Lecture 20: Exactness & projective modules, III.

0) Recap & road map.
1) Nakayama lemma.
2) Projective modules over local rings.
Ref: [AM], Sec 2.5.
BONUS: Why to care about Serre's Thm, briefly.

0) Recap & road map.

In this lecture we complete the proof of the following.

Thm (Serre): Let P be a finitely presented A-module. TFAE
1) P is projective
2) P is locally free, i.e. $P_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ $\forall$ max. ideal $\mathfrak{m} \subset A$.

We have proved 2) $\Rightarrow$ 1) last time and will prove 1) $\Rightarrow$ 2) today. Here's how this is going to work.

By the initial definition: P is projective $\Leftrightarrow$ $P \oplus P' \simeq A^{\oplus I}$ for some A-module P'. Since $(P\oplus P')_{\mathfrak{m}} \simeq P_{\mathfrak{m}} \oplus P'_{\mathfrak{m}}$ & $(A^{\oplus I})_{\mathfrak{m}} \simeq (A_{\mathfrak{m}})^{\oplus I}$ (this is true for the localization in general, Sec 1.2 of Lec 10), we see that $P_{\mathfrak{m}}$ is projective over $A_{\mathfrak{m}}$. Also since P is finitely generated over A, we see that $P_{\mathfrak{m}}$ is finitely generated over A.

Also recall, Sec 2 of Lec 10, that $A_{\mathfrak{m}}$ is local, i.e. has the unique maximal ideal (which equals $\mathfrak{m}_{\mathfrak{m}}$). So, the proof of is reduced to:

Proposition: Every finitely generated projective module over a local ring is free.

We will prove this proposition in Sec 2 based on Nakayama lemma – a very important theorem about modules over local rings.

1) Nakayama lemma.

Let A be a local ring with maximal ideal $\mathfrak{m}$. Here is the most fundamental result about modules over A.

Thm (Nakayama lemma). Let M be a finitely generated A-module. If $\mathfrak{m}M = M$, then $M = \{0\}$.

Remark: One needs A to be local for the theorem to be true: take a field $\mathbb{F}$ & $A := \mathbb{F}\times\mathbb{F}$, $\mathfrak{m}$, $M = \{(x,0) \mid x\in\mathbb{F}\}$. Then $\mathfrak{m}M = M$.

1.1) Cayley-Hamilton type lemma.

Here's a lemma to be used in the proof of Nakayama lemma & on several other occasions later in the course. Let A be a comm've ring.

Lemma: Let M be a finitely generated A-module, $I \subset A$ an ideal, $\varphi: M \to M$ A-linear map s.t $\varphi(M) \subset IM$. Then there is a polynomial $f(x) \in A[x]$ of the form

(*) $f(x) = x^n + a_1 x^{n-1} + \ldots + a_n$ with $a_k \in I^k$ $\forall k$

s.t $f(\varphi)=0$.

Proof: Note that M upgrades to an $A[x]$-module w. x acting by φ. Pick generators $m_1, \dots m_n \in M$. We have elements $a_{ij} \in I$, $i=1,\dots n$ s.t.

$$(1) \qquad x m_i = \sum_{j=1}^{n} a_{ij} m_j$$

Form the matrix $X = xI - (a_{ij})$. Then $\det(X) \in A[x]$. Note that $\det(X)$ is a polynomial $f(x)$ as in condition (*) (exercise: hint - use that $\det(X) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}$ & $x_{ij} \in \delta_{ij} x + I$). Also note that $\det(X)$ acts by $f(\varphi)$ on M. So it's enough to show that $\det(X)$ acts by 0.

Let $\vec{m} = (m_1, \dots m_n)$ viewed as a column vector. Then $X\vec{m} = \vec{0}$ by (1). Consider the "adjoint" matrix $X' = (x'_{ij})$ w. $x'_{ij} = (-1)^{i+j} \cdot \det$ (the matrix obtained from X by removing row #i & column #j) so that $X'X = \det(X) \cdot \mathrm{Id}$. Then $X\vec{m} = \vec{0} \Rightarrow \det(X)\vec{m} = X'X\vec{m} = \vec{0} \Rightarrow$

$$(2) \qquad \det(X) m_i = 0 \ \forall i.$$

Since $m_1, \dots m_n$ span the A- (and hence $A[x]$-) module M, (2) $\Rightarrow f(\varphi)m = \det(X)m = 0 \ \forall m \in M$. This finishes the proof $\square$

1.2) Proof of Nakayama lemma.

Proof of Thm: We apply Lemma to $B := A$, $I := \mathfrak{m}$, $\varphi = id$: the condition $M = \mathfrak{m}M$ means $\varphi(M) \subset IM$. We conclude $0 = f(\varphi) = (1 + a_1 + \dots + a_n)\varphi$. Set $a = a_1 + \dots + a_n \in \mathfrak{m}$. Note that

$1 + a \notin \mathfrak{m} \Leftrightarrow (1+a) \not\subset \mathfrak{m} \Leftrightarrow$ [$\mathfrak{m}$ is the unique max. ideal] $(1+a) = A$

$\Leftrightarrow 1+a$ is invertible. So $(1+a)\varphi = 0 \Rightarrow \varphi = 0$. But $\varphi = id_M$, so $M = \{0\}$. $\square$

1.3) Corollary of Nakayama lemma.

