Lecture 16: Tensor products, II.

1) Further discussion of tensor products.
2) Tensor-Hom adjunction.
Ref: [AM], Section 2.7

1) Further discussion of tensor products.

1.1) Tensor products of linear maps & functoriality.

Let M_1, M_1', M_2, M_2' be A-modules & $\varphi_i \in \operatorname{Hom}_A(M_i, M_i')$, $i=1,2$.

Goal: define A-linear map $\varphi_1 \otimes \varphi_2 : M_1 \otimes_A M_2 \to M_1' \otimes_A M_2'$.

Consider: $M_1 \times M_2 \longrightarrow M_1' \otimes_A M_2'$, $(m_1, m_2) \mapsto \varphi_1(m_1) \otimes \varphi_2(m_2)$

Exercise: This map is A-bilinear.

So it gives rise to an A-linear map $\varphi_1 \otimes \varphi_2 : M_1 \otimes_A M_2 \to M_1' \otimes_A M_2'$ uniquely characterized by $\varphi_1 \otimes \varphi_2 (m_1 \otimes m_2) = \varphi_1(m_1) \otimes \varphi_2(m_2)$ $\forall m_i \in M_i$.

Properties of tensor products of maps:

- $id_{M_1} \otimes id_{M_2} = id_{M_1 \otimes_A M_2}$.
- Compositions: $M_1 \xrightarrow{\varphi_1} M_1' \xrightarrow{\varphi_1'} M_1''$, $M_2 \xrightarrow{\varphi_2} M_2' \xrightarrow{\varphi_2'} M_2''$
  $(\varphi_1' \varphi_1) \otimes (\varphi_2' \varphi_2) = (\varphi_1' \otimes \varphi_2')(\varphi_1 \otimes \varphi_2)$ b/c they coincide on generators (Sec 2.4 of Lec 15) $m_1 \otimes m_2$ of $M_1 \otimes_A M_2$.

So: we have the tensor product functor

$$A\text{-Mod} \times A\text{-Mod} \longrightarrow A\text{-Mod}$$

Important exercise: Prove that $(\varphi_1, \varphi_2) \mapsto \varphi_1 \otimes \varphi_2$:
$\mathrm{Hom}_A(M_1, M_1') \times \mathrm{Hom}_A(M_2, M_2') \longrightarrow \mathrm{Hom}_A(M_1 \otimes_A M_2, M_1' \otimes_A M_2')$
is A-bilinear (hint: check on generators of $M_1 \otimes_A M_2$)

1.2) "Algebra properties" of tensor products.

Theorem: Let M_1, M_2, M_3 be A-modules. Then:

1) There is a unique isomorphism $(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A (M_2 \otimes_A M_3)$ s.t. $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$. (i.e. tensor product is associative).
2) $\exists!$ isom'm $M_1 \otimes_A M_2 \xrightarrow{\sim} M_2 \otimes_A M_1$ w. $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$.
3) $\exists!$ isom'm $M_1 \otimes_A (M_2 \oplus M_3) \xrightarrow{\sim} M_1 \otimes_A M_2 \oplus M_1 \otimes_A M_3$ w. $m_1 \otimes (m_2, m_3) \mapsto (m_1 \otimes m_2, m_1 \otimes m_3)$
4) $\exists!$ unique isom'm $A \otimes_A M \xrightarrow{\sim} M$ s.t. $a \otimes m \mapsto am$.

Proof: (1)

We want an A-linear map

$\tilde{\beta}: (M_1 \otimes_A M_2) \otimes_A M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$, $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$

i.e. want a bilinear map $\beta: (M_1 \otimes_A M_2) \times M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$
$(m_1 \otimes m_2, m_3) \mapsto m_1 \otimes (m_2 \otimes m_3)$.

Fix $m_3 \rightsquigarrow$ a linear map $M_2 \to M_2 \otimes_A M_3$, $m_2 \mapsto m_2 \otimes m_3$. Define $\beta_{m_3}: M_1 \otimes_A M_2 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$ to be the tensor product of id_{M_1} & $[m_2 \mapsto m_2\otimes m_3]$ so $\beta_{m_3}(m_1\otimes m_2) = m_1\otimes(m_2\otimes m_3)$

Note that β_{m_3} depends linearly on m_3 (e.g. $\beta_{am_3} = a\beta_{m_3}$)

$\rightsquigarrow$ A-bilinear map $\beta: (M_1\otimes_A M_2)\times M_3 \longrightarrow M_1\otimes_A(M_2\otimes_A M_3)$,

$\beta(x, m_3) := \beta_{m_3}(x) \rightsquigarrow \tilde\beta$ as needed.

$\tilde\beta$ is an isom'm: have $\tilde\beta': M_1\otimes_A(M_2\otimes_A M_3)\longrightarrow (M_1\otimes_A M_2)\otimes M_3$

$m_1\otimes(m_2\otimes m_3)\mapsto(m_1\otimes m_2)\otimes m_3$. It's inverse of $\tilde\beta$ b/c $\tilde\beta'\circ\tilde\beta = id$ &

$\tilde\beta\circ\tilde\beta' = id$ on generators (tensor monomials). $\square$ of (1).

