Lecture 13: categories, functors & functor morphisms, III.

1) Yoneda Lemma vs compositions & inverses.
2) Objects representing functors.
3) Products in categories.

Refs: [RT], Secs 2.1-2.3; [HS], Sec II.5.

BONUS: category equivalences.

1) Yoneda Lemma vs compositions & inverses.

1.0) Reminder.

Let $F, G: C \to D$ be functors. By a functor morphism $\varphi: F \Rightarrow G$, we mean a collection $\varphi_x \in \text{Hom}_D(F(x), G(x)), x \in \text{Ob}(C)$ s.t. $\forall f \in \text{Hom}_C(X, Y)$, the following is commutative:

$$
\begin{align*}
F(X) & \xrightarrow{F(f)} F(Y) \\
\downarrow \varphi_X & \quad \downarrow \varphi_Y \\
G(X) & \xrightarrow{G(f)} G(Y)
\end{align*}
$$

We write $\text{Hom}_{\text{Fun}}(F, G)$ for the collection of functor morphisms $F \Rightarrow G$.

In Sec 2.2 of Lec 12 we have constructed the following example. For $X \in \text{Ob}(C)$, consider the Hom functor $F_X = \text{Hom}_C(X, \cdot): C \to \text{Sets}$. Then for $g \in \text{Hom}_C(X, Y)$ we have $\varphi^g: F \Rightarrow F_X, \varphi^g: \text{Hom}_C(X, Y) \to \text{Hom}_C(X, Y), \varphi^g(\psi) = \psi \circ g.
Thm (Yoneda Lemma): \( g \mapsto \gamma^g : \text{Hom}_C(X',X) \to \text{Hom}_F(X, X') \)

As we’ve seen in the proof, the inverse is given by
\[
\text{Hom}_F(X, X') \to \text{Hom}_C(X',X)(= F_{X'}(x)), \quad \gamma \mapsto \gamma^g(x)
\]

Remark: More generally, for arbitrary \( F : C \to \text{Sets} \)
\[
\text{Hom}_F(X, F) \to F(x), \quad \gamma \mapsto \gamma^g(x)
\]
The inverse map is given by \( \alpha \mapsto \gamma^\alpha \), \( \gamma^\alpha : \text{Hom}_C(X,Y) \to F(Y) \),
given by \( \gamma \mapsto [F(\gamma)](\alpha) \). Details are a premium exercise.

1.1) Yoneda lemma vs compositions.

- Every functor \( F : C \to D \), have the identity morphism \( \text{id}_F : F \Rightarrow F \).
- \( D = \text{Sets} \) & \( F = F_X \), then \( \text{id}_F = \gamma^{1_X} \).
- Also, we can take compositions of functor morphisms
  \( \tau : G \Rightarrow H, \quad \gamma : F \Rightarrow G \Rightarrow \tau \circ \gamma : F \Rightarrow H, \quad (\tau \circ \gamma)_X = \tau_X \circ \gamma_X \)

Exercise: Check that \( \tau \circ \gamma \) is indeed a functor morphism.
(hint: stack the commutative diagrams).

If \( F = F_X, \ G = F_{X'}, \ H = F_{X''} \) for \( X,X',X'' \in \text{Ob}(C) \), & \( g \in \text{Hom}_C(X,X') \),
\( g' \in \text{Hom}_C(X',X) \), then
\[
(1) \quad \gamma^g \circ \gamma^{g'} = \gamma^{g \circ g'}
\]
equality of functor morphisms \( F_X \Rightarrow F_{X''} \). Indeed, for \( \psi \in \text{Hom}_C(X,Y) \)
\[
\gamma^\psi \circ \gamma^g(\psi) = \gamma^{\psi \circ g}(\psi \circ g) = (\psi \circ g) \circ \gamma^g \gamma^\psi = \gamma^{\psi \circ g} \gamma^g(\psi).
\]
Remark: If \( C \) is small (\( \text{Ob}(C) \) is a set), then the collection of functor morphisms \( F \Rightarrow G \), denoted \( \text{Hom}_{\text{Fun}}(F, C) \), is a set. The category of functors \( \text{Fun}(C, D) \): objects are functors, morphisms = morphisms of functors.

