Lecture 16: Tensor products, II.

1) Further discussion of tensor products.
2) Tensor-Hom adjunction.

Ref: [AM], Section 2.7

1) Further discussion of tensor products.

1.1) Tensor products of linear maps & functoriality.

Let $M_i, M'_i, M_j, M'_j$ be $A$-modules & $\phi_i \in \text{Hom}_A(M_i, M'_i), i = 1, 2$.

Goal: define $A$-linear map $\phi_1 \otimes \phi_2 : M_1 \otimes_A M_2 \to M'_1 \otimes_A M'_2$.

Consider: $M_1 \times M_2 \to M'_1 \otimes_A M'_2, (m_1, m_2) \mapsto \phi_1(m_1) \otimes \phi_2(m_2)$.

Exercise: This map is $A$-bilinear.

So it gives rise to an $A$-linear map $\phi_1 \otimes \phi_2 : M_1 \otimes_A M_2 \to M'_1 \otimes_A M'_2$ uniquely characterized by $(\phi_1 \otimes \phi_2)(m_1 \otimes m_2) = \phi_1(m_1) \otimes \phi_2(m_2)$ for $m_i \in M_i$.

Properties of tensor products of maps:

- $\text{id}_{M_1} \otimes \text{id}_{M_2} = \text{id}_{M_1 \otimes_A M_2}$.

- Compositions: $M_1 \xrightarrow{\phi_1} M'_1 \xrightarrow{\phi_2} M''_1$ and $M_2 \xrightarrow{\phi_2} M'_2 \xrightarrow{\phi_2} M''_2$.

  $(\phi_1 \otimes \phi_2)(\phi_2 \otimes \phi_2) = (\phi_1 \otimes \phi_2)(\phi_1 \otimes \phi_2)$ since they coincide on generators (Sec 2.4 of Lec 15).

  $m_1 \otimes m_2$ of $M_1 \otimes_A M_2$. 


So: we have the tensor product functor
\[ A \text{-Mod} \times A \text{-Mod} \to A \text{-Mod} \]

**Important exercise:** Prove that \((q_1, q_2) \mapsto q_1 \otimes q_2: \text{Hom}_A(M_1, M'_1) \times \text{Hom}_A(M_2, M'_2) \to \text{Hom}_A(M_1 \otimes_A M_2, M'_1 \otimes_A M'_2)\) is \(A\)-bilinear (hint: check on generators of \(M_1 \otimes_A M_2\)).

1.2) "Algebra properties" of tensor products.

**Theorem:** Let \(M_1, M_2, M_3\) be \(A\)-modules. Then:

1) There is a unique isomorphism \((M_1 \otimes_A M_2) \otimes_A M_3 \cong M_1 \otimes_A (M_2 \otimes_A M_3)\) s.t. \((m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)\). (i.e. tensor product is associative.)

2) \(\exists! \) isomorphism \(M_1 \otimes_A M_2 \cong M_2 \otimes_A M_1\) w. \(m_1 \otimes m_2 \mapsto m_2 \otimes m_1\).

3) \(\exists! \) isomorphism \(M_1 \otimes_A (M_2 \otimes_A M_3) \cong M_1 \otimes_A M_2 \otimes_A M_3\) w. \(m_1 \otimes (m_2, m_3) \mapsto (m_1 \otimes m_2, m_1 \otimes m_3)\).

4) \(\exists! \) unique isomorphism \(A \otimes_A M \cong M\) s.t. \(a \otimes m \mapsto am\).

