Lecture 15: tensor products, I

1) Definition of tensor products of modules.
2) Construction.

Ref: [AM], Section 2.7.

BONUS: Tensor products over noncommutative rings.

1) Definition of tensor products of modules

1.0) Bilinear maps: Let A be a comm've ring, M_1, M_2, N be A-modules. Recall (Sec 2.3 of Lec 3) that $\beta: M_1\times M_2 \to N$ is A-bilinear if $\beta(m_1, ?): M_2 \to N$ is A-linear $\forall m_1\in M_1$ & $\beta(?, m_2)$: $M_1 \to N$ is A-linear $\forall m_2 \in M_2$. Consider the set.

$$\operatorname{Bilin}_A(M_1\times M_2, N) := \{A\text{-bilinear maps } M_1\times M_2 \to N\}.$$

Digression: why should we care about bilinear maps – b/c they are everywhere!

1) Linear algebra: for an $\mathbb{F}$-vector space V can talk about bilinear forms := bilinear maps $V\times V \to \mathbb{F}$, fundamentally important in Linear algebra & beyond.

2) if M is an A-module $\Rightarrow$ mult'n map $A\times M \to M$ is A-bilinear.

3) the composition map $\operatorname{Hom}_A(L,M)\times \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(L,N)$, where L, M, N are A-modules is A-bilinear, Problem 6 in HW1.

4) If B is an A-algebra, then the product map $B\times B \to B$ is A-bilinear.

Observation: $F_{M_1,M_2} := \mathrm{Bilin}_A(M_1\times M_2, \bullet)$ is actually a functor $A\text{-Mod} \longrightarrow \text{Sets}$: to $\psi \in \mathrm{Hom}_A(N,N')$ we assign

$$\begin{array}{ccc} F_{M_1,M_2}(\psi): \mathrm{Bilin}_A(M_1\times M_2, N) & \longrightarrow & \mathrm{Bilin}_A(M_1\times M_2, N') \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \beta & \longmapsto & \psi\circ\beta \end{array}$$

Exercise: Show $\psi\circ\beta$ is A-bilinear & F_{M_1,M_2} is indeed a functor $A\text{-Mod} \to \text{Sets}$.

1.1) Definition of tensor product:

Definition: By the tensor product $M_1 \otimes_A M_2$ we mean a representing object for $\mathrm{Bilin}_A(M_1\times M_2, \bullet)$ i.e. want a functor isomorphism $\mathrm{Hom}_A(M_1\otimes_A M_2, \bullet) \xrightarrow[\eta]{\sim} \mathrm{Bilin}_A(M_1\times M_2, \bullet)$

Similarly to products (Sec 3 of Lec 13) we can equivalently define tensor products via a universal property: this is an A-module $M_1\otimes_A M_2$ w. a bilinear map $M_1\times M_2 \longrightarrow M_1\otimes_A M_2$, $(m_1,m_2) \longmapsto m_1\otimes m_2$, w. the following universal property:

$\forall$ A-module N & A-bilinear map $\beta: M_1\times M_2 \to N$ $\exists!$ A-linear map $\tilde\beta: M_1\otimes_A M_2 \longrightarrow N$ s.t. $\beta(m_1,m_2) = \tilde\beta(m_1\otimes m_2)$ i.e. the following is commutative

$$\begin{tikzcd} M_1\times M_2 \arrow[d, "(m_1{,}m_2) \mapsto m_1\otimes m_2"'] \arrow[dr, "\beta"] & \\ M_1\otimes_A M_2 \arrow[r, dashed, "\tilde\beta"] & N \end{tikzcd}$$

In terms of η, this universal bilinear map is $\eta_{M_1\otimes_A M_2}(1_{M_1\otimes_A M_2})$.

Thx to either of the equivalent definitions, the tensor product is unique (if exists) in the following sense: if $M_1 \otimes'_A M_2$ is another tensor product w. bilinear map $(m_1, m_2) \mapsto m_1 \otimes' m_2$, then $\exists!$ A-module isomorphism $\iota: M_1 \otimes_A M_2 \xrightarrow{\sim} M_1 \otimes'_A M_2 \mid \iota(m_1 \otimes m_2) = m_1 \otimes' m_2 \ \forall m_i \in M_i$.

2) Construction

The main goal of this section is to give a constructive proof of

Theorem: $M_1 \otimes_A M_2$ exists for all A-modules M_1, M_2.

2.1) Case 1: M_1 is free.