We then can consider a functor \( C^{\text{op}} \to \text{Fun}(C, \text{Set}) \)

\[ X \mapsto F_X \quad \forall X \in \text{Ob}(C), g \in \text{Hom}_{C^{\text{op}}}(X, Y) = \text{Hom}_C(Y, X) \mapsto \gamma^g : F_X \Rightarrow F_Y. \]

The Yoneda lemma says that

\[ \text{Hom}_{C^{\text{op}}}(X, Y) \to \text{Hom}_{\text{Fun}(C, \text{Sets})}(F_X, F_Y) \]

Functors that induce isomorphisms between the morphism sets have a name: fully faithful (faithful refers to the injectivity, and "full" to the surjectivity).

1.2) Yoneda vs functor isomorphisms

Let \( C, D \) be categories & \( F, C : C \to D \) be functors.

By def'n, \( \gamma : F \Rightarrow G \) is a functor isomorphism if it has a (two-sided) inverse: \( \gamma^{-1} : G \Rightarrow F \) s.t. \( \gamma \circ \gamma^{-1} = \text{id}_G \), \( \gamma^{-1} \circ \gamma = \text{id}_F \), which, as usual, is unique if it exists.

Lemma: 1) \( \gamma \) is an isomorphism \( \iff \gamma_Y \in \text{Hom}_D(F(Y), C(Y)) \) is an isomorphism for all \( Y \in \text{Ob}(C) \).

2) Let \( D = \text{Sets}, F = F_X, C = F_X, g \in \text{Hom}_C(X, X) \). Then \( \gamma^g : F_X \Rightarrow F_X \) is an isomorphism (of functors) \( \iff g \) is an isomorphism (of objects).

Proof: 1) \( \Rightarrow \): from def'n.

\( \Leftarrow\): define \( (\gamma^{-1})_Y := \hat{\delta}_Y \).
Exercise: check the axiom of a functor morphism.

Then \( y^{-1} \) is automatically inverse of \( y \).

2): exercise (from Yoneda lemma & \( y^g \circ g = y^g \circ y^g \)). □

2) Objects representing functors

Definition: Let \( F: C \to \text{Sets} \) be a functor. We say \( X \in \text{Ob}(C) \) represents \( F \) if \( F \) is isomorphic to \( F_X \).

A representing object may fail to exist (this is the case for the forgetful functor from finite groups to sets (cf. HW4). If a representing object exists, we say that \( F \) is representable.

Lemma: An object \( X \), representing \( F \) is unique up to isomorphism if it exists.

Proof:

Let \( X, X' \in \text{Ob}(C) \) represent \( F: F_X \cong F \leq F_{X'}, \sim \) a functor isomorphism \( y: \cong F_X \cong F_{X'} \). By 2) of Lemma in Sec 1.2, \( X, X' \) are isomorphic. □

Example: The forgetful functor \( \text{For}: C = \text{Groups} \to \text{Sets} \) is represented by \( \mathbb{Z} \). Indeed, for any group \( G \) we have a bijection (of sets) \( \varphi_G: \text{Hom}_{\text{Groups}}(\mathbb{Z}, G) \to G, \varphi \mapsto \varphi(1) \).

\( \varphi_G \) is a functor morphism – what we need to check is that \( \forall \) group homomorphism \( f: C \to H \), the diagram
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Since $y_G$ is bijective to $G$, by (2) of Lemma in Sec 1.2, is a functor isomorphism implying our claim.

Now let's use the Yoneda Lemma to compute $\text{End}(\text{For}) := \text{Hom}_{\text{Fun}}(\text{For}, \text{For})$ as a monoid w.r.t. composition. By Yoneda, it's $\text{End}_{\text{Groups}}(\mathbb{Z})$ (with multiplication $gg' := g \circ g$). We have $\text{End}_{\text{Groups}}(\mathbb{Z}) \cong \text{For}(\mathbb{Z}) = \mathbb{Z}$. to $m \in \mathbb{Z}$ we assign $g_m : \mathbb{Z} \rightarrow \mathbb{Z}, x \mapsto mx$. We conclude that $\text{End}_{\text{Fun}}(\text{For})$ is the monoid $\mathbb{Z}$ w.r.t. multiplication.