**Proof:** (1)

We want an \(A\)-linear map
\[ \beta: (M_1 \otimes_A M_2) \otimes_A M_3 \to M_1 \otimes_A (M_2 \otimes_A M_3), (m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3) \]
i.e. want a bilinear map \(\beta: (M_1 \otimes_A M_2) \times M_3 \to M_1 \otimes_A (M_2 \otimes_A M_3)\)
\[(m_1 \otimes m_2, m_3) \mapsto m_1 \otimes (m_2 \otimes m_3).\]

Fix \(m_3 \mapsto a\) linear map \(M_2 \to M_2 \otimes_A M_3, m_2 \mapsto m_2 \otimes m_3\). Define \[\beta_{m_3}: M_1 \otimes_A M_2 \to M_1 \otimes_A (M_2 \otimes_A M_3)\] to be the tensor product
of \( \text{id}_M \) & \([m_2 \mapsto m_2 \otimes m_3]\) so \( \beta_{m_2}(m_1 \otimes m_2) = m_1 \otimes (m_2 \otimes m_3) \).

Note that \( \beta_{m_2} \) depends linearly on \( m_2 \) (e.g. \( \beta_{m_2} = \alpha \beta_{m_3} \)).

\( A \) bilinear map \( \beta: (M_i \otimes A) \times M_j \rightarrow M_i \otimes (M_j \otimes A) M_k \),

\( \beta(x, m_k) = \beta_{m_k}(x) \rightarrow \tilde{\beta} \) as needed.

\( \tilde{\beta} \) is an isomorphism: have \( \tilde{\beta}: M_i \otimes (M_j \otimes A) M_k \rightarrow (M_i \otimes M_j) \otimes M_k \)

\( m_i \otimes (m_j \otimes m_k) \mapsto (m_i \otimes m_j) \otimes m_k \). It's inverse of \( \tilde{\beta} / \alpha \) \( \tilde{\beta} \circ \tilde{\beta} = \text{id} \) & \( \tilde{\beta} \circ \tilde{\beta} = \text{id} \) on generators (tensor monomials). \( \square \) of (1).

(2) commutativity is an exercise & (4) unit follows from our construction.

**Proof of (3) distributivity**: consider the projection \( \Pi_i: M_i \rightarrow M_i, i = 1, 2, 3 \); & inclusion \( \iota_i: M_i \hookrightarrow M_i \).

\( \iota_i \otimes \Pi_i: M_i \otimes (M_j \otimes M_k) \leftrightarrow M_i \otimes M_j \otimes M_k \).

\( (\iota_i \otimes \Pi_i, \iota_i \otimes \Pi_i): M_i \otimes (M_j \otimes M_k) \leftrightarrow M_i \otimes M_j \otimes M_k \).

\( \iota_i \otimes \Pi_i(x) \mapsto \iota_i \otimes \Pi_i(y) \leftrightarrow 1(x, y) \).

**Exercise**: check that these maps are mutually inverse. \( \square \)

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2) Tensor-Hom adjunction.

The goal of this section is to prove that tensor product functors are left adjoint to Hom functors.

2.1) Basic setting.

Let \( L \) be an \( A \)-module. We can consider the following functors:

\( A-\text{Mod} \rightarrow A-\text{Mod} \):

1) \( L \otimes_A \) - that sends an \( A \)-module \( M \) to \( L \otimes_A M \) & an \( A \)-linear map
\( \psi: M \to M' \) to \( \text{id}_A \otimes \psi: \bigotimes_A M \to \bigotimes_A M' \), also \( A \)-linear.

2) \( \text{Hom}_A (L, \cdot) \) defined exactly as \( \text{Hom}_A (L, \cdot): A \text{-Mod} \to \text{Sets} \) but viewed as a functor to \( A \text{-Mod} \), which makes sense by for an \( A \)-linear map \( \psi: M \to M' \), the map \( \psi_\cdot: \text{Hom}_A (L, M) \to \text{Hom}_A (L, M') \) is \( A \)-linear (Prob 6 in HW1). Formally, \( \text{Hom}_A (L, \cdot) \Rightarrow \text{For} \circ \text{Hom}_A (L, \cdot) \), where \( \text{For} \) is the forgetful functor \( A \text{-Mod} \to \text{Sets} \).