Lemma: For any set I, $A^{\oplus I} \otimes_A M \xrightarrow{\sim} M^{\oplus I}$ w.

$$(a_i)_{i \in I} \otimes m = (a_i m)_{i \in I} \ \forall (a_i) \in A^{\oplus I}, m \in M.$$

Proof:

As for A-linear map from free modules, $\beta \in Bilin_A(A^{\oplus I}, M; N)$ is uniquely determined by $\beta_i \in Hom_A(M, N)$, $\beta_i(m) := \beta(e_i, m)$, via $\beta((a_i), m) = \sum_{i \in I} a_i \beta_i(m)$. Note that the r.h.s. is well-defined b/c $\{i \in I \mid a_i \neq 0\}$ is finite. This gives a bijection $Bilin_A(A^{\oplus I}, M; N) \xrightarrow{\sim} Hom_A(M, N)^{\times I}$, $\beta \mapsto (\beta(e_i, \cdot))$. On the other hand, by Sec 1.2 in Sec 4, have bijection $Hom_A(M, N)^{\times I} \longrightarrow Hom_A(M^{\oplus I}, N)$, $(\beta_i) \mapsto [(m_i)_{i \in I} \mapsto \sum_{i \in I} \beta_i(m_i)]$. Consider the composition,

$$\eta'_N: Bilin_A(A^{\oplus I}, N) \xrightarrow{\sim} Hom_A(M^{\oplus I}, N), \beta \mapsto [(m_i) \mapsto \sum \beta(e_i, m_i)].$$

These bijections constitute a functor isomorphism (exercise). This shows that $M^{\oplus I}$ is indeed $A^{\oplus I} \otimes_A M$ (take $\eta = (\eta')^{-1}$). Note that

$\beta((a_i),m) := (a_i m)_{i\in I}$ is in $\mathrm{Bilin}(A^{\oplus I}, M; M^{\oplus I})$ & satisfies $\eta'(\beta) = 1_{M^{\oplus I}}$ (exercise) so indeed, $(a_i)\otimes m = (a_i m)$. $\square$

2.2) Step 2: arbitrary M_1.

Let M_1', M_2 be A-modules s.t. $M_1'\otimes_A M_2$ exists. Let $K_1 \subset M_1'$ be an A-submodule $\rightsquigarrow$ $M_1 := M_1'/K_1$ & $\pi_1 : M_1' \twoheadrightarrow M_1$. Inside $M_1'\otimes_A M_2$ consider submodule $K := \mathrm{Span}_A(k_1\otimes m_2 \mid k_1 \in K_1, m_2 \in M_2)$.
$\rightsquigarrow$ projection $\pi : M_1'\otimes_A M_2 \twoheadrightarrow M_1'\otimes_A M_2/K$.

Claim: $M_1'\otimes_A M_2/K$ is the tensor product $M_1\otimes_A M_2$ & for $m_1 = \pi_1(m_1') \in M_1$ & $m_2 \in M_2 \Rightarrow m_1\otimes m_2 := \pi(m_1'\otimes m_2)$.

How this implies Thm: M_1 is a quotient of $A^{\oplus I}$ for some I (if $(m_i)_{i\in I}$ are generators of M_1, then $\varphi_{\underline{m}} : A^{\oplus I} \to M_1$, $(a_i) \mapsto \sum_{i\in I} a_i m_i$, is surjective). $A^{\oplus I}\otimes_A M$ exists by Step 1. Now in Claim take $M_1' := A^{\oplus I}$ to see that $M_1\otimes_A M_2$ exists.

Proof of Claim. Exercise: $m_1\otimes m_2$ is well-defined (independent of choice of m_1') & gives a bilinear map $M_1\times M_2 \to M_1'\otimes_A M_2/K$.

Now we only need to check univ'l property (II): $\forall$ bilinear $\beta : M_1\times M_2 \to N$ $\exists!$ linear $\tilde\beta : M_1'\otimes_A M_2/K \to N$ s.t. $\beta(m_1,m_2) = \tilde\beta(m_1\otimes m_2)$.

Define $\beta': M_1' \times M_2 \to N$ by $\beta'(m_1', m_2) = \beta(\pi_1(m_1'), m_2)$ so β' is bilinear $\rightsquigarrow \exists!\ \tilde{\beta}': M_1' \otimes_A M_2 \to N$ s.t. $\tilde{\beta}'(m_1' \otimes m_2) = \beta'(m_1', m_2)$. Note that $\tilde{\beta}'(k_1 \otimes m_2) = \beta'(k_1, m_2) = \beta(0, m_2) = 0$ so $\tilde{\beta}'(K) = 0$. So $\exists!\ \tilde{\beta}: M_1' \otimes_A M_2 / K \to N$ s.t. $\tilde{\beta}' = \tilde{\beta} \circ \pi$. This is precisely the cond'n $\tilde{\beta}(m_1 \otimes m_2) = \beta(m_1, m_2)$ $\square$

2.3) Examples.

1) Tensor product of free modules: $A^{\oplus I} \otimes_A A^{\oplus J} =$ [Step 1] $= (A^{\oplus J})^{\oplus I} \simeq A^{\oplus (I \times J)}$ w. basis $e_i \otimes e_j$ $(i \in I, j \in J)$.

