Lecture 14: Categories, functors & functor morphisms IV.

1) Coproducts.
2) Adjoint functors.
BONUS: Adjunction unit & counit.

Refs: [R], Section 4.1; [HS], Sections II.5, II.7

1) Coproducts.

Let $\mathcal{C}$ be a category.

Definition: Let $X_1, X_2 \in Ob(\mathcal{C})$. Their coproduct (that we denote by $X_1 * X_2$) is the product in $\mathcal{C}^{opp}$. I.e.

(I) $F_{X_1 * X_2} \xrightarrow{\sim} F_{X_1} \times F_{X_2}$, where for $X \in Ob(\mathcal{C})$ we write F_X for the Hom functor $Hom_{\mathcal{C}}(X, \cdot): \mathcal{C} \longrightarrow Sets$.

(II) equivalently, there are morphisms $X_i \xrightarrow{\iota_i} X_1 * X_2$, $i=1,2$, s.t. $\forall Y \in Ob(\mathcal{C})$ & $X_i \xrightarrow{\varphi_i} Y$, $i=1,2$, $\exists! \ \varphi: X_1 * X_2 \to Y \mid \varphi_i = \varphi \circ \iota_i$

The equivalence of (I) & (II) follows from Lemma in Sec 3 of Lec 13 (where we replace $\mathcal{C}$ w. $\mathcal{C}^{opp}$).

Examples: 1) Let $\mathcal{C} = Sets$. Then $X_1 * X_2 = X_1 \sqcup X_2$ (and ι_i is the natural inclusion). (II) is manifest.

2) Let $\mathcal{C} = A\text{-mod}$. Then $X_1 * X_2 = X_1 \oplus X_2$: for any A-module Y, have a natural isomorphism

$\eta_Y: \mathrm{Hom}_A(X_1 \oplus X_2, Y) \xrightarrow{\sim} \mathrm{Hom}_A(X_1, Y) \times \mathrm{Hom}_A(X_2, Y)$

see Sec 1.2 of Lec 4. To check (η_Y) is a functor morphism is an exercise.

Later on we will describe the coproduct in the category of commutative A-algebras (this will be the tensor product).

2) Adjoint functors.

Let $\mathcal{C}, \mathcal{D}$ be categories. Being "adjoint" is the most important relationship that a functor $\mathcal{C} \to \mathcal{D}$ can have with a functor $\mathcal{D} \to \mathcal{C}$.

2.1) Definition

Let $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ be functors.

Definition: F is left adjoint to G (and G is right adjoint to F) if:

$\forall X \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D})$ $\exists$ bijection $\eta_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ s.t.

(1) $\forall$ $X, X' \in Ob(\mathcal{C})$, $Y \in Ob(\mathcal{D})$, $X' \xrightarrow{\varphi} X$ $(\rightsquigarrow F(X') \xrightarrow{F(\varphi)} F(X))$ the following is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\eta_{X,Y}", "\sim"'] \arrow[d, "?\circ F(\varphi)"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \arrow[d, "?\circ\varphi"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X'),Y) \arrow[r, "\eta_{X',Y}", "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(X',G(Y))
\end{tikzcd}$$

(2) $\forall\, Y, Y' \in Ob(\mathcal{D})$, $Y \xrightarrow{\psi} Y'$, $X \in Ob(\mathcal{C})$, the following is comm've

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\eta_{X,Y}", "\sim"'] \arrow[d, "\psi\circ ?"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \arrow[d, "G(\psi)\circ ?"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X),Y') \arrow[r, "\eta_{X,Y'}", "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y'))
\end{tikzcd}$$

For us the main reason to consider adjoint functors is that we can get interesting functors as adjoints to boring (e.g. forgetful) functors.

2.2) Examples.

Example 1:

Let A be a comm've ring. Let G be $\mathrm{For}: A\text{-}\mathrm{Mod} \to \mathrm{Sets}$
$F := \mathrm{Free}: \mathrm{Sets} \to A\text{-}\mathrm{Mod}$ (see Sec 1 of Lec 12), $\mathrm{Free}(I) = A^{\oplus I}$
& for $f: I \to J$ (map of sets): $\mathrm{Free}(f)(e_i) = e_{f(i)}$.