(2) - commutativity - is an exercise & (4) - unit - follows from our construction.

Proof of (3) - distributivity: consider the projection

$\pi_i: M_2\oplus M_3 \longrightarrow M_i$, $i=2,3$; & inclusion $\iota_i: M_i \hookrightarrow M_2\oplus M_3$

$\rightsquigarrow id_{M_1}\otimes\pi_i: M_1\otimes_A(M_2\oplus M_3) \leftrightarrows M_1\otimes_A M_i : id_{M_1}\otimes\iota_i$

$(id_{M_1}\otimes\pi_2, id_{M_1}\otimes\pi_3): M_1\otimes_A(M_2\oplus M_3) \leftrightarrows M_1\otimes_A M_2\oplus M_1\otimes_A M_3 : (id_{M_1}\otimes\iota_2, id_{M_1}\otimes\iota_3)$

$id_{M_1}\otimes\iota_2(x) + id_{M_1}\otimes\iota_3(y) \longleftarrow (x,y)$

Exercise: check that these maps are mutually inverse. $\square$

2) Tensor-Hom adjunction.

The goal of this section is to prove that tensor product functors are left adjoint to Hom functors.

2.1) Basic setting.

Let L be an A-module. We can consider the following functors A-Mod $\longrightarrow$ A-Mod:

1) $L\otimes_A\bullet$ that sends an A-module M to $L\otimes_A M$ & an A-linear map

$\psi: M\to M'$ to $id_L\otimes\psi: L\otimes_A M\to L\otimes_A M'$, also A-linear.

2) $\underline{Hom}_A(L,\cdot)$ defined exactly as $Hom_A(L,\cdot): A\text{-Mod}\to Sets$ but viewed as a functor to A-Mod, which makes sense b/c for an A-linear map $\psi: M\to M'$, the map $\psi\circ ?: Hom_A(L,M)\to Hom_A(L,M')$ is A-linear (Prob 6 in HW1). Formally, $Hom_A(L,\cdot)\xrightarrow{\sim} For\circ \underline{Hom}_A(L,\cdot)$, where For is the forgetful functor $A\text{-Mod}\to Sets$.

Preliminary Thm (tensor-Hom adjunction): $L\otimes_A\cdot$ is left adjoint to $\underline{Hom}_A(L,\cdot)$ (as functors $A\text{-Mod}\to A\text{-Mod}$).

2.2) General setting

It turns out that the same method gives left (and right) adjoint functors to pullback functors $\varphi^*: B\text{-Mod}\to A\text{-Mod}$ (Sec 1.1 of Lec 12) for $\varphi: A\to B$, a homomorphism of commutative rings. These adjoints are important so we need to explore a more general setup.

Let L be a B-module (so also an A-module) & M be an A-module $\rightsquigarrow$ A-module $L\otimes_A M$.

Lemma: 1) There is a unique B-module str're on $L\otimes_A M$ s.t.
$$b(l\otimes m)=(bl)\otimes m \ \forall\ b\in B, l\in L, m\in M$$
2) If $\psi: M\to M'$ is an A-linear map, then $id_L\otimes\psi$ is a B-linear map $L\otimes_A M\to L\otimes_A M'$.

Proof: 1) Consider the map $\beta_b : L\times M \to L\otimes_A M$, $(\ell,m)\mapsto (b\ell)\otimes m$. It's A-bilinear (exercise) so $\exists!$ A-linear map $\tilde\beta_b : L\otimes_A M \to L\otimes_A M$ s.t. $\tilde\beta_b(\ell\otimes m) = (b\ell)\otimes m$ ($\forall\, b\in B, \ell\in L, m\in M$). Define a map

$$B\times (L\otimes_A M) \longrightarrow L\otimes_A M,\ (b,x)\mapsto \tilde\beta_b(x).$$

We claim that it defines a B-module structure on $L\otimes_A M$. This is a boring check of axioms using that $\tilde\beta_b$ is A-linear & $\mathrm{Span}_A(\ell\otimes m) = L\otimes_A M$ (Sec 2.4 of Lec 15). For example, to check associativity, $(b_1b_2)x = b_1(b_2x)$ it's enough to assume that $x=\ell\otimes m$. Then $(b_1b_2)x = (b_1b_2\ell)\otimes m = b_1(b_2(\ell\otimes m)) = b_1(b_2x)$.

2) is left as an exercise. $\square$

This lemma gives us a functor $L\otimes_A\cdot : A\text{-Mod}\to B\text{-Mod}$. On the other hand, we have a functor $\varphi^*\underline{\mathrm{Hom}}_B(L,\cdot): B\text{-Mod}\to A\text{-Mod}$.

Thm (Tensor-Hom adjunction): The functor $L\otimes_A\cdot : A\text{-Mod}\to B\text{-Mod}$ is left adjoint to $\varphi^*\underline{\mathrm{Hom}}_B(L,\cdot): B\text{-Mod}\to A\text{-Mod}$.

