Lecture 19: Exactness & projective modules, II.

1) Projective & flat modules.
2) Projective vs locally free.

Refs: [E], A.3.2, 6.1, 6.3.

BONUSES: 1) 5-lemma
2) Injective modules.

1) Projective & flat modules.

1.1) Projective modules, revisited.

Let P be an A-module. We know that the functor $\underline{Hom_A}(P,\cdot): A\text{-Mod} \rightarrow A\text{-Mod}$ is left exact, Sec 2.3 of Lec 18. A natural question is when (= for which P) it's exact, equivalently, (modulo the left exactness, Sec 2.4 of Lec 18), when:

$$Hom_A(P,M) \twoheadrightarrow Hom_A(P,N) \ \forall\ M \twoheadrightarrow N \text{ (surj've } A\text{-linear).}$$

Recall (Sec 0.2 of Lec 5) that P is projective if $\exists$ A-module P' s.t. $P \oplus P'$ is free.

Thm: TFAE

(1) P is projective.

(2) $\forall$ A-linear surjection $\pi: M \twoheadrightarrow P$ $\exists$ A-linear $\iota: P \rightarrow M$ s.t. $\pi \circ \iota = id_P$ (say, π "splits")

(3) $\underline{Hom_A}(P,\cdot)$ is exact.

Proof: Let's do easier parts first.

$(3)\Rightarrow(2)$: $\mathrm{Hom}_A(P,M) \xrightarrow{\pi\circ ?} \mathrm{Hom}_A(P,P)$ is surjective $\Rightarrow$ $\exists\, \iota\in \mathrm{Hom}_A(P,M)$ s.t. $\pi\circ\iota = id_P$, which is (2).

$(2)\Rightarrow(1)$: Pick $\iota: M\to P$ w. $\pi\circ\iota = id_P \Rightarrow \iota$ is injective, so $P\simeq \mathrm{im}\,\iota$.

Exercise: $\pi\circ\iota = id_P \Rightarrow M = \ker\pi \oplus \mathrm{im}\,\iota$ (hint: $m = \underbrace{(m-\iota(\pi(m))}_{\in \ker \pi} + \underbrace{\iota(\pi(m)))}_{\in \mathrm{im}\,\iota}$)

We apply this to $\pi: M := A^{\oplus I} \twoheadrightarrow P$, $(a_i)_{i\in I}\mapsto \sum_{i\in I} a_i p_i$, for some generators $p_i, i\in I$, of P to get (1) w. $P' = \ker \pi$.

$(1)\Rightarrow(3)$: Let $P_i, i\in I$, be some A-modules. We claim that

$(*)$ $\underline{\mathrm{Hom}}_A(\bigoplus_{i\in I} P_i, \cdot)$ is exact $\Leftrightarrow$ $\underline{\mathrm{Hom}}_A(P_i,\cdot)$ is exact $\forall i\in I$.

Recall, Rem in Sec 1.2 of Lec 4, that we have a natural isomorphism

$$\eta_M: \mathrm{Hom}_A(\bigoplus_{i\in I} P_i, M) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_A(P_i, M),\ \tau\mapsto (\tau|_{P_i})_{i\in I}.$$

These isomorphisms form a functor (iso)morphism, in particular, for $\varphi: M\twoheadrightarrow N$, the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Hom}_A(\bigoplus_{i\in I} P_i, M) \arrow[r, "\sim"] \arrow[d, "\varphi\circ ?"] & \prod_{i\in I} \mathrm{Hom}_A(P_i, M) \arrow[d, "(\varphi\circ ?)_{i\in I}"] \\ \mathrm{Hom}_A(\bigoplus_{i\in I} P_i, N) \arrow[r, "\sim"] & \prod_{i\in I} \mathrm{Hom}_A(P_i, N) \end{tikzcd}$$

It follows that the left arrow is surjective iff the right is surjective iff $\forall\, i$ the map $\mathrm{Hom}_A(P_i, M) \to \mathrm{Hom}_A(P_i, N)$ is surjective. $(*)$ follows.