Claim: F is left adjoint to G

Below we write Maps for $\mathrm{Hom}_{\mathrm{Sets}}$ (& Hom_A for $\mathrm{Hom}_{A\text{-}\mathrm{Mod}}$)

• construct $\eta_{I,M}: \mathrm{Hom}_A(A^{\oplus I}, M) \xrightarrow{\sim} \mathrm{Maps}(I, M)$

$$\tau \overset{\eta_{I,M}}{\longmapsto} [i \mapsto \tau(e_i)]$$

• check comm've diagram (1): $\forall$ maps $\varphi: I \to J$

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus J}, M) \arrow[r, "\eta_{J,M}", "\sim"'] \arrow[d, "? \circ \mathrm{Free}(\varphi)"] & \mathrm{Maps}(J, M) \arrow[d, "? \circ \varphi"] \\
\mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\eta_{I,M}", "\sim"'] & \mathrm{Maps}(I, M)
\end{tikzcd}$$

$\downarrow \longrightarrow$: $\tau \longmapsto [\text{unique } \tau': A^{\oplus I} \to M \text{ s.t. } \tau'(e_i) := \tau(e_{\varphi(i)})]$

$$\longmapsto [i \mapsto \tau(e_{\varphi(i)})]$$

$\longrightarrow \downarrow$: $\tau \longmapsto [j \mapsto \tau(e_j)] \longmapsto [i \mapsto \tau(e_{\varphi(i)})]$

Check (2): for $\psi \in \mathrm{Hom}_A(M, N)$, the following is commutative

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\eta_{I,M}", "\sim"'] \arrow[d, "\psi \circ ?"] & \mathrm{Maps}(I, M) \arrow[d, "\psi \circ ?"] \\
\mathrm{Hom}_A(A^{\oplus I}, N) \arrow[r, "\eta_{I,N}", "\sim"'] & \mathrm{Maps}(I, N)
\end{tikzcd}$$

where now ψ is viewed as map of sets

Both $\longrightarrow \downarrow$ & $\downarrow \longrightarrow$ send τ to $[i \mapsto \psi(\tau(i))]$.

The adjunction is established

Example 2: Let A be a commutative ring, $S\subset A$ multiplicative subset $\rightsquigarrow A[S^{-1}]$ w. ring homomorphism $\iota: A\longrightarrow A[S^{-1}]$. So we get functors $F:=\bullet[S^{-1}]: A\text{-Mod}\longrightarrow A[S^{-1}]\text{-Mod}$ and $G:=\iota^*: A[S^{-1}]\text{-Mod}\longrightarrow A\text{-Mod}$ (pullback = forgetful functor). We *claim* that F is left adjoint to G.

For $M\in Ob(A\text{-Mod})$, $N\in Ob(A[S^{-1}]\text{-Mod})$, we have a bijection

$$\eta_{M,N}: \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}],N) \xrightarrow{\sim} \mathrm{Hom}_A(M,N) \quad \text{(we omit } \iota^* \text{ from the notation)}$$
$$\psi \longmapsto \psi\circ\iota_M$$

(where $\iota_M: M\to M[S^{-1}]$, $m\mapsto \frac{m}{1}$), this is an equivalent way to state the universal property of localization from Sec 2.2 of Lec 9.

Now we need to show that diagrams (1) and (2) from Sec 2.1 commute. Let's check (1): for $\tau\in\mathrm{Hom}_A(M_1,M_2)$ need to show

$$\begin{tikzcd}
\mathrm{Hom}_{A[S^{-1}]}(M_2[S^{-1}],N) \arrow[r, "?\circ\iota_{M_2}"] \arrow[d, "?\circ\tau[S^{-1}]"] & \mathrm{Hom}_A(M_2,N) \arrow[d, "?\circ\tau"] \\
\mathrm{Hom}_{A[S^{-1}]}(M_1[S^{-1}],N) \arrow[r, "?\circ\iota_{M_1}"] & \mathrm{Hom}_A(M_1,N)
\end{tikzcd}$$

commutes

$\to\downarrow$ gives $?\circ\overbrace{\iota_{M_2}\circ\tau}^{M_1\to N}$, and $\downarrow\to$ gives $?\circ\overbrace{\tau[S^{-1}]\circ\iota_{M_1}}^{M_1\to N}$; for $m\in M_1$, have $\iota_{M_2}\circ\tau(m)=\frac{\tau(m)}{1}$, $\tau[S^{-1}]\circ\iota_{M_1}(m)=\tau[S^{-1}](\frac{m}{1})=\frac{\tau(m)}{1}$. So $\iota_{M_2}\circ\tau=\tau[S^{-1}]\circ\iota_{M_1}$, and the diagram indeed commutes.