Proof: Let M be an A-module & N be a B-module. Our goal is to construct a natural bijection:

$$\eta_{M,N}: \mathrm{Hom}_B(L\otimes_A M, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(L,N)).$$

Take $\tilde\tau\in \mathrm{Hom}_B(L\otimes_A M, N)$. For $m\in M$, define $\psi_{\tilde\tau}(m): L\to N$, $\ell\mapsto \tilde\tau(\ell\otimes m)$; $\psi_{\tilde\tau}(m)$ is B-linear, e.g. $[\psi_{\tilde\tau}(m)](b\ell) =$ [def'n of $\tilde\tau$] $= \tilde\tau((b\ell)\otimes m) =$ [def'n of B-action on $L\otimes_A M$] $= \tilde\tau(b(\ell\otimes m)) =$ [$\tilde\tau$ is B-linear] $= b\tilde\tau(\ell\otimes m) = b([\psi_{\tilde\tau}(m)](\ell))$. This gives a map $M\to \mathrm{Hom}_B(L,N)$, $m\mapsto \psi_{\tilde\tau}(m)$, that is A-linear (exercise) so $\varphi_{\tilde{\tau}} \in Hom_A(M, Hom_B(L,N))$. Define $\eta_{M,N}$ by $\tilde{\tau} \mapsto \varphi_{\tilde{\tau}}$.

Now we produce an inverse. Take $\varphi \in Hom_A(M, Hom_B(L,N))$. Consider the map $\tau_\varphi : L \times M \to N$ by $\tau_\varphi(\ell, m) = [\varphi(m)](\ell)$. Then $\exists !$ A-linear $\tilde{\tau}_\varphi : L \otimes_A M \to N$. We claim that $\tilde{\tau}_\varphi$ is actually B-linear: $\tilde{\tau}_\varphi(bx) = b\,\tilde{\tau}_\varphi(x)$ $\forall b \in B$, $x \in L \otimes_A M$. Since $Span_A(\ell \otimes m) = L \otimes_A M$ & $\tilde{\tau}_\varphi$ is A-linear, it's enough to assume $x = \ell \otimes m \Rightarrow \tilde{\tau}_\varphi(b(\ell \otimes m)) = \tilde{\tau}_\varphi((b\ell) \otimes m) = \tau_\varphi(b\ell, m) = [\varphi(m)](b\ell) = b([\varphi(m)](\ell)) = b\tau_\varphi(\ell, m) = b\tilde{\tau}_\varphi(\ell \otimes m)$. So $\tilde{\tau}_\varphi \in Hom_B(L \otimes_A M, N)$.

Now we show that the maps $\tilde{\tau} \mapsto \varphi_{\tilde{\tau}}$ & $\varphi \mapsto \tilde{\tau}_\varphi$ are inverse to each other:

$\tilde{\tau}_{\varphi_{\tilde{\tau}}} = \tilde{\tau} \Leftrightarrow \tilde{\tau}_{\varphi_{\tilde{\tau}}}(\ell \otimes m) = \tilde{\tau}(\ell \otimes m)$ $\forall \ell \in L, m \in M$; $\tilde{\tau}_{\varphi_{\tilde{\tau}}}(\ell \otimes m) = \tau_{\varphi_{\tilde{\tau}}}(\ell, m) =$
$= [\varphi_{\tilde{\tau}}(m)](\ell) = \tilde{\tau}(\ell \otimes m)$ ✓

$\varphi_{\tilde{\tau}_\varphi} = \varphi \Leftrightarrow [\varphi_{\tilde{\tau}_\varphi}(m)](\ell) = [\varphi(m)](\ell)$ $\forall \ell \in L, m \in M$; $[\varphi_{\tilde{\tau}_\varphi}(m)](\ell) =$
$\tilde{\tau}_\varphi(\ell \otimes m) = \tau_\varphi(\ell, m) = [\varphi(m)](\ell)$ ✓

Let's check that the bijections η_{MN}'s make one diagram in the definition of adjoint functors (Sec 2.1 of Lec 14) commutative (the other is an exercise). Pick $\zeta \in Hom_A(M, M')$. We need to show the following is commutative

$$\begin{array}{ccc} Hom_B(L \otimes_A M', N) & \xrightarrow{\eta_{M',N}} & Hom_A(M', Hom_B(L,N)). \\ \downarrow {\scriptstyle ?\circ(id_L \otimes \zeta)} & & \downarrow {\scriptstyle ?\circ\zeta} \\ Hom_B(L \otimes_A M, N) & \xrightarrow{\eta_{M,N}} & Hom_A(M, Hom_B(L,N)). \end{array}$$

$\downarrow \longrightarrow : \tilde{\tau} \mapsto [m \mapsto [\ell \mapsto \tilde{\tau} \circ (id_L \otimes \zeta)(\ell \otimes m) = \tilde{\tau}(\ell \otimes \zeta(m)]]$

$\|$

$\longrightarrow \downarrow : \tilde{\tau} \mapsto [m \mapsto [\eta_{M;N}(\tilde{\tau})](\zeta(m)]$ $\square$

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