2) $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) =$ [quotient of $M_1' \otimes_{\mathbb{Z}} M_2 = \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m\mathbb{Z}$ by $K = \mathrm{Span}_{\mathbb{Z}}(nz \otimes x \mid z \in \mathbb{Z}, x \in \mathbb{Z}/m\mathbb{Z})$ by Step 2. Under identification from Step 1, $z \otimes x \mapsto zx$, $K \subset \mathbb{Z}/m\mathbb{Z}$ becomes $n(\mathbb{Z}/m\mathbb{Z})$] $= (\mathbb{Z}/m\mathbb{Z})/n(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/GCD(n,m)\mathbb{Z}$.

3) $A = \mathbb{F}[x,y]$ ($\mathbb{F}$ is field), ideal $I = (x,y)$, want to compute $I \otimes_A I$.

Step 1: we present I as a quotient of a free module: $A^{\oplus 2} \twoheadrightarrow I$, $(a,b) \mapsto (ax+by)$.
Compute the kernel $K_1 = \{(a,b) \mid ax = -by\}$. For the same reason as in Problem 8 of HW1, $K_1 = \mathrm{Span}_A((y,-x))$ (exercise).

Step 2: We now describe $I \otimes_A I$ as a quotient of $A^2 \otimes_A I = I^{\oplus 2}$.

By Section 2.2, $I\otimes_A I = A^2\otimes_A I/\mathrm{Span}_A(f(y,-x)\otimes h \mid f\in A, h\in I) = I^{\oplus 2}/\{(yh,-xh) \mid h\in I\}$

Exercise: Use the presentation of I as a quotient of $A^{\oplus 2}$ to present $I\otimes_A I$ as the quotient of $A^{\oplus 4}$ by the submodule $\mathrm{Span}_A((y,-x,0,0),(0,0,y,-x),(y,0,-x,0),(0,y,0,-x))$.

2.4) Generators of tensor product.

Not every element of $M_1\otimes_A M_2$ has the form $m_1\otimes m_2$ (we'll call such elements **elementary tensors** or **tensor monomials**.)

Exercise: $A=\mathbb{F}$ (a field), $M_1=\mathbb{F}^{\oplus k}$, $M_2=\mathbb{F}^{\oplus \ell}$. By Example 1, $M_1\otimes_{\mathbb{F}} M_2 \xrightarrow{\sim} \{\sum_{i=1}^{\ell}\sum_{j=1}^{\ell} a_{ij}\, e_i\otimes e_j\} \cong \{k\times\ell\text{-matrices}\}$. Show that under this identification, the tensor monomials correspond to rk 1 matrices.

Lemma: If $M_k=\mathrm{Span}_A(m_k^i \mid i\in I_k)$ $k=1,2$, then $M_1\otimes_A M_2 = \mathrm{Span}_A(m_1^i\otimes m_2^j \mid i\in I_1, j\in I_2)$.

Sketch of proof: First handle the case $M_1=A^{\oplus I_1}$, $m_i^1=e_i$, then use the construction in Sec 2.2 to handle the general case (details are an exercise. $\square$

BONUS: Tensor products over noncommutative rings.

Let A be a comm've ring & R be an A-algebra (associative but perhaps non-commutative). Recall that it makes sense to talk about left & right R-modules & also about bimodules. Also (compare to Bonus of Lec 3) for two left R-modules M_1, M_2, the Hom set $Hom_R(M_1, M_2)$ is only an A-module, not an R-module.

As for tensor products, we can tensor left R-modules w. right R-modules. Namely, let M be a left R-module & N be a right R-module. For an A-module L consider the set $Bilin_R(N\times M, L)$ consisting of all A-bilinear maps $\varphi: N\times M \to L$ s.t. in addition $\varphi(nr, m) = \varphi(n, rm)$ $\forall r\in R$, $n\in N, m\in M$.

Definition: $N\otimes_R M \in Ob(A\text{-Mod})$ represents the functor $Bilin_R(N\times M, \bullet): A\text{-Mod} \longrightarrow Sets$.

Important exercise: If R is comm've, then this definition gives the same as the definition in Lec 15.

To construct $N\otimes_R M$ we can use the same construction as we did in the lecture. Alternatively, $N\otimes_R M$ is the quotient of $N\otimes_A M$ by the A-submodule
$$Span_A(nr\otimes m - n\otimes rm \mid n\in N, m\in M, r\in R).$$

Now suppose we have 2 more A-algebras, S and T. Let N be an S-R-bimodule & M be an R-T-bimodule.

Important exercise: $\exists!$ S-T-bimodule str're on $N \otimes_R M$ s.t. $s(n \otimes m) = sn \otimes m$, $(n \otimes m)t = n \otimes mt$.