Let A be a local ring w. maximal ideal $\mathfrak{m}$ ($\Rightarrow A/\mathfrak{m}$ is a field).

Corollary: Let M be a finitely generated A-module & $m_1, \dots m_k \in M$. Let $\bar{m}_1, \dots, \bar{m}_k$ be the images of $m_1, \dots, m_k$ in $M/\mathfrak{m}M$. If $\bar{m}_1, \dots, \bar{m}_k$ span the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$, then $m_1, \dots, m_k$ span A-module M.

Proof: Set $N := Span_A(m_1, \dots m_k)$. Note that the composed map $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is surjective $\Leftrightarrow M = N + \mathfrak{m}M \Leftrightarrow \forall m \in M \ \exists a_1, \dots a_k \in \mathfrak{m}, m_1, \dots m_k \in M \mid m - \sum_{i=1}^{k} a_i m_i \in N \Leftrightarrow \mathfrak{m}(M/N) = M/N$. The A-module M/N is finitely generated. Applying the Nakayama lemma, we get $M/N = \{0\} \Leftrightarrow M = N$. $\square$

Exercise: Let M_1, M_2 be finitely generated A-modules & $\varphi \in Hom_A(M_1, M_2)$. Then φ is surjective iff the induced map $M_1/\mathfrak{m}M_1 \to M_2/\mathfrak{m}M_2$ is surjective.

2) Projective modules over local rings

We will use the Nakayama lemma to prove the proposition from Section 0:

Prop'n: Every finitely generated projective module over a local ring A is free.

Proof: Let $\mathfrak{m} \subset A$ denote the maximal ideal, so $P/\mathfrak{m}P$ is a vector space over the field $A/\mathfrak{m}$. Since P is fin. generated over A, the vector space $P/\mathfrak{m}P$ is fin. dimensional. Let $\overline{m}_1, \ldots \overline{m}_\ell$ be a basis, and let $m_1, \ldots m_\ell$ be preimages of these elements in P (under $P \twoheadrightarrow P/\mathfrak{m}P$). By Corollary in Section 1.3, $P = \mathrm{Span}_A(m_1, \ldots, m_\ell)$, equivalently the homomorphism

$$\pi: A^{\oplus \ell} \longrightarrow P, \quad (a_1, \ldots, a_\ell) \mapsto \sum_{i=1}^{\ell} a_i m_i$$

is surjective. We want to show it's an isomorphism.

Note that $A^{\oplus \ell}/\mathfrak{m}A^{\oplus \ell}$ is naturally (in particular, $A/\mathfrak{m}$-linearly) identified w. $(A/\mathfrak{m})^{\oplus \ell}$. The homomorphism $(A/\mathfrak{m})^{\oplus \ell} \longrightarrow P/\mathfrak{m}P$ induced by π sends the standard basis e_i to the basis element $\overline{m}_i$, so is an isomorphism.

Since P is projective, Thm from Sec. 3.1 of Lec 21 (and its proof) shows that $A^{\oplus \ell} \simeq P \oplus P'$ w. $P' := \ker \pi$. It follows that

$$(A/\mathfrak{m})^{\oplus \ell} \simeq A^{\oplus \ell}/\mathfrak{m}A^{\oplus \ell} \simeq (P \oplus P')/\mathfrak{m}(P \oplus P') \simeq P/\mathfrak{m}P \oplus P'/\mathfrak{m}P'.$$

But $(A/\mathfrak{m})^{\oplus \ell}$ & $P/\mathfrak{m}P$ are isomorphic dim ℓ vector spaces over $A/\mathfrak{m}$. So $P'/\mathfrak{m}P' = \{0\}$. The A-module P' admits a surjective homomorphism from $A^{\oplus \ell}$. So, it's finitely generated.

Applying Nakayama lemma to $P' = \mathfrak{m}P'$ we see that $P' = \{0\}$. So π is an isomorphism. $\square$

BONUS: Why to care about Serre's Thm, briefly.

Being projective is an important "niceness" property from the point of category theory (Hom from such object behaves well). It turns out that being locally free is important geometrically - it says that this module corresponds to a "vector bundle". Below is a short account on this, a more detailed account will follow in a bonus Algebraic geometry lecture.

A crucially important class of commutative rings is rings (often $\mathbb{R}$- or $\mathbb{C}$-algebras) of functions (w. some conditions - C^∞, holomorphic or - which is closest to us - algebraic) on "spaces", which can mean for example, manifolds. In fact, any commutative ring can be thought of as some kind of ring of functions on its "spectrum".

In various settings, maximal ideals in the ring of functions are in bijection w. the points of the space (to a point we assign the ideal of all functions vanishing at that point, compare to Sec 2.2 in Lec 2). In the algebraic setting, the localization $A_{\mathfrak{m}}$ means the "functions" defined in some neighborhood of the point corresponding to $\mathfrak{m}$ - which agrees with the name "local".

"Vector bundles" on manifolds is one of the most fundamental objects in Differential geometry (examples later). They enjoy the property of being "locally trivial". In the algebraic setting this is formalized via being "locally free".