Now we get back to proving $(1) \Rightarrow (3)$. Recall $\underline{\mathrm{Hom}}_A(A,\cdot) \xrightarrow{\sim}$ $\mathrm{id}_{A\text{-Mod}}$, in particular, exact. By using $\Leftarrow$ of $(*)$ w. any set $I := J$ & $P_j = A\ \forall j \in J$, we see that $\underline{\mathrm{Hom}}_A(A^{\oplus J}, \bullet)$ is exact. Now by using $\Rightarrow$ of $(*)$ w. $A^{\oplus J} = P \oplus P'$ $(|I| = 2)$, we see that $\underline{\mathrm{Hom}}_A(P, \cdot)$ is exact. $\square$

1.2) Flat modules.

Definition: An A-module F is *flat* if $F \otimes_A \bullet : A\text{-Mod} \to A\text{-Mod}$ is exact ($\Leftrightarrow$ sends injections to injections since, in general, $F \otimes_A \cdot$ is right exact).

Examples:

(I) $A^{\oplus I}$ is flat b/c $A^{\oplus I} \otimes_A \bullet \xrightarrow{\sim} \bullet^{\oplus I}$, this follows from Sec 2.1 of Lec 12. If $N \hookrightarrow M$, then $N^{\oplus I} \hookrightarrow M^{\oplus I}$. So $A^{\oplus I}$ is flat.

(II) If $M_1 \oplus M_2$ is flat, then M_1, M_2 are flat, this is similar to the analogous part in the proof of the Thm. This & (I) imply that projective modules are flat.

(III) Let $S \subset A$ be a multiplicative subset. Since $A[S^{-1}] \otimes_A \bullet \xrightarrow{\sim} \bullet[S^{-1}]$ (Sec 1.1 of Lec 17) & $\bullet[S^{-1}]$ is exact (Sec 2.3 of Lec 18), $A[S^{-1}] \otimes_A \bullet$ is exact, so $A[S^{-1}]$ is a flat A-module.

Rem: In a nice class of A-modules, all flat modules are projective. Namely, recall that an A-module M is finitely presented if $\exists\ \varphi: A^{\oplus k} \to A^{\oplus \ell}$ (for some k, ℓ) s.t. $M \simeq A^{\oplus \ell}/\operatorname{im}\varphi$. In particular, finitely presented $\Rightarrow$ finitely generated. If A is Noetherian, then $\Leftarrow$ holds: if $\pi: A^{\oplus \ell} \twoheadrightarrow M$, then $\ker \pi$ is finitely generated, so $A^{\oplus k} \twoheadrightarrow \ker\pi$ (for some k). For φ take the composition $A^{\oplus k} \twoheadrightarrow \ker\pi \hookrightarrow A^{\oplus \ell}$.

Now, we have the following fact:

Fact: Any finitely presented flat module is projective.

A proof can be found in [E], Corollary 6.6.

2) Projective vs locally free.

Here A is a commutative ring.

Our goal for the rest of the topic is to give yet another characterization of finitely presented projective modules. Recall that for prime (in particular, maximal) ideal $\mathfrak{p} \subset A$ we write $\bullet_{\mathfrak{p}} := \bullet[(A \setminus \mathfrak{p})^{-1}]$. Recall the following def'n (from HW3):

Definition: An A-module M is locally free if $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module $\forall$ max. ideal $\mathfrak{m} \subset A$.

Thm (Serre): Let P be a finitely presented A-module. TFAE

1) P is projective
2) P is locally free.

Unlike the previous theorem whose proof is quite formal, this one requires several new ideas. Today we'll do 2) $\Rightarrow$ 1) proving

Prop'n: If $P_{\mathfrak{m}}$ is projective $A_{\mathfrak{m}}$-module $\forall$ max. ideal $\mathfrak{m} \subset A$, then P is a projective A-module.

The proof crucially uses the categorical characterization of projectives (Sec 1.1): P is projective $\Leftrightarrow$ $\underline{\mathrm{Hom}_A}(P, \cdot)$ is exact (the original definition of projectives doesn't help).

