Lecture 17: tensor products, III.

1) Tensor-Hom adjunction, cont'd.
2) Tensor products of algebras.

Refs: [AM], Secs 2.8, 2.11
BONUS: (Co)induction of group representations.

1) Tensor-Hom adjunction, cont'd.

Recall the theorem from Lec 16. Let A, B be commutative rings & $\varphi: A \to B$ a homomorphism. Let L be a B-module.

Thm (Tensor-Hom adjunction): The functor $L\otimes_A \bullet: A\text{-Mod} \to B\text{-Mod}$ is left adjoint to $\underbrace{\varphi^*}_{\text{pullback}}\underline{\mathrm{Hom}}_B(L,\bullet): B\text{-Mod} \to A\text{-Mod}$.

1.1) Base change

Take $L=B$. Then $\underline{\mathrm{Hom}}_B(B,N)$ is naturally isomorphic to N for any B-module N, i.e. $\underline{\mathrm{Hom}}_B(B,\bullet)$ is isomorphic to the identity endo-functor of B-Mod. We arrive at the following:

Corollary: The functor $B\otimes_A \bullet: A\text{-Mod} \to B\text{-Mod}$ (base change or induction functor) is left-adjoint to $\varphi^*: B\text{-Mod} \to A\text{-Mod}$.

We encounter base change first when in Linear algebra we replace vector spaces over a field $\mathbb{F}$ w. vector spaces over the algebraic closure $\overline{\mathbb{F}}$ (e.g. $\mathbb{F}=\mathbb{R}$, $\overline{\mathbb{F}}=\mathbb{C}$), this is technically done by applying $\overline{\mathbb{F}}\otimes_{\mathbb{F}}\bullet$. Here's another appearance of base change from this course.

Proposition: Let $S\subset A$ be a multiplicative subset. Then the functor $A[S^{-1}]\otimes_A\bullet : A\text{-Mod}\longrightarrow A[S^{-1}]\text{-Mod}$ is isomorphic to the localization functor $\bullet[S^{-1}]$.

Proof:

By Example 2 in Sec 2.2 in Lec 14, $\bullet[S^{-1}]$ is left adjoint to the pullback functor $A[S^{-1}]\text{-Mod}\rightarrow A\text{-Mod}$. By Corollary above, so is $A[S^{-1}]\otimes_A\bullet$. Now the uniqueness of adjoints (Sec 2.3 of Lec 14) guarantees $\bullet[S^{-1}]\xrightarrow{\sim}A[S^{-1}]\otimes_A\bullet$ $\square$

Rem 1) One can also prove the proposition w/o the categorical language, but this is lengthier. One shows that $\forall$ A-module M, $\exists!$ $A[S^{-1}]$-linear map $\varphi: A[S^{-1}]\otimes_A M\longrightarrow M[S^{-1}]$ w. $\frac{a}{s}\otimes m\mapsto\frac{am}{s}$. To construct an inverse, one applies the universal property of localization of modules (Sec 2.2 of Lec 9) to $M\rightarrow A[S^{-1}]\otimes_A M$, $m\mapsto 1\otimes m$ to show $\exists!$ $\varphi': M[S^{-1}]\rightarrow A[S^{-1}]\otimes_A M$ w. $\frac{m}{s}\mapsto\frac{1}{s}\otimes m$. Details are an exercise. And to show this isomorphism $A[S^{-1}]\otimes_A M\xrightarrow{\sim} M[S^{-1}]$ agrees w. one in Proposition is a premium exercise.

2*) The modules are important in Algebraic Geometry & so is the base change functor. More on this should be expected as a bonus to later lectures.

1.2) Coinduction

Now we briefly discuss the <u>right</u> adjoint of φ^*, again, in a more general form of tensor-Hom adjunction.

Let L be a B-module and N be an A-module. Then $Hom_A(L,N)$ becomes a B-module via $[b\psi](\ell)=\psi(b\ell)$ $\forall \psi\in Hom_A(L,N), \ell\in L$ (exercise). So we get a functor $\underline{Hom}_A(L,\cdot): A\text{-Mod}\to B\text{-Mod}$, to check this rigorously is also an exercise.

Thm: The functor $\varphi^*(L\otimes_B \cdot): B\text{-Mod}\to A\text{-Mod}$ is left adjoint to $\underline{Hom}_A(L,\cdot): A\text{-Mod}\to B\text{-Mod}$.

The proof is morally similar to that of Thm in Sec 2.2 of Lec 16 and is left as a premium exercise.

Now consider $L=B$. Then, by Sec 1.2 of Lec 16, $B\otimes_B M \xrightarrow{\sim} M$ $(b\otimes m\mapsto m)$ $\forall$ B-module M. So, $\varphi^*(B\otimes_B \cdot)\xRightarrow{\sim}\varphi^*$. We arrive at:

Corollary: The functor $\underline{Hom}_A(B,\cdot): A\text{-Mod}\to B\text{-Mod}$ is right adjoint to $\varphi^*: B\text{-Mod}\to A\text{-Mod}$.