Diagram (2) becomes: for $\zeta\in\mathrm{Hom}_{A[S^{-1}]}(N_1,N_2)$:

$$\begin{tikzcd}
\mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}],N_1) \arrow[r, "\eta_{M,N_1}=?\circ\iota_{M_1}"] \arrow[d, "\zeta\circ ?"] & \mathrm{Hom}_A(M,N_1) \arrow[d, "\zeta\circ ?"] \\
\mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}],N_1) \arrow[r, "\eta_{M,N_2}=?\circ\iota_{M_1}"] & \mathrm{Hom}_A(M,N_2)
\end{tikzcd}$$

[$\iota^*(\zeta)=\zeta$ b/c ι^* is forgetful]

It commutes.

2.3) Uniqueness.

Proposition: If $F^1, F^2: \mathcal{C} \to \mathcal{D}$ are left adjoint to $G: \mathcal{D} \to \mathcal{C}$, then $F^2 \xRightarrow{\sim} F^1$.

Proof: Suppose we have $\eta^i_{X,Y}: Hom_{\mathcal{D}}(F^i(X), Y) \xrightarrow{\sim} Hom_{\mathcal{C}}(X, G(Y))$ that make (1) & (2) comm've $\rightsquigarrow$
$\eta_{XY} := (\eta^2_{X,Y})^{-1} \circ \eta^1_{X,Y}: Hom_{\mathcal{D}}(F^1(X), Y) \xrightarrow{\sim} Hom_{\mathcal{D}}(F^2(X), Y)$ that make the following analogs of (1) and (2) commutative (exercise)

(1) $\forall\ X' \xrightarrow{\varphi} X$:

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F^1(X), Y) \arrow[r, "\cancel{\eta_{X,Y}}\ ?\circ\tau_X"] \arrow[d, "?\circ F^1(\varphi)"] & Hom_{\mathcal{D}}(F^2(X), Y) \arrow[d, "?\circ F^2(\varphi)"] \\
Hom_{\mathcal{D}}(F^1(X'), Y) \arrow[r, "\cancel{\eta_{X',Y}}\ ?\circ\tau_{X'}"] & Hom_{\mathcal{D}}(F^2(X'), Y)
\end{tikzcd}$$

(2) $\forall\ Y \xrightarrow{\psi} Y'$

$$\begin{tikzcd}
Hom_{\mathcal{D}}(F^1(X), Y) \arrow[r, "\cancel{\eta_{X,Y}}\ ?\circ\tau_X"] \arrow[d, "\psi\circ ?"] & Hom_{\mathcal{D}}(F^2(X), Y) \arrow[d, "\psi\circ ?"] \\
Hom_{\mathcal{D}}(F^1(X), Y') \arrow[r, "\cancel{\eta_{X,Y'}}\ ?\circ\tau_X"] & Hom_{\mathcal{D}}(F^2(X), Y')
\end{tikzcd}$$

Fix X, look at (2): it tells us that $\eta_{X,\bullet}$ is a functor morphism (and hence isomorphism - b/c each $\eta_{X,Y}$ is bijection) between $Hom_{\mathcal{D}}(F^1(X), \bullet)$ & $Hom_{\mathcal{D}}(F^2(X), \bullet)$. By Yoneda Lemma, have the unique isomorphism $\tau_X \in Hom_{\mathcal{D}}(F^2(X), F^1(X))$ s.t. $\eta_{X,Y} = ? \circ \tau_X$. Plug this into diagrams (1) & (2).

We now show that τ is a functor morphism $F^2 \Rightarrow F^1$ (hence an isom'm b/c each τ_X is an iso): we need to show the diagram

$$(*)\quad \begin{tikzcd} F^2(X') \arrow[r, "\tau_{X'}"] \arrow[d, "F^2(\varphi)"] & F^1(X') \arrow[d, "F^1(\varphi)"] \\ F^2(X) \arrow[r, "\tau_X"] & F^1(X) \end{tikzcd}$$

is commutative. Indeed, (1) is commutative, so
$\psi \circ (\tau_X \circ F^2(\varphi)) = \psi \circ (F^1(\varphi) \circ \tau_{X'})$ $\forall\, Y \in Ob(\mathcal{D})$, $\psi \in Hom_{\mathcal{D}}(F^1(X), Y)$.
Take $Y = F^1(X)$, $\psi = 1_{F^1(X)}$ & get that $(*)$ is commutative. $\square$

2.4) Remarks.

1) Fix X & consider composition of functors

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{Hom_{\mathcal{C}}(X,\cdot)} Sets$$

If F is left adj't to G, then $F(X)$ represents this composition via isomorphism $\eta_{X,\cdot}$, see Diagram (2) in Sec 2.1.