Proof: We need to show: $\forall$ surjective $\varphi \in \mathrm{Hom}_A(M, N)$, the map $\tilde{\varphi}: \mathrm{Hom}_A(P, M) \xrightarrow{\varphi \circ ?} \mathrm{Hom}_A(P, N)$ is surjective. The proof is in two steps:

I) Prove $\tilde{\varphi}_{\mathfrak{m}}: \mathrm{Hom}_A(P, M)_{\mathfrak{m}} \twoheadrightarrow \mathrm{Hom}_A(P, N)_{\mathfrak{m}}$ $\forall$ max. ideal $\mathfrak{m} \subset A$.

II) Prove that if $\tilde{M}, \tilde{N}$ are A-modules, & $\tilde{\varphi} \in \mathrm{Hom}_A(\tilde{M}, \tilde{N})$ are s.t. $\tilde{\varphi}_{\mathfrak{m}}$ is surjective $\forall$ max. ideal $\mathfrak{m} \subset A$, then $\tilde{\varphi}$ is surjective.

Then we set $\tilde{M} = \mathrm{Hom}_A(P, M)$, $\tilde{N} = \mathrm{Hom}_A(P, N)$ & take $\tilde{\varphi}$ as above getting that $\tilde{\varphi}$ is surjective $\Leftrightarrow$ P is projective.

I: Recall (Prob. 2.2 in HW 3) that for any finitely presented A-module P & multiplicative subset $S \subset A$, have

$$\mathrm{Hom}_A(P, M)[S^{-1}] \xrightarrow{\sim} \mathrm{Hom}_{A[S^{-1}]}(P[S^{-1}], M[S^{-1}]),\ \frac{\tau}{s} \mapsto \left[\frac{p}{t} \mapsto \frac{\tau(p)}{ts}\right]$$

($\tau \in \mathrm{Hom}_A(P, M)$). We apply this to $S = A \setminus \mathfrak{m}$.

The following diagram is commutative (exercise):

$$\begin{tikzcd}
\mathrm{Hom}_A(P,M)_{\mathfrak{m}} \arrow[r, "\sim"] \arrow[d, "\tilde{\varphi}_{\mathfrak{m}}"] & \mathrm{Hom}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}, M_{\mathfrak{m}}) \arrow[d, "\varphi_{\mathfrak{m}}\circ ?"] \\
\mathrm{Hom}_A(P,N)_{\mathfrak{m}} \arrow[r, "\sim"] & \mathrm{Hom}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}, N_{\mathfrak{m}})
\end{tikzcd}$$

If φ is surjective, so is $\varphi_{\mathfrak{m}}$ (Sec 1.3 in Lec 10). Since $P_{\mathfrak{m}}$ is projective, the right arrow is surjective. So is the left arrow. This gives (I).

II: Let $K = \tilde{N}/\mathrm{im}\,\tilde{\varphi}$. Since $\mathrm{im}(\tilde{\varphi}_{\mathfrak{m}}) = (\mathrm{im}\,\tilde{\varphi})_{\mathfrak{m}}$ we see $K_{\mathfrak{m}} = \{0\}$ $\forall$ max. ideals $\mathfrak{m} \subset A$. Pick $k \in K$, we want to show $k=0$. The equality $K_{\mathfrak{m}}=0$ gives: $\exists$ $f(\mathfrak{m}) \notin \mathfrak{m}$ s.t. $f(\mathfrak{m})k=0$. Let $I := \mathrm{Ann}_A(k) = \{a \in A \mid ak=0\} \Rightarrow f(\mathfrak{m}) \in I$. Have $k \neq 0 \Leftrightarrow 1 \notin I \Leftrightarrow I$ is contained in a max. ideal, say $\mathfrak{m}$. But $f(\mathfrak{m}) \in I \setminus \mathfrak{m}$, contradiction that gives $k=0$ and finishes the proof of II and of Proposition $\square$

BONUS 1: 5-lemma.

This important result sometimes allows to check whether a module homomorphism is an isomorphism.