2) Tensor product of algebras.

2.1) Construction.

Let A be a commutative ring, B, C be A-algebras (& so A-modules) $\rightsquigarrow$ A-module $B\otimes_A C$.

Proposition: $\exists!$ A-algebra str're on $B\otimes_A C$ s.t. $(b_1\otimes c_1)\cdot(b_2\otimes c_2)=b_1b_2\otimes c_1c_2$ $\forall b_1,b_2\in B,\ c_1,c_2\in C$ (w. unit $1\otimes 1$).

Proof: Uniqueness will follow b/c $B\otimes_A C=\mathrm{Span}_A(b\otimes c\mid b\in B, c\in C)$ & any bilinear map is uniquely determined by images of generators.

Now we need to show existence. The product map $B\times B\to B$ is A-bilinear $\rightsquigarrow$ $\exists!$ A-linear $\mu_B: B\otimes_A B\to B$, w. $b_1\otimes b_2\mapsto b_1b_2$. Similarly, we have $\mu_C: C\otimes_A C\to C$ $\rightsquigarrow$

$$\begin{tikzcd}
 & \mu_B\otimes\mu_C:(B\otimes_A B)\otimes_A(C\otimes_A C) \arrow[r] & B\otimes_A C \\
x\otimes y\in & (B\otimes_A C)\otimes_A(B\otimes_A C) \arrow[u, "\wr"] \arrow[ur, "(b_1\otimes c_1)\otimes(b_2\otimes c_2)\mapsto(b_1b_2)\otimes(c_1c_2)"'] & \\
(x,y)\in & (B\otimes_A C)\times(B\otimes_A C) \arrow[u] \arrow[uur] &
\end{tikzcd}$$

assoc. & commut. of $\otimes$ $\to$ $\wr$
Sec 1.2 of Lec 16

our multiplication map

So we've shown existence of A-bilinear product map. Associativity & unit axioms can be checked on tensor monomials, e.g. here is a part of unit axiom:
$(1\otimes 1)(b\otimes c)=(1\otimes b)\otimes(1\otimes c)=b\otimes c.$ $\square$

Rem: B, C are commutative $\Rightarrow$ so is $B\otimes_A C$.

2.2) Coproduct.

Theorem: Let B, C be commutative. Then $B\otimes_A C$ is the coproduct of B & C in $\mathcal{E} := A$-CommAlg (cat'y of commutative A-algebras), i.e. the following functors are isomorphic

$$Hom_{\mathcal{E}}(B\otimes_A C, \cdot),\ Hom_{\mathcal{E}}(B,\cdot)\times Hom_{\mathcal{E}}(C,\cdot): \mathcal{E}\to Sets.$$

equiv'y: $\exists$ A-algebra homom'ms $\iota^B: B\to B\otimes_A C$, $\iota^C: C\to B\otimes_A C$ s.t. $\forall$ alg. homom's $\varphi^B: B\to \mathcal{D}$, $\varphi^C: C\to\mathcal{D}$, where $\mathcal{D}$ is a commutative A-algebra, $\exists!$ A-alg. homom. $\varphi: B\otimes_A C\to\mathcal{D}$ w. $\varphi^B = \varphi\circ\iota^B (: B\to\mathcal{D})$ & $\varphi^C = \varphi\circ\iota^C: C\to\mathcal{D}$.

Proof: Constr'n of ι^B, ι^C: $\iota^B(b) := b\otimes 1$, $\iota^C(c) := 1\otimes c$. The conditions on φ are $\varphi(b\otimes 1) = \varphi^B(b)$, $\varphi(1\otimes c) = \varphi^C(c) \Leftrightarrow [b\otimes c = (b\otimes 1)(1\otimes c)]$

(*) $$\varphi(b\otimes c) = \varphi^B(b)\varphi^C(c).$$

We need to show $\exists!$ A-algebra homom'm $\varphi: B\otimes_A C\to\mathcal{D}$ satisf. (*). The map $B\times C\to\mathcal{D}$, $(b,c)\mapsto \varphi^B(b)\varphi^C(c)$ is A-bilinear, so $\exists!$ <u>A-linear</u> φ satisfying (*).

