Lecture 15: tensor products, I

1) Definition of tensor products of modules.

2) Construction.

Ref: [AM], Section 2.7.

BONUS: Tensor products over noncommutative rings.

1) Definition of tensor products of module

1.0) Bilinear maps: Let $A$ be a commutative ring, $M_1, M_2, N$ be $A$-modules. Recall (Sec 2.3 of Lec 3) that $\beta: M_1 \times M_2 \rightarrow N$ is $A$-bilinear if $\beta(m_1, m_2): M_1 \rightarrow N$ is $A$-linear $\forall m_1 \in M_1$ & $\beta(m_1, m_2): M_2 \rightarrow N$ is $A$-linear $\forall m_2 \in M_2$. Consider the set $\text{Bilin}_A(M_1 \times M_2, N) = \{ A$-bilinear maps $M_1 \times M_2 \rightarrow N \}$.

Digression: Why should we care about bilinear maps - b/c they are everywhere!

1) Linear algebra: for an $\mathbb{F}$-vector space $V$ can talk about bilinear forms := bilinear maps $V \times V \rightarrow \mathbb{F}$, fundamentally important in linear algebra & beyond.

2) if $M$ is an $A$-module $\Rightarrow$ mult'n map $A \times M \rightarrow M$ is $A$-bilinear.

3) the composition map $\text{Hom}_A(M, L) \times \text{Hom}_A(N, M) \rightarrow \text{Hom}_A(L, N)$, where $L, M, N$ are $A$-modules is $A$-bilinear, Problem 6 in HW 1.

4) If $B$ is an $A$-algebra, then the product map $B \times B \rightarrow B$ is $A$-bilinear.
Observation: \( F_{M_0, M_1} : \text{Bilin}_A(M_0 \times M_1, \cdot) \) is actually a functor \( A \text{-Mod} \to \text{Sets} \): to \( \psi \in \text{Hom}_A(N, N') \) we assign
\[
F_{M_0, M_1}(\psi) : \text{Bilin}_A(M_0 \times M_1, N) \to \text{Bilin}_A(M_0 \times M_1, N')
\]

Exercise: Show \( \psi \circ \beta \) is \( A \)-bilinear & \( F_{M_0, M_1} \) is indeed a functor \( A \text{-Mod} \to \text{Sets} \).

1.1) Definition of tensor product:
Definition: By the tensor product \( M_1 \otimes_A M_2 \) we mean a representing object for \( \text{Bilin}_A(M_1 \times M_2, \cdot) \) i.e. want a functor isomorphism \( \text{Hom}_A(M_1 \otimes_A M_2, \cdot) \cong \text{Bilin}_A(M_1 \times M_2, \cdot) \).

Similarly to products (Sec 3 of Lec 13) we can equivalently define tensor products via a universal property: this is an \( A \)-module \( M_1 \otimes_A M_2 \) w. a bilinear map \( M_1 \times M_2 \to M_1 \otimes_A M_2 \), \((m_1, m_2) \mapsto m_1 \otimes m_2\), w. the following universal property:

forall \( A \)-module \( N \) \& \( A \)-bilinear map \( \beta : M_1 \times M_2 \to N \) \exists!
\( A \)-linear map \( \beta' : M_1 \otimes_A M_2 \to N \) s.t. \( \beta'(m_1 \otimes m_2) = \beta(m_1, m_2) \)
i.e. the following is commutative
\[
\begin{array}{ccc}
(m_1, m_2) & \mapsto & (m_1 \otimes m_2) \\
\downarrow & & \downarrow \\
M_1 \times M_2 & \xrightarrow{\beta} & M_1 \otimes_A M_2 \\
\downarrow & & \downarrow \\
M_1 \otimes_A M_2 & \xrightarrow{\beta'} & N
\end{array}
\]

In terms of \( \psi \), this universal bilinear map is \( \psi^{M_1 \otimes_A M_2}(m_1 \otimes m_2) \).
The tensor product is unique (if exists) in the following sense: if \( M_1 \otimes_A M_2 \) is another tensor product w. bilinear map \((m_1, m_2) \mapsto m_1 \otimes m_2\), then \( \exists! \) \( A \)-module isomorphism \( I : M_1 \otimes_A M_2 \rightarrow M_1 \otimes_A M_2 \) s.t. \((m_1 \otimes m_2) = m_1 \otimes m_2 \forall m_1 \in M_1\).

2) Construction

The main goal of this section is to give a constructive proof of

**Theorem:** \( M_1 \otimes_A M_2 \) exists for all \( A \)-modules \( M_1, M_2 \).

2.1) Case 1: \( M_1 \) is free.