Theorem: Suppose we have a commutative diagram of A-modules & their homomorphisms:

$$\begin{tikzcd}
M_1 \arrow[r, "\tau_1"] \arrow[d, "\varphi_1"] & M_2 \arrow[r, "\tau_2"] \arrow[d, "\varphi_2", "\wr"'] & M_3 \arrow[r, "\tau_3"] \arrow[d, "\varphi_3"] & M_4 \arrow[r, "\tau_4"] \arrow[d, "\varphi_4", "\wr"'] & M_5 \arrow[d, hook, "\varphi_5"] \\
M_1' \arrow[r, "\tau_1'"] & M_2' \arrow[r, "\tau_2'"] & M_3' \arrow[r, "\tau_3'"] & M_4' \arrow[r, "\tau_4'"] & M_5'
\end{tikzcd}$$

Assume both rows are exact, φ_2, φ_4 are isomorphisms, φ_1 is surjective and φ_5 is injective. Then φ_3 is an isomorphism.

Proof: Let's prove φ_3 is surjective. The proof is by "diagram chase." Pick $m_3' \in M_3'$. We want to show $m_3' \in \operatorname{im} \varphi_3$.

Set $m_4' := \tau_3'(m_3')$, $m_4 := \varphi_4^{-1}(m_4')$, $m_5 := \tau_4(m_4)$. Note that, since the bottom row is exact, $\tau_4'(m_4') = \tau_4'(\tau_3'(m_3)) = 0$. Since the rightmost square is commutative, $\varphi_5(m_5) = \varphi_5(\tau_4(m_4)) = \tau_4'(\varphi_4(m_4)) = \tau_4'(m_4') = 0$. Since φ_5 is injective, $m_5 = 0$. So $m_4 \in \ker \tau_4 = \operatorname{im} \tau_3$. Pick $m_3 \in M_3$ w. $\tau_3(m_3) = m_4$. Since the 2nd square from the right is commutative, $m_4' = \varphi_4(\tau_3(m_3)) = \tau_3'(\varphi_3(m_3))$. So $\tau_3'(m_3' - \varphi_3(m_3)) = 0$ $\Rightarrow$ $m_3' - \varphi_3(m_3) \in \ker \tau_3' = \operatorname{im} \tau_2'$. Take $m_2' \in M_2'$ w. $\tau_2'(m_2') = m_3' - \varphi_3(m_3)$. Set $m_2 = \varphi_2^{-1}(m_2')$. Since the 2nd square from the left is commutative, we get $\varphi_3(\tau_2(m_2)) = \tau_2'(\varphi_2(m_2)) = \tau_2'(m_2') = m_3' - \varphi_3(m_3)$.

So $m_3' = \varphi_3(\tau_2(m_2)) + \varphi_3(m_3) \in \operatorname{im}\varphi_3$.

The proof that φ_3 is injective is similar and is left as an exercise. $\square$

Let's provide an application of the 5 lemma and get an alternative proof of $\cdot[S^{-1}] \xrightarrow{\sim} A[S^{-1}]\otimes_A \cdot$, see Sec 1.1 of Lec 17. Let M be an A-module. Apply the universal property of localization to $M \longrightarrow A[S^{-1}]\otimes_A M$, $m \mapsto 1\otimes m$. We get an $A[S^{-1}]$-linear map $\eta = M[S^{-1}] \longrightarrow A[S^{-1}]\otimes_A M$ w. $\frac{m}{s} \mapsto \frac{1}{s}\otimes m$. Note that for $A=M$, under the identification $A[S^{-1}]\otimes_A A \xrightarrow{\sim} A[S^{-1}]$, $b\otimes a \mapsto b\frac{a}{1}$, η_A becomes the identity. So η_A is an isomorphism. Similarly, $\eta_{A^{\oplus I}}$ is an isomorphism, as both functors send direct sums to direct sums (details are an exercise).