**Exercise:** Let $y[m]$ be the endomorphism $\text{For} \Rightarrow \text{For}$ corresponding to $m \in \mathbb{Z}$. Show that $y[m]_G$ is given by $y[m]_G(x) = x^m$, $\forall x \in G$.

3) **Products in categories.**

The concept of a representing object allows to carry constructions from the category of sets to a general category. Here we consider a basic such construction - products.

Recall that in our usual categories: Sets, Groups, Rings, $A$-mod we have the notion of direct product. In all of them, this is characterized by universal property. E.g. if $A_i, A_\ell$ are rings, then $A \times A_\ell$ is a ring w. ring homomorphisms $\pi_i : A \times A_\ell \rightarrow A_i$ s.t. $\forall$ rings $B$ w. homomorphisms $\varphi : B \rightarrow A_i$ $\exists! \varphi : B \rightarrow A \times A_\ell$ w.
Define their product \( F_1 \times F_2 \) by

- Sending \( X \in \text{Ob}(C) \) to \( F_1(X) \times F_2(X) \)
- Sending \( \phi \in \text{Hom}_C(X,Y) \) to \( F_1(\phi) \times F_2(\phi) : F_1(X) \times F_2(X) \rightarrow F_1(Y) \times F_2(Y) \).

\( F_1 \times F_2 \) is a functor, to check the axioms is an exercise.

Now take \( X_1, X_2 \in \text{Ob}(C) \) and let \( F_i^\text{op} = F_i^\text{op} \) be the \( \text{Hom} \) functor \( \text{Hom}^\text{op}(X, \cdot) \) (= \( \text{Hom}(\cdot, X) \)) : \( C^\text{op} \rightarrow \text{Sets} \).

**Definition:** If \( X \in \text{Ob}(C) \) represents \( F_1 \times F_2 \), then we say that \( X \) is the **product** \( X_1 \times X_2 \).

The to Lemma in Sec 2, \( X_1 \times X_2 \) is unique (up to iso) if it exists (that may fail to be the case).

Here's an alternative characterization of products.

**Lemma:** 1) There are \( \pi_i : X \rightarrow \text{Hom}_C(X, X_i) \) s.t.

\( (*) \quad \forall Y \in \text{Ob}(C), \phi_i \in \text{Hom}_C(Y, X_i), i = 1, 2, \exists! \phi \in \text{Hom}_C(Y, X) | \phi_i = \pi_i \circ \phi. \)

2) Conversely, let \( (X, \pi_1, \pi_2) \) satisfy \( (*) \). Then \( X = X_1 \times X_2 \).

Note that \( (*) \) is the usual universal property of direct products.

In particular, in our usual categories: Sets, Groups, Rings, \( A \)-Mod products are just direct products - and they exist \( \forall X_1, X_2 \).
Proof (of Lemma): 1) Let \( y : F^\text{op}_{x} \Rightarrow F^\text{op}_{x} \times F^\text{op}_{x} \Rightarrow \) for \( x \in \text{Ob}(C) \)
\( y_{x} : \text{Hom}_{C}(Y, X) \Rightarrow \text{Hom}_{C}(Y, X) \times \text{Hom}_{C}(Y, X) \). We define \((y^{x}, y^{x}) \in \text{Hom}_{C}(X, X) \times \text{Hom}_{C}(X, X)\) as \( y_{x}(1_{x}) \). As in Step 3 of the proof of Yoneda Lemma in Sec 12, if \( \varphi \in \text{Hom}_{C}(Y, X) \), we have commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{C}(X, X) & \xrightarrow{? \circ \varphi} & \text{Hom}_{C}(Y, X) \\
 y_{x} \downarrow & & \downarrow y_{x} \\
 \text{Hom}_{C}(X, X) \times \text{Hom}_{C}(X, X) & \xrightarrow{(? \circ \varphi, ? \circ \varphi)} & \text{Hom}_{C}(Y, X) \times \text{Hom}_{C}(Y, X)
\end{array}
\]
which we apply to \( 1_{x} \) getting: \( y_{x}(\varphi) = (y^{x} \circ \varphi, y^{x} \circ \varphi) \). (*) follows b/c \( y_{x} \) is a bijection: \( \forall \varphi, \psi \in Y_{X}, \exists ! \varphi \odot \psi \) w. \( y_{x}(\varphi) = (\varphi, \psi) \)