**Preliminary Thm (tensor-Hom adjunction):** \( \bigotimes_A \cdot \) is left adjoint to \( \text{Hom}_A (L, \cdot) \) (as functors \( A \text{-Mod} \to A \text{-Mod} \)).

2.2) General setting

It turns out that the same method gives left (and right) adjoint functors to pullback functors \( \phi^*: B \text{-Mod} \to A \text{-Mod} \) (Sec 1.1 of Lec 12) for \( \phi: A \to B \), a homomorphism of commutative rings. These adjoints are important so we need to explore a more general setup.

Let \( L \) be a \( B \)-module (so also an \( A \)-module) & \( M \) be an \( A \)-module \( \rightsquigarrow A \)-module \( \bigotimes_A M \).

**Lemma:** 1) There is a unique \( B \)-module st. \( \otimes \) on \( \bigotimes_A M \) s.t.

\[ b(l \otimes m) = (bl) \otimes m \neq 6 \in B, l \in L, m \in M \]

2) If \( \psi: M \to M' \) is an \( A \)-linear map, then \( \text{id}_A \otimes \psi \) is a \( B \)-linear map \( \bigotimes_A M \to \bigotimes_A M' \).
Proof: 1) Consider the map $\beta_b : \mathfrak{L}M \rightarrow \mathfrak{L}A, (\zeta, m) \mapsto (bl) \otimes m$.

It's $A$-bilinear (exercise) so $\exists!$ $A$-linear map $\tilde{\beta}_b : \mathfrak{L}A M \rightarrow \mathfrak{L}A M$ s.t. $\tilde{\beta}_b (l \otimes m) = (bl) \otimes m \ (\forall \ b \in B, \ l \in \mathfrak{L}, m \in M)$. Define a map

$$B \times (\mathfrak{L}A M) \rightarrow \mathfrak{L}A M, \ (b, x) \mapsto \tilde{\beta}_b(x).$$

We claim that it defines a $B$-module structure on $\mathfrak{L}A M$. This is a boring check of axioms using that $\tilde{\beta}_b$ is $A$-linear & $\text{Span}_A (l \otimes m) = \mathfrak{L}A M$ (Sec 2.4 of Lec 15). For example, to check associativity,

$$(b_1 b_2) x = b_1 (b_2 x) \text{ it's enough to assume that } x = l \otimes m \text{ Then } (b_1 b_2) x = (b_1 b_2) l \otimes m = b_1 (b_2 l \otimes m) = b_1 (b_2 x).$$

2) is left as an exercise. \qed

This lemma gives us a functor $\mathfrak{L}A : A\text{-Mod} \rightarrow B\text{-Mod}$. On the other hand, we have a functor $\psi^* \text{Hom}_B (\zeta, \cdot) : B\text{-Mod} \rightarrow A\text{-Mod}.$

**Thm (Tensor-Hom adjunction):** The functor $\mathfrak{L}A : A\text{-Mod} \rightarrow B\text{-Mod}$ is left adjoint to $\psi^* \text{Hom}_B (\zeta, \cdot) : B\text{-Mod} \rightarrow A\text{-Mod}.$

**Proof:** Let $M$ be an $A$-module & $N$ be a $B$-module. Our goal is to construct a natural bijection:

$$\psi_{M,N} : \text{Hom}_B (\mathfrak{L}A M, N) \cong \text{Hom}_A (M, \text{Hom}_B (\zeta, N)).$$