**Lemma:** For any set \( I \), \( A^{\oplus I} \otimes_A M \rightarrow M^{\oplus I} \) w.

\[(a_i)_{i \in I} \otimes m = (a_i \cdot m)_{i \in I} \forall (a_i) \in A^{\oplus I}, m \in M.\]

**Proof:**

As for \( A \)-linear map from free modules, \( \beta \in \text{Bilin}_A(A^{\oplus I}, M; N) \) is uniquely determined by \( \beta_\alpha \in \text{Hom}_A(M, N) \), \( \beta_\alpha(m) = \beta(e_\alpha, m) \), via \( \beta((a_\alpha)_{\alpha \in I}) = \sum_{\alpha \in I} a_\alpha \beta_\alpha(m) \). Note that the r.h.s. is well-defined if \( \{\alpha \in I | a_\alpha \neq 0\} \) is finite. This gives a bijection \( \text{Bilin}_A(A^{\oplus I}, M; N) \rightarrow \text{Hom}_A(M, N)^{\times I} \), \( \beta \mapsto (\beta(e_\alpha, \cdot)) \). On the other hand, by Sec 1.2 in Sec 4, have bijection \( \text{Hom}_A(M, N)^{\times I} \rightarrow \text{Hom}_A(M^{\oplus I}, N), (\beta_\alpha) \mapsto [(m_i)_{i \in I} \mapsto \sum_{i \in I} \beta_\alpha(m_i)] \). Consider the composition,

\[\varphi' : \text{Bilin}_A(A^{\oplus I}, N) \rightarrow \text{Hom}_A(M^{\oplus I}, N), \beta \mapsto [(m_i) \mapsto \sum \beta(e_\alpha, m_i)]\]

These bijections constitute a functor isomorphism (**exercise**). This shows that \( M^{\oplus I} \) is indeed \( A^{\oplus I} \otimes_A M \) (take \( \varphi = (\varphi')^{-1} \)). Note that
\[ \beta((a_i), m) = (a_i m)_{i \in I} \text{ is in } \text{Bilin}(A^{\otimes I}, M; M^{\otimes I}) \] & satisfies \( \varphi'(\beta) = \iota_{M^{\otimes I}} \) (exercise) so indeed, \( (a_i) \otimes m = (a_i m). \) \( \Box \)

2.2) Step 2: arbitrary \( M'_1. \)

Let \( M'_1, \) \( M'_2 \) be \( A \)-modules s.t. \( M'_1 \otimes A M'_2 \) exists. Let \( K'_1 \subset M'_1 \) be an \( A \)-submodule \( \sim M'_1 = M'_1 / K'_1 \) & \( \phi'_1 : M'_1 \rightarrow M'_2. \) Inside \( M'_1 \otimes A M'_2 \) consider submodule \( K_1 = \text{Span}_A (k \otimes m_2 | k \in K'_1, m_2 \in M'_2) \)

Projection \( \phi_2 : M'_1 \otimes A M'_2 \rightarrow M'_1 \otimes A M'_2 / K. \)

**Claim:** \( M'_1 \otimes A M'_2 / K \) is the tensor product \( M'_1 \otimes A M'_2 \) & for \( m'_2 = \phi_2^{-1}(m'_2) \subset M'_2 \) \& \( m_2 \in M'_2 \Rightarrow m'_1 \otimes m'_2 = \phi_1(m'_1 \otimes m'_2). \)

How this implies Thm: \( M'_2 \) is a quotient of \( A^{\otimes I} \) for some \( I \) (i.e. \( (m_i)_{i \in I} \) are generators of \( M'_2 \), then \( \psi : A^{\otimes I} \rightarrow M'_2, (a_i) \mapsto \sum_{i \in I} a_i m_i, \) is surjective) \( A^{\otimes I} \otimes A M \) exists by Step 1. Now in Claim take \( M'_2 = A^{\otimes I} \) to see that \( M'_1 \otimes A M'_2 \) exists.

**Proof of Claim. Exercise:** \( m_1 \otimes m_2 \) is well-defined (independent of choice of \( m'_1 \)) \& gives a bilinear map \( M'_1 \times M'_2 \rightarrow A^{\otimes I} \otimes M'_2 / K. \)

Now we only need to check univ. property (II): \( \forall \) bilinear \( \beta : M'_1 \times M'_2 \rightarrow N \exists! \) linear \( \bar{\beta} : M'_1 \otimes A M'_2 / K \rightarrow N \) s.t. \( \beta(m'_1, m'_2) = \bar{\beta}(m'_1 \otimes m'_2). \)
Define \( \beta^*: M_1 \times M_2 \to N \) by \( \beta^*(m_1', m_2) = \beta (\gamma(m'_1), m_2) \) so \( \beta^* \) is bilinear \( \exists! \tilde{\beta}^*: M_1^* \otimes A M_2 \to N \) s.t. \( \tilde{\beta}^*(m_1' \otimes m_2) = \beta^*(m_1', m_2) \). Note that \( \tilde{\beta}^*(\xi \otimes m_2) = \beta^*(\xi, m_2) = \beta(\xi, m_2) = 0 \) so \( \tilde{\beta}^*(0) = 0 \). So \( \exists! \tilde{\beta}^*: M_1^* \otimes A M_2 / K \to N \) s.t. \( \tilde{\beta}^* = \tilde{\beta} \circ \Phi \). This is precisely the cond'n \( \tilde{\beta}^*(m_1 \otimes m_2) = \beta(m_1, m_2) \). □

2.3) Examples.