What remains to check is: φ respects ring multipl'n (unit is clear), enough to do this on tensor monomials

$$\varphi(b_1\otimes c_1\cdot b_2\otimes c_2) = \varphi(b_1b_2\otimes c_1c_2) = \varphi^B(b_1b_2)\varphi^C(c_1c_2) =$$
$$= \varphi^B(b_1)\varphi^B(b_2)\varphi^C(c_1)\varphi^C(c_2) = [\mathcal{D} \text{ is comm've}] = (\varphi^B(b_1)\varphi^C(c_1))\cdot$$
$$(\varphi^B(b_2)\varphi^C(c_2)) = \varphi(b_1\otimes c_1)\varphi(b_2\otimes c_2)$$ $\square$

Example: $B = A[x_1,\dots,x_k]/(f_1,\dots f_{k'})$, $C = A[y_1,\dots y_\ell]/(g_1,\dots g_{\ell'})$. Then $B\otimes_A C \simeq A[x_1,\dots x_k, y_1,\dots y_\ell]/(\underbrace{f_1,\dots f_{k'}}_{\text{on } x_1,\dots x_k}, \underbrace{g_1,\dots, g_{\ell'}}_{\text{on } y_1,\dots y_\ell})$, denote the right hand side by $\mathcal{D}$.

Will show isomorphism of functors: $F_{\mathcal{D}} \overset{\sim}{\Rightarrow} F_B \times F_C$ (where $F_{\mathcal{D}} = \mathrm{Hom}_{\mathcal{E}}(\mathcal{D}, \cdot)$: $\mathcal{E} \to \mathrm{Sets}$ & F_B, F_C are defined similarly), then we are done by the uniqueness of representing object, Sec 2 of Lec 13.

Define another functor $F'_B : \mathcal{E} \to \mathrm{Sets}$ sending a comm've A-algebra R to $\{(r_1, \dots, r_k) \in R^k \mid f_i(r_1, \dots, r_k) = 0, i = 1, \dots k'\}$ and an A-algebra homomorphism $\varphi: R^1 \to R^2$ to $F'_B(\varphi): F'_B(R^1) \to F'_B(R^2)$, $(r_1, \dots, r_k) \mapsto (\varphi(r_1), \dots, \varphi(r_k))$.
-well-defined map b/c $\varphi(r_1) \dots \varphi(r_k)$ satisfy the rel'ns of $r_1, \dots r_k$.

Then $F_B \overset{\sim}{\Rightarrow} F'_B$: $\varphi \in \mathrm{Hom}_{A\text{-}Alg}(B, R)$ is sent to $(\varphi(\bar{x}_1), \dots, \varphi(\bar{x}_r)) \in R^k$ here $\bar{x}_i$ = image of x_i in B; $(\varphi(\bar{x}_1), \dots, \varphi(\bar{x}_k)) \in F'_B(R)$ b/c $f_i(\varphi(\bar{x}_1), \dots \varphi(\bar{x}_k)) = \varphi(f_i(x_1, \dots x_k)) = 0$, the map $\eta_R : \varphi \mapsto (\varphi(\bar{x}_1), \dots, \varphi(\bar{x}_r))$ is a bijection (by the description of homomorphisms from algebras given by generators & relations, Exercise 2 in Sec 0 of Lec 2). To show (η_R) constitute a functor (iso)morphism is an exercise.

Similarly, we have $F_C \overset{\sim}{\Rightarrow} F'_C$, $F_{\mathcal{D}} \overset{\sim}{\Rightarrow} F'_{\mathcal{D}}$. That $F'_{\mathcal{D}} \overset{\sim}{\Rightarrow} F'_B \times F'_C$ is an exercise. This completes the example.

Exercise: Let g_i^B be the image of $g_i \in A[x_1, \dots x_\ell]$ in $B[x_1, \dots x_\ell]$. Note the $B \otimes_A C$ is a B-algebra via ι^B. Show that
$$B \otimes_A C \simeq B[x_1, \dots x_\ell]/(g_1^B, \dots, g_{\ell'}^B)$$

Bonus: (co)induction of group representations.

This bonus is aimed at students who took Math 353 in Spring 2023 (or know relevant representation theory). It's also based on Bonuses to Lecs 3 and 15.

Let A, B be general (associative unital) rings & $\varphi: A \to B$ be a homomorphism. Then it still makes sense to consider functors $B \otimes_A \bullet: A\text{-Mod} \longrightarrow B\text{-Mod}$ & $\operatorname{Hom}_A(B, \bullet): A\text{-Mod} \to B\text{-Mod}$ (for an A-module M, the B-module structure on $\operatorname{Hom}_A(B, M)$ is given by $[b\varphi](b') = \varphi(b'b)$). They are left & right adjoint of φ^*.

An interesting situation is as follows. Let $H \subset G$ be finite groups. Let $\mathbb{F}$ be a field. Set $A = \mathbb{F}H$, $B = \mathbb{F}G$ and let φ be the inclusion $A \hookrightarrow B$. It turns out that the functors $B \otimes_A \bullet$ & $\operatorname{Hom}_A(B, \bullet)$ in this case are isomorphic, both are referred to as the induction of group representations. The claim that these functors are adjoint to the pullback functor (a.k.a. the restriction functor) is known as the Frobenius reciprocity. This was discussed in Lectures 14 & 15 of MATH 353.