi"] \arrow[d, "\varphi\mapsto\varphi(1)"] & \mathrm{Hom}_{Groups}(\mathbb{Z}, H) \arrow[d, "\psi\mapsto\psi(1)"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

is commutative, which is left as an exercise.

Since η_G is bijective $\forall\, G$, by (2) of Lemma in Sec 1.2, is a functor isomorphism, implying our claim.

Now let's use the Yoneda lemma to compute $\mathrm{End}_{Fun}(For)$ $(:= \mathrm{Hom}_{Fun}(For, For))$ as a monoid w.r.t. composition. By Yoneda, it's $\mathrm{End}_{Groups}(\mathbb{Z})$ (with multiplication $gg' := g'\circ g$). We have $\mathrm{End}_{Groups}(\mathbb{Z}) \xrightarrow{\sim} For(\mathbb{Z}) = \mathbb{Z}$; to $m\in\mathbb{Z}$ we assign $g_m: \mathbb{Z}\to\mathbb{Z}, x\mapsto mx$. We conclude that $\mathrm{End}_{Fun}(For)$ is the monoid $\mathbb{Z}$ w.r.t. multiplication.

Exercise: Let $\eta[m]$ be the endomorphism $For \Rightarrow For$ corresponding to $m\in\mathbb{Z}$. Show that $\eta[m]_G$ is given by $\eta[m]_G(x) = x^m$, $\forall\, x\in G$.

3) Products in categories.

The concept of a representing object allows to carry constructions from the category of sets to a general category. Here we consider a basic such construction - products.

Recall that in our usual categories: Sets, Groups, Rings, A-mod we have the notion of direct product. In all of them, this is characterized by universal property. E.g. if A_1, A_2 are rings, then $A_1\times A_2$ is a ring w. ring homomorphisms $\pi_i: A_1\times A_2 \to A_i$ s.t. $\forall$ rings B w. homomorphisms $\varphi_i: B\to A_i$ $\exists!\ \varphi: B\to A_1\times A_2$ w.

$\varphi_i = \pi_i \circ \varphi$.

Now let $\mathcal{C}$ be a category and $F_1, F_2 : \mathcal{C} \to Sets$ be functors. Define their product $F_1 \times F_2$ by

- Sending $X \in Ob(\mathcal{C})$ to $F_1(X) \times F_2(X)$
- Sending $\varphi \in Hom_{\mathcal{C}}(X,Y)$ to $F_1(\varphi) \times F_2(\varphi) : F_1(X) \times F_2(X) \to F_1(Y) \times F_2(Y)$.

$F_1 \times F_2$ is a functor, to check the axioms is an exercise.

Now take $X_1, X_2 \in Ob(\mathcal{C})$ and let $F_i := F_{X_i}^{opp}$ be the Hom functor $Hom_{\mathcal{C}^{opp}}(X, \bullet)\ (= Hom_{\mathcal{C}}(\bullet, X)) : \mathcal{C}^{opp} \to Sets$.

Definition: If $X \in Ob(\mathcal{C})$ represents $F_{X_1}^{opp} \times F_{X_2}^{opp}$, then we say that X is the product $X_1 \times X_2$.

Thx to Lemma in Sec 2, $X_1 \times X_2$ is unique (up to iso) if it exists (that may fail to be the case).

Here's an alternative characterization of products.

Lemma: 1) There are $\pi_i \in Hom_{\mathcal{C}}(X, X_i)$ s.t.

(*) $\forall\, Y \in Ob(\mathcal{C}), \varphi_i \in Hom_{\mathcal{C}}(Y, X_i), i=1,2, \exists!\, \varphi \in Hom_{\mathcal{C}}(Y,X) \mid \varphi_i = \pi_i \circ \varphi$.

2) Conversely, let (X, π_1, π_2) satisfy (*). Then $X = X_1 \times X_2$

Note that (*) is the usual universal property of direct products. In particular, in our usual categories: Sets, Groups, Rings, A-Mod products are just direct products - and they exist $\forall\, X_1, X_2$.

Proof (of Lemma): 1) Let $\eta: F_X^{opp} \xrightarrow{\sim} F_{X_1}^{opp}\times F_{X_2}^{opp} \rightsquigarrow$ for $Y\in Ob(\mathcal{C})$ $\eta_Y: Hom_{\mathcal{C}}(Y,X) \xrightarrow{\sim} Hom_{\mathcal{C}}(Y,X_1)\times Hom_{\mathcal{C}}(Y,X_2)$. We define $(\pi_1,\pi_2)\in Hom_{\mathcal{C}}(X,X_1)\times Hom_{\mathcal{C}}(X,X_2)$ as $\eta_X(1_X)$. As in Step 3 of the proof of Yoneda lemma in Lec 12, $\forall\, \varphi\in Hom_{\mathcal{C}}(Y,X)$, we have comm've diagram

$$\begin{tikzcd}
Hom_{\mathcal{C}}(X,X) \arrow[r, "?\circ\varphi"] \arrow[d, "\eta_X"', "\wr"] & Hom_{\mathcal{C}}(Y,X) \arrow[d, "\wr"', "\eta_Y"] \\
Hom_{\mathcal{C}}(X,X_1)\times Hom_{\mathcal{C}}(X,X_2) \arrow[r, "(?\circ\varphi,\, ?\circ\varphi)"] & Hom_{\mathcal{C}}(Y,X_1)\times Hom_{\mathcal{C}}(Y,X_2)
\end{tikzcd}$$

which we apply to 1_X getting: $\eta_Y(\varphi)=(\pi_1\circ\varphi, \pi_2\circ\varphi)$. (*) follows b/c η_Y is a bijection: $\forall\, \varphi_1,\varphi_2\ \exists!\ \varphi$ w. $\eta_Y(\varphi)=(\varphi_1,\varphi_2)$