Take $\tilde{\psi} \in \text{Hom}_B (\mathfrak{L}A M, N)$. For $m \in M$, define $\psi_{\tilde{\psi}} (m) : \zeta \rightarrow N, \ z \mapsto \tilde{\psi} (l \otimes m)$; $\psi_{\tilde{\psi}} (m)$ is $B$-linear, e.g. $[\psi_{\tilde{\psi}} (m)] (bl) = [\text{defn of } \tilde{\psi}] = \tilde{\psi} (l \otimes m) = [\text{defn of } B\text{-action on } \mathfrak{L}A M] = \tilde{\psi} (b (l \otimes m)) = [\tilde{\psi} \text{ is } B\text{-linear}] = b \tilde{\psi} (l \otimes m) = b ([\psi_{\tilde{\psi}} (m)](l)).$ This gives a map $M \rightarrow \text{Hom}_B (\zeta, N), \ m \mapsto \psi_{\tilde{\psi}} (m).$
that is $A$-linear (exercise) so $\varphi_\Lambda \in \text{Hom}_A(M, \text{Hom}_B(L, N))$. Define $\varphi_{MN}$ by $\widetilde{\tau} \mapsto \varphi_\Lambda$.

Now we produce an inverse. Take $\varphi \in \text{Hom}_A(M, \text{Hom}_B(L, N))$. Consider the map $\tilde{\varphi}_\Lambda : L \otimes M \to N$ by $\tilde{\varphi}_\Lambda (l, m) = [\varphi(m)](l)$. Then $\exists!$ $A$-linear $\tilde{\varphi} : L \otimes A \otimes M \to N$. We claim that $\tilde{\varphi}_\Lambda$ is actually $B$-linear: $\tilde{\varphi}_\Lambda (b l) = b \tilde{\varphi}_\Lambda (l) + b \in B$, $x \in L \otimes A \otimes M$. Since $\text{Span}_A (\otimes A \otimes m) = \otimes A \otimes N$ & $\tilde{\varphi}$ is $A$-linear, it's enough to assume $x = \otimes A \otimes m \Rightarrow \tilde{\varphi}_\Lambda (b l \otimes A \otimes m) = \tilde{\varphi}_\Lambda (b l \otimes A \otimes m) = \tilde{\varphi}_\Lambda (b l, m) = [\varphi(m)](b l) = b ([\varphi(m)](l)) = b \tilde{\varphi}_\Lambda (l, m) = b \tilde{\varphi}_\Lambda (l \otimes A \otimes m)$. So $\tilde{\varphi} \in \text{Hom}_B(L \otimes A \otimes M, N)$.

Now we show that the maps $\tilde{\tau} \mapsto \varphi_\Lambda$ & $\varphi \mapsto \tilde{\varphi}_\Lambda$ are inverse to each other:

$\tilde{\varphi}_\Lambda \circ \tilde{\tau} = \tilde{\tau} \Leftrightarrow \tilde{\varphi}_\Lambda (l \otimes A \otimes m) = \tilde{\tau} (l \otimes A \otimes m) \forall l \in L, m \in M; \tilde{\varphi}_\Lambda (l \otimes A \otimes m) = \tilde{\varphi}_\Lambda (l, m) = [\varphi_\Lambda (m)](l) = \tilde{\tau} (l \otimes A \otimes m) \checkmark$

$\varphi_\Lambda \circ \tilde{\tau} = \varphi \Leftrightarrow [\varphi_\Lambda (m)](l) = [\varphi(m)](l) \forall l \in L, m \in M; [\varphi_\Lambda (m)](l) = \tilde{\varphi}_\Lambda (l \otimes A \otimes m) = \tilde{\varphi}_\Lambda (l, m) = [\varphi(m)](l) \checkmark$

Let's check that the bijections $\varphi_{MN}$'s make one diagram in the definition of adjoint functors (Sec 2.1 of Lec 14) commutative (the other is an exercise). Pick $f \in \text{Hom}_A(M, M')$. We need to show the following is commutative:

$$
\begin{array}{ccc}
\text{Hom}_B(L \otimes A \otimes M, N) & \xrightarrow{\varphi_{MN}} & \text{Hom}_A(M, \text{Hom}_B(L, N)) \\
\downarrow \circ \text{id}_{L \otimes A \otimes M} & & \downarrow \circ f \\
\text{Hom}_B(L \otimes A \otimes M, N) & \xrightarrow{\varphi_{MN}} & \text{Hom}_A(M, \text{Hom}_B(L, N)).
\end{array}
$$