2) We essentially reverse the argument. Define
\( y_{x} : \text{Hom}_{C}(Y, X) \rightarrow \text{Hom}_{C}(Y, X) \times \text{Hom}_{C}(Y, X) \), \( \varphi \rightarrow (y^{x} \circ \varphi, y^{x} \circ \varphi) \)
By (*), \( y_{x} \) is a bijection. To check that \( y_{x} \) constitute a functor morphism is an exercise. So \( y = (y_{x}) \) is a functor isomorphism \( \square \)

Remark: Note that we use the Yoneda Lemma when we talk about representing objects incl. products: it guarantees (via its consequence in Sec 1.2) that representing objects are uniquely determined if they exist.
**BONUS: Category equivalences.**

Our question here: when are two categories the “same”? Turns out, functor isomorphisms play an important role in answering this question.

Before we address this, we should discuss an easier question: when are two sets the same? Well, they are literally the same if they consist of the same elements. But this definition is quite useless: sets arising from different constructions won't be the same in this sense. Of course, we use isomorphic instead of being literally the same.

Now back to categories. Again, being the same is useless.

How about being isomorphic? Turns out, this is not useful. Let's see why. Let $C, D$ be categories. We say that either $C, D$ are isomorphic if there are functors $F: C \to D$, $G: D \to C$ such that $FG \cong \text{Id}_D$, $GF \cong \text{Id}_C$. The issue is: two functors obtained by different constructions are never the same (compare to sets). The solution: replace “equal” with “isomorphic” (as functors).

**Definition:** • Functors $F: C \to D$, $G: D \to C$ are quasi-inverse if $FG \cong \text{Id}_D$, $GF \cong \text{Id}_C$ (isomorphic).

• We say $C, D$ are equivalent if there are quasi-inverse functors (called equivalences) $F: C \to D$, $G: D \to C$. 
Now we are going to state a general result. For this we need another definition.

**Definitions:** A functor $F: C \to D$ is called
- **fully faithful** if $\forall X, X' \in \text{Ob}(C) \Rightarrow f \mapsto F(f)$ is a bijection $\text{Hom}_C(X, X') \cong \text{Hom}_D(F(X), F(X'))$
- **essentially surjective** if $\forall Y \in \text{Ob}(D) \exists X \in \text{Ob}(C)$ such that $F(X)$ is isomorphic to $Y$

**Thm:** A functor $F: C \to D$ is an equivalence $\iff$ $F$ is fully faithful & essentially surjective.

We won't prove this, but we will give an example - that illustrates how the proof works in general.

**Example:** Consider the category $D = \text{F-Vect}_k$ of finite dimensional vector spaces over a field $k$ and its full subcategory $C$ w. objects $F^n(n \geq 0)$. We claim that the inclusion functor $F: C \to D$ is an equivalence. It's fully faithful by def'n and the claim that it's essentially surjective.

Now we produce a quasi-inverse functor $G$. In each $V \in \text{Ob}(D)$ we fix a basis, which leads to an isomorphism $\rho_V: V \to F^n$. We define $G(V)$ as $F^n$. For a linear map $f: U \to V$ w.
\[ \dim U = m, \dim V = n \] we set \( g(f) := y^* \circ f \circ y \).

**Exercise:** Check \( g \) is a functor.

Now we are going to simplify our life a bit & assume that \( y_{\text{id}} : \text{id} \to \text{id} \) is the identity.

**Exercise:** \( GF : C \to C \) is the identity functor (not just isomorphic to it).

Now we produce a functor isomorphism \( \gamma : \text{id} \Rightarrow GF \).
So we need to have \( \gamma_V : V \to \text{id} \circ \text{id} \circ V \) and this is the isomorphism from above.

**Exercise:** prove that \( \gamma \) is indeed a functor morphism.

Then \( \gamma \) is an isomorphism of functors. So \( F \) is indeed a category equivalence.

Another exercise: prove that the duality functor \( \ast \) is an equivalence \( \text{FinVect}_{Fd} \to \text{FinVect}_{Fd}^{\text{op}} \).