Note that the maps η_M constitute a functor morphism (another exercise). Now to see that η_M is an isomorphism, we include M into an exact sequence of the form $A^{\oplus J} \to A^{\oplus I} \to M \to 0$, this can be done because every module admits a surjection from a free module. Both functors $\mathcal{F} := \cdot[S^{-1}]$ & $\mathcal{G} = A[S^{-1}]\otimes_A \cdot$ are right exact so we get exact sequences

$$\mathcal{F}(A^{\oplus J}) \longrightarrow \mathcal{F}(A^{\oplus I}) \longrightarrow \mathcal{F}(M) \longrightarrow 0 \quad \text{and}$$
$$\mathcal{G}(A^{\oplus J}) \longrightarrow \mathcal{G}(A^{\oplus I}) \longrightarrow \mathcal{G}(M) \longrightarrow 0$$

Since η is a functor morphism, these sequences fit into a a commutative diagram:

$$\begin{tikzcd}
\mathcal{F}(A^{\oplus J}) \arrow[r] \arrow[d, "\wr", "\eta_{A^{\oplus J}}"'] & \mathcal{F}(A^{\oplus I}) \arrow[r] \arrow[d, "\wr", "\eta_{A^{\oplus I}}"'] & \mathcal{F}(M) \arrow[r] \arrow[d, "\eta_M"] & 0 \arrow[r] \arrow[d, "\wr"] & 0 \arrow[d, "\wr"] \\
G(A^{\oplus J}) \arrow[r] & G(A^{\oplus I}) \arrow[r] & G(M) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We can view this as the commutative diagram in the 5 lemma (by adding another 0). Now we use the 5 lemma to conclude that η_M is an isomorphism.

BONUS 2: injective modules.

Let A be a commutative ring.

Definition: An A-module I is injective if $\mathrm{Hom}_A(\cdot, I)$: $A\text{-Mod}^{opp} \longrightarrow A\text{-Mod}$ is exact (equivalently, for an inclusion $N \hookrightarrow M$ the induced homomorphism

$$\mathrm{Hom}_A(I, M) \longrightarrow \mathrm{Hom}_A(I, N) \text{ is surjective}).$$

The definition looks very similar to that of projective modules, however the properties of injective & projective modules are very different! Projective modules – especially finitely generated ones – are nice, but injective modules are quite ugly, they are almost never finitely generated.

The simplest ring is $\mathbb{Z}$. Let's see what being injective means for $\mathbb{Z}$.

Definition: An abelian group M is divisible if $\forall\, m \in M, a \in \mathbb{Z}$ $\exists\, m' \in M$ s.t. $am' = m$.

Example: The abelian group $\mathbb{Q}$ is divisible. So is $\mathbb{Q}/\mathbb{Z}$.

Proposition 1: For an abelian group M TFAE:

(a) M is injective

(b) M is divisible

Sketch of proof: (a) $\Rightarrow$ (b): apply

$$N \hookrightarrow M \Rightarrow \mathrm{Hom}_A(I, M) \twoheadrightarrow \mathrm{Hom}_A(I, N) \qquad (*)$$

to $M = \mathbb{Z}$, $N = a\mathbb{Z}$.

(b) $\Rightarrow$ (a) is more subtle. The first step is to show that if $(*)$ holds for $N \subset M$, then it holds for $N + \mathbb{Z}m \subset M$ $\forall\, m \in M$. So $(*)$ holds for all fin. gen'd submodules $N \subset M$. Then a clever use of transfinite induction yields $(*)$ for all submodules of M. $\square$

We can get examples of injective modules for more general rings as follows. Note that for an abelian group M, the group $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ is an A-module. The proof of the following is based on $\underline{\mathrm{Hom}_{\mathbb{Z}}}(A, \cdot)$ being right adjoint to the forgetful functor $A\text{-Mod} \to \mathbb{Z}\text{-Mod}$. With this, the proof of the following is a premium exercise.

Proposition 2: If M is injective as an abelian group, then $Hom_{\mathbb{Z}}(A, M)$ is an injective A-module.

Finally, using this proposition one can show that every A-module embeds into an injective one (the corresponding statement for projectives – that every module admits a surjection from a projective module – is easy b/c every free module is projective). This claim is important for Homological algebra.