2) We essentially reverse the argument. Define
$\eta_Y: Hom_{\mathcal{C}}(Y,X)\longrightarrow Hom_{\mathcal{C}}(Y,X_1)\times Hom_{\mathcal{C}}(Y,X_2),\ \varphi\mapsto(\pi_1\circ\varphi,\pi_2\circ\varphi)$
By (*), η_Y is a bijection. To check that η_Y constitute a functor morphism is an exercise. So $\eta=(\eta_Y)$ is a functor isomorphism. $\square$

Rem: Note that we use the Yoneda lemma when we talk about representing objects incl. products: it guarantees (via its consequence in Sec 1.2) that representing objects are uniquely determined if they exist.

BONUS: Category equivalences.

Our question here: when are two categories the "same"? Turns out, functor isomorphisms play an important role in answering this question.

Before we address this, we should discuss an easier question: when are two sets the same? Well, they are literally the same if they consist of the same elements. But this definition is quite useless: sets arising from different constructions won't be the same in this sense. Of course, we use isomorphic instead of being literally the same.

Now back to categories. Again, being the same is useless. How about being isomorphic? Turns out, this is not useful. Let's see why. Let $\mathcal{C}, \mathcal{D}$ be categories. We say that either $\mathcal{C}, \mathcal{D}$ are isomorphic if there are functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ such that $FG = Id_{\mathcal{D}}$, $GF = Id_{\mathcal{C}}$. The issue is: two functors obtained by different constructions are never the same (compare to sets). The solution: replace "equal" w. "isomorphic" (as functors).

Definition: • Functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ are *quasi-inverse* if $FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$, $GF \overset{\sim}{\Rightarrow} Id_{\mathcal{C}}$ (isomorphic).

• We say $\mathcal{C}, \mathcal{D}$ are *equivalent* if there are quasi-inverse functors (called *equivalences*) $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$.

Now we are going to state a general result. For this we need another definition.

Definitions: A functor $F: \mathcal{C} \to \mathcal{D}$ is called

- fully faithful if $\forall\, X, X' \in Ob(\mathcal{C}) \Rightarrow$
$$f \mapsto F(f) \text{ is a bijection } Hom_{\mathcal{C}}(X, X') \xrightarrow{\sim} Hom_{\mathcal{D}}(F(X), F(X'))$$
- essentially surjective if $\forall\, Y \in Ob(\mathcal{D})\ \exists\ X \in Ob(\mathcal{C})$ such that $F(X)$ is isomorphic to Y.

Thm: A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence $\Leftrightarrow$ F is fully faithful & essentially surjective.

We won't prove this, but we will give an example - that illustrates how the proof works in general.

Example: Consider the category $\mathcal{D} = \mathbb{F}\text{-}Vect_{fd}$ of finite dimen'l vector spaces over a field $\mathbb{F}$ and its full subcategory $\mathcal{C}$ w. objects $\mathbb{F}^n$ $(n \geq 0)$. We claim that the inclusion functor $F: \mathcal{C} \hookrightarrow \mathcal{D}$ is an equivalence. It's fully faithful by def'n and the claim that it's essentially surjective.

Now we produce a quasi-inverse functor, G. In each $V \in Ob(\mathcal{D})$ we fix a basis, which leads to an isomorphism $\eta_V : V \xrightarrow{\sim} \mathbb{F}^n$. We define $G(V)$ as $\mathbb{F}^n$. For a linear map $f: U \to V$ (w.

$\dim U = m$, $\dim V = n$) we set $G(f) := \eta_V^{-1} \circ f \circ \eta_U$.

Exercise: Check G is a functor.

Now we are going to simplify our life a bit & assume that $\eta_{\mathbb{F}^n} : \mathbb{F}^n \xrightarrow{\sim} \mathbb{F}^n$ is the identity.

Exercise: $GF : \mathcal{C} \longrightarrow \mathcal{C}$ is the identity functor (not just isomorphic to it).

Now we produce a functor isomorphism $\eta : Id_{\mathcal{D}} \overset{\sim}{\Rightarrow} FG$
So we need to have $\eta_V : V \to \mathbb{F}^{\dim V}$ and this is the isomorphism from above.

Exercise: prove that η is indeed a functor morphism

Then η is an isomorphism of functors. So F is indeed a category equivalence.

Another exercise: prove that the duality functor $\cdot^*$ is an equivalence $\mathbb{F}\text{ Vect}_{fd} \longrightarrow \mathbb{F}\text{-Vect}_{fd}^{opp}$.