2) We can view $Hom_{\mathcal{C}}(\cdot, ?)$ as a functor $\mathcal{C}^{opp} \times \mathcal{C} \longrightarrow Sets$
Similarly for $\mathcal{D} \rightsquigarrow$ compositions $\mathcal{C}^{opp} \times \mathcal{D} \longrightarrow Sets$
$Hom_{\mathcal{D}}(F(\cdot), ?)$, $Hom_{\mathcal{C}}(\cdot, G(?))$
Diagrams (1) & (2) combine to show that [F is left adj't to G] $\Longleftrightarrow$ the two functors above are isomorphic (via $\eta_{\cdot,?}$)

3) Many categorical notions (including adjunction) have parallels

in Linear Algebra. Let $\mathbb{F}$ be a field. There's a distinguished vector space, $\mathbb{F}$. For a finite dimensional vector space V, we can consider its dual, V^*. Have a vector space pairing $\langle\cdot,\cdot\rangle: V^*\times V \to \mathbb{F}$, $\langle\alpha, v\rangle := \alpha(v)$. And for a linear map $A: V\to W$ we can consider its adjoint, the unique linear map $A^*: W^*\to V^*$ s.t. $\langle\beta, Av\rangle =$ $= \langle A^*\beta, v\rangle$.

Here are analogs of this for categories. An analog of $\mathbb{F}$ is Sets. An analog of passing from V to V^* is passing from a category $\mathcal{C}$ to the category $\mathcal{C}^{opp}$. An analog of linear maps $U\to V$ is functors $\mathcal{C}\to\mathcal{D}$. An analog of the pairing $V^*\times V\to\mathbb{F}$ is $\mathrm{Hom}_{\mathcal{C}}(\cdot,?): \mathcal{C}^{opp}\times\mathcal{C}\to$ Sets. Finally an analog of $\langle A^*\beta, v\rangle =$ $\langle\beta, Av\rangle$ is our definition of adjoint functors.

There are differences as well. First, a functor $\mathcal{C}\to\mathcal{D}$ is the same thing as a functor $\mathcal{C}^{opp}\to\mathcal{D}^{opp}$ but there's no way to get a linear map $V^*\to W^*$ from $V\to W$. Also adjunction of functors is very sensitive to the sides (the left adjoint of G may not be isomorphic to the right adjoint - moreover exactly one of those may fail to exist), while for linear maps this issue doesn't arise.

BONUS: adjunction unit & counit.

Let $F: \mathcal{C} \to \mathcal{D}$ be left adjoint to $G: \mathcal{D} \to \mathcal{C}$. We claim that this gives rise to functor morphisms: the adjunction unit $\varepsilon: Id_{\mathcal{C}} \Longrightarrow GF$ & counit $\eta: FG \Longrightarrow Id_{\mathcal{D}}$.

We construct ε and leave η as an exercise.

Consider $X_1, X_2 \in Ob(\mathcal{C})$. Then we have the bijection

$$\eta_{X_1, F(X_2)}: Hom_{\mathcal{D}}(F(X_1), F(X_2)) \xrightarrow{\sim} Hom_{\mathcal{C}}(X_1, GF(X_2))$$

Note that F gives rise to a map $Hom_{\mathcal{C}}(X_1, X_2) \to Hom_{\mathcal{D}}(F(X_1), F(X_2))$. Composing this map w. the bijection $\eta_{X_1, F(X_2)}$ we get

$$\varepsilon_{X_1, X_2}: Hom_{\mathcal{C}}(X_1, X_2) \longrightarrow Hom_{\mathcal{C}}(X_1, GF(X_2)).$$

Now we can argue as in the proof of Proposition 1.3 to see that $\exists!\ \varepsilon: Id_{\mathcal{C}} \Rightarrow GF$ s.t. $\varepsilon_{X_1, X_2}(\varphi) = \varepsilon_{X_2} \circ \varphi$.

A natural question to ask is: for two functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ & functor morphisms $\varepsilon: Id_{\mathcal{C}} \Rightarrow GF$, $\eta: FG \Rightarrow Id_{\mathcal{D}}$ when is F left adjoint to G (& ε, η unit & counit).

Very Premium Exercise: TFAE

a) F is left adjoint to G w. unit ε & counit η

b) The composed morphisms $F \Rightarrow FGF \Rightarrow F$, $G \Rightarrow GFG \Rightarrow G$ induced by ε, η (cf. Problem 8 in HW3) are the identity endomorphisms (of F & G).

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