1) Tensor product of free modules: \( A^I \otimes_A A^J = \) [Step 1]
\[ = \left( A^J \right)^I \cong A^{(I \times J)} \text{ w. basis } e_i \otimes e_j \ (i \in I, j \in J). \]

2) \( (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) = \) [quotient of \( M_1' \otimes M_2 = \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m\mathbb{Z} \) by \( K = \text{ Span}_\mathbb{Z} \{nz \otimes x \mid z \in \mathbb{Z}, x \in \mathbb{Z}/m\mathbb{Z} \} \) by Step 2. Under identification from Step 1, \( z \otimes x \mapsto zx \), \( K \subset \mathbb{Z}/m\mathbb{Z} \) becomes \( n(\mathbb{Z}/m\mathbb{Z}) \)]
\[ = (\mathbb{Z}/m\mathbb{Z}) / n(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z} / \text{ GCD}(n, m) \mathbb{Z}. \]

3) \( A = \mathbb{F}[x, y] \) (\( \mathbb{F} \) is field), ideal \( I = (x, y) \), want to compute \( I \otimes_A I \).

Step 1: we present \( I \) as a quotient of a free module: \( A^2 \to I, (a, b) \mapsto (ax + by) \).
Compute the kernel \( K_1 = \{(a, b) \mid ax = -by\} \). For the same reason as in Problem 8 of HW9, \( K_1 = \text{ Span}_A \{(y, -x)\} \) (exercise).

Step 2: We now describe \( I \otimes_A I \) as a quotient of \( A^2 \otimes_A I = I^2 \).
By Section 2.2, \( I \otimes A = A^2 \otimes A / \text{Span}_A (f(y, -x) \otimes h \mid f \in A, h \in I) = I^{\oplus 2} / \{(y_h, -x_h) \mid h \in I \} \)

**Exercise:** Use the presentation of \( I \) as a quotient of \( A^{\oplus 2} \) to present \( I \otimes A \) as the quotient of \( A^{\oplus 3} \) by the submodule \( \text{Span}_A ((y, -x, 0), (0, y, -x), (y, 0, -x), (0, y, 0, -x)) \)

2.4) **Generators of tensor product.**

Not every element of \( M_1 \otimes M_2 \) has the form \( m_1 \otimes m_2 \) (we'll call such elements *elementary tensors* or *tensor monomials*.)

**Exercise:** \( A = \mathbb{F} \) (a field), \( M_1 = \mathbb{F}^{\oplus k} \), \( M_2 = \mathbb{F}^{\oplus l} \). By Example 1, \( M_1 \otimes M_2 \sim \{ \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} e_i \otimes e_j \} = \{ k \times l \text{-matrices} \} \).

Show that under this identification, the tensor monomials correspond to \( k \times 1 \) matrices.

**Lemma:** If \( M_k = \text{Span}_A (m_k^i \mid i \in I_k) \) \( k = 1, 2 \), then \( M_1 \otimes A M_2 = \text{Span}_A (m_1^i \otimes m_2^j \mid i \in I_1, j \in I_2) \).

**Sketch of proof:** First handle the case \( M_i = \mathbb{F}^{\oplus I_i} \), \( m_i^i = e_i \), then use the construction in Sec 2.2 to handle the general case (details are an exercise.)

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**BONUS:** Tensor products over noncommutative rings.

Let $A$ be a commutative ring & $R$ be an $A$-algebra (associative but perhaps non-commutative). Recall that it makes sense to talk about left & right $R$-modules & also about bimodules. Also (compare to Bonus of Lec 3) for two left $R$-modules $M_1$, $M_2$, the Hom set $\text{Hom}_R(M_1, M_2)$ is only an $A$-module, not an $R$-module.

As for tensor products, we can tensor left $R$-modules with right $R$-modules. Namely, let $M$ be a left $R$-module & $N$ be a right $R$-module. For an $A$-module $L$ consider the set $\text{Bil}_R(N \times M, L)$ consisting of all $A$-bilinear maps $g: N \times M \rightarrow L$ s.t. in addition $g(nr, m) = g(n, rm) \forall r \in R, n \in N, m \in M$.

**Definition:** $N \otimes_R M \in \mathcal{Ob}(A\text{-Mod})$ represents the functor $\text{Bil}_R(N \times M, \ast): A\text{-Mod} \rightarrow \text{Sets}$.

**Important exercise:** If $R$ is commutative, then this definition gives the same as the definition in Lec 15.

To construct $N \otimes_R M$ we can use the same construction as we did in the lecture. Alternatively, $N \otimes_R M$ is the quotient of $N \otimes_A M$ by the $A$-submodule $\text{Span}_A (nr \otimes m - nr \otimes m | n \in N, m \in M, r \in R)$.
Now suppose we have 2 more $A$-algebras, $S$ and $T$. Let $N$ be an $S$-$R$-$b$-module & $M$ be an $R$-$T$-$b$-module.

**Important exercise:** \( \exists ! \) $S$-$T$-$b$-module $N \otimes_R M$ s.t. $s \cdot (n \otimes m) = s n \otimes m$, $(n \otimes m) t = n \